

MATHEMATICAL PAPERS

BY

WILLIAM KINGDON CLIFFORD.

EDITED BY ROBERT TUCKER,

WITH AN INTRODUCTION BY H. J. STEPHEN SMITH.

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"If he had lived we might have known something."

London:
MACMILLAN AND CO.

1882

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Cambridge;

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MY DEAR MRS CLIFFORD,

When towards the end of April 1879 I called upon you in answer to an urgent request to do so, it was with much surprise that I heard from you that it was one of your lamented Husband's last requests in regard to his work that I should be one of two persons to be asked to see the remaining Manuscript of his admirable book on *Dynamic* through the press. Though much gratified by his kind approval of what I had written upon the portion already in print, I could hardly look upon myself as qualified to undertake such a task—who indeed could hope to succeed in fitly editing fragments which had not received his last energizing touches? Whatever diffidence may have at first deterred me, yielded on further reflection to your pressing solicitation.

At this same interview you told me that it had been suggested, I believe by Mr Spottiswoode and Mr Macmillan, that a collected edition of Clifford's Mathematical Works, including the Memoirs already printed as well as such posthumous papers as might

be found in a state fit for publication, should be brought out under the auspices of a double editorship, one editor to be a resident at Cambridge and the other to be myself. If I hesitated to undertake the editing of the Manuscript of the *Dynamic*, you may believe I was even more afraid to undertake so great a responsibility as the editing of the Papers. Finding however at an interview I had with Dr Jack that it was desired that the papers should be brought out with all speed, and learning from Mr Spottiswoode that the plan of a double editorship had fallen through, I consented to do the best I could, feeling the less hesitation in making the attempt as I was able to obtain the assistance of Mr Spottiswoode himself and of Professors Cayley, Henrici and Smith.

Thus much of explanation I feel to be due from me to show that I did not engage in so great an undertaking of my own mere motion, but that I entered upon my task with no slight idea of its magnitude, although the actuality has far surpassed my expectation.

Your own assistance and kind interest in the progress of the work have been most valuable and most encouraging to me ; and I can now only wish that the result of my efforts may deserve your approval, and may be not unworthy of the illustrious mathematician, your much loved Husband.

“ If I have done well, it is that which I desired : but if slenderly and meanly, it is that which I could attain unto.”

I need not enter into any account of the causes which have operated to bring about so tardy a completion of my labours. Sufficient it is for me to know that you have understood and allowed for the many hindrances which have arisen from circumstances connected with the fragmentary condition of some of the papers, and from the pressing claims upon my own time and upon that of the eminent mathematicians, whose advice and assistance have so ungrudgingly been extended to me.

Thanking you, my dear Mrs Clifford, for the honour you have conferred upon me in entrusting to me the work of raising this monument to the memory of one you love so well,

I remain,

Yours faithfully,

R. TUCKER.

December 30th, 1881.

DIRECTIONS TO BINDER.

| | PAGE |
|--|------|
| Lithographed sheet to face Title-Page. | |
| Plato i. figs. 1—11 to face | 54 |
| „ ii. „ 12—17 „ | 152 |
| „ iii. „ 18—22 „ | 176 |
| „ iv. „ 23—30 „ | 228 |
| „ v. „ 31—50 „ | 264 |
| „ vi. „ 51—59 „ | 414 |
| „ vii. „ 60—74 „ | 488 |
| „ viii. „ 75—82 „ | 494 |
| „ ix. „ 83—94 „ | 504 |
| „ x. „ 95—102 „ | 514 |
| „ xi. „ 104—110 „ (there is no fig. 103) | 522 |
| „ xii. „ 111—119 „ | 558 |
| „ xiii. „ 120—126 „ | 642 |

Papers now published for the first time are distinguished by an * prefixed to the number indicating their order in this volume.

CORRIGENDA.

On p. 22, for xv. read xvii.

„ 114, 6 up, read convertical.

„ 134, in connexion with the formula, refer to *Quarterly Journal of Mathematics*, No. 25. For this reference I am indebted to Mr J. F. Walker, who also suggests the following corrections “on p. 185, 4 up, for

κ read $\frac{1}{\kappa}$; p. 180, omit $\left[\frac{1}{\kappa}\right]$, l. 10, - 2, l. 11.”

„ 205, r is v in Clifford's MS. The paper is printed as in the *London Math. Society's Proceedings*.

„ 621, l. 12, read vol. xxxv, p. 21.

CONTENTS.

| | PAGE |
|--|------|
| PREFACE BY R. TUCKER | xlii |
| i. BIOGRAPHICAL | xv |
| ii. BIBLIOGRAPHICAL | xx |
| INTRODUCTION BY PROF. H. J. S. SMITH | xxxi |

PAPERS.

| | |
|--|----|
| I. ON THE TYPES OF COMPOUND STATEMENT INVOLVING FOUR CLASSES <i>Mem. Lit. and Phil. Soc., Manchester, Jan. 9, 1877.</i> | 1 |
| *II. ENUMERATION OF THE TYPES OF COMPOUND STATEMENTS. [1877] | 14 |
| III. ON SOME PORISMATIC PROBLEMS <i>Camb. Phil. Soc. Proc., Nov. 9, 1868.</i> | 17 |
| IV. PROOF THAT EVERY RATIONAL EQUATION HAS A ROOT <i>Camb. Phil. Soc. Proc., Feb. 21, 1870.</i> | 20 |
| V. ON THE SPACE-THEORY OF MATTER <i>Camb. Phil. Soc. Proc., Feb. 21, 1870.</i> | 21 |
| VI. ON JACOBIANS AND POLAR OPPOSITES <i>Oxf., Camb. and Dub. Messenger of Math. Vol. II, 229—239.</i> [1863] | 23 |
| VII. ON THE PRINCIPAL AXES OF A RIGID BODY <i>Oxf., Camb. and Dub. Messenger of Math. Vol. IV, 78—81.</i> [1867] | 34 |
| VIII. SYNTHETIC PROOF OF MIQUEL'S THEOREM <i>Oxf., Camb. and Dub. Messenger of Math. Vol. V, 124—141. [1870]</i> | 38 |
| IX. ON THE HYPOTHESES WHICH LIE AT THE BASES OF GEOMETRY <i>Nature. Vol. VIII, Nos. 183, 4. [1873]</i> | 55 |
| X. ANALOGUES OF PASCAL'S THEOREM <i>Q. J. of Pure and Applied Math. No. 23, 210—222. [1863]</i> | 72 |
| XI. ANALYTICAL METRICS <i>Q. J. of Pure and Applied Math. Nos. 25, 29, 30. [1864]</i> | 80 |

| PAPER | PAGE |
|--|------|
| XII. ON THE GENERAL THEORY OF ASYMPTOTES | 110 |
| <i>Proc. L. M. S.</i> Vol. II. 3-6. (1906) | |
| XIII. ON A GENERALIZATION OF THE THEORY OF FOLDS | 116 |
| <i>Proc. L. M. S.</i> Vol. II. 116-119. (1907) | |
| XIV. ON HYPOTHETIC RELATIONS AMONG THE FORMS OF LIEBOW QUANTUM | 119 |
| <i>Proc. L. M. S.</i> Vol. III. 9-12. (1907) | |
| *XV. ON HYPOTHETIC RELATIONS CONNECTED WITH FORMS OF LIEBOW QUANTUM. (1909) | 123 |
| *XVI. ON THE THEORY OF DISTANCE. (1909) | 130 |
| NOTE BY PROF. CAVLEY | 157 |
| ON THE THEORY OF DISTANCE | 161 |
| <i>Brit. Assoc. Report, 1909, p. 20</i> | |
| XVII. ON A CASE OF EVALUATION IN THE THEORY OF A HYPOTHETIC | 165 |
| <i>Proc. L. M. S.</i> Vol. III. 60-62. (1909) | |
| XVIII. ON A THEOREM RELATIVE TO PENTAGONAL, AND OTHER TO M _n | 168 |
| COTTERILL'S THEOREM ON PLANE POLYGONS | 168 |
| <i>Proc. L. M. S.</i> Vol. IV. 17-18. (1910) | |
| XIX. GEOMETRY ON AN ELLIPSOID | 177 |
| <i>Proc. L. M. S.</i> Vol. IV. 215-217. (1910) | |
| XX. PRELIMINARY SKETCH OF BIPARTITION | 181 |
| <i>Proc. L. M. S.</i> Vol. IV. 191-192. (1910) | |
| XXI. GRAPHIC REPRESENTATION OF THE HYPOTHETIC CONNECTION OF A PERIODIC MOTION | 201 |
| <i>Proc. L. M. S.</i> Vol. V. 11-14. (1910) | |
| XXII. ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS | 205 |
| <i>Proc. L. M. S.</i> Vol. VII. 23-29. (1915) | |
| XXIII. NOTES ON THE COMMUTATION BETWEEN THE TWO TRANSFORMATION OF ELLIPTIC FUNCTIONS | 216 |
| <i>Proc. L. M. S.</i> Vol. VII. 27-29. (1915) | |
| *ADDITIONS TO XXII. | 224 |
| *XXIV. ON IRREDUCIBLE POLYNOMIALS. (1916) | 229 |
| XXV. ON A CANONICAL FORM OF BERNARDI. HANSEN'S | 234 |
| <i>Brit. Assoc. Report, 1911, p. 10</i> | |
| XXVI. ON THE FREE MOTION UNDER NO FORCES OF A BODY SYSTEM IN AN n -FOLD HOMOLOGY. (Provisional Method) | 240 |
| <i>Proc. L. M. S.</i> Vol. VII. 52-53. (1916) | |
| XXVII. ON THE CANONICAL FORM AND DISSECTION OF A HANSEN'S BERNARDI | 241 |
| <i>Proc. L. M. S.</i> Vol. VIII. 272-281. (1917) | |

CONTENTS.

xi

| PAPER | PAGE |
|--|------|
| XXVIII. REMARKS ON THE CHEMICO-ALGEBRAICAL THEORY | 255 |
| <i>Amer. Journal of Math.</i> Vol. i. 126—8. [1878] | |
| *XXIX. NOTES ON QUANTICS OF ALTERNATE NUMBERS, USED AS A MEANS FOR DETERMINING THE INVARIANTS AND CO-VARIANTS OF QUANTICS IN GENERAL | 258 |
| <i>Proc. L. M. S.</i> Vol. x. 124—9. [1878] | |
| XXX. APPLICATIONS OF GRASSMANN'S EXTENSIVE ALGEBRA | 266 |
| <i>Amer. Journal of Math.</i> Vol. i. 350—8. [1878] | |
| *XXXI. BINARY FORMS OF ALTERNATE VARIABLES | 277 |
| <i>Proc. L. M. S.</i> Vol. x. 214—221. [1878]. | |
| XXXII. ON MR SPOTTISWOODE'S CONTACT PROBLEMS | 287 |
| <i>Phil. Trans.</i> Vol. CLXIV. Pt. 2. [1878] | |
| XXXIII. ON THE CLASSIFICATION OF LOGI | 305 |
| <i>Phil. Trans.</i> Pt. 2. 663—81. [1878] | |
| ABSTRACT | 320 |
| <i>Proc. of Royal Society</i> , No. 187. | |
| *XXXIV. ON THE POWERS OF SPHERES, [1868] | 332 |
| *XXXV. A FRAGMENT ON MATRICES. [1875] | 337 |
| *XXXVI. ON TRICIRCULAR SEXTICS. [1876] | 342 |
| *XXXVII. ON BESSEL'S FUNCTIONS, [1878] | 346 |
| *XXXVIII. ON GROUPS OF PERIODIC FUNCTIONS [1877] | 350 |
| *XXXIX. THEORY OF MARKS OF MULTIPLE THETA-FUNCTIONS. [1878] | 356 |
| *XL. ON THE DOUBLE THETA-FUNCTIONS [1876] | 368 |
| *XLI. MOTION OF A SOLID IN ELLIPTIC SPACE. [1874] | 378 |
| *XLII. FURTHER NOTE ON BICQUATERNIONS. [1876] | 385 |
| *NOTES ON BICQUATERNIONS, | 395 |
| *XLIII. ON THE CLASSIFICATION OF GEOMETRIC ALGEBRAS. [1870] | 397 |
| *XLIV. ON THE THEORY OF SCREWS IN A SPACE OF CONSTANT POSITIVE CURVATURE. [1876] | 402 |
| XLV. REMARKS ON A THEORY OF THE EXPONENTIAL FUNCTION DE- RIVED FROM THE EQUATION $\frac{du}{dt} = pu$ | 406 |
| <i>Proc. L. M. S.</i> Vol. iv. 111. [1872] | |
| XLVI. NOTES ON VORTEX-MOTION, ON THE TRIPLE-GENERATION OF THREE- BAR CURVES, AND ON THE MASS-CENTRE OF AN OCTAHEDRON | 407 |
| <i>Proc. L. M. S.</i> Vol. ix. 26—8. [1877] | |

| PAPER | | PAGE |
|---------|---|------|
| XLVII. | GEOMETRICAL THEOREM <i>Oxf., Camb. and Dub. Messenger of Math.</i> Vol. III. 31, 32. [1864] | 410 |
| XLVIII. | ON TRIANGULAR SYMMETRY <i>Math. from Ed. Times.</i> Vol. IV. 88, 9. [1865] | 412 |
| XLIX. | ON SOME EXTENSIONS OF THE FUNDAMENTAL PROPOSITION IN M. CHABLER'S THEORY OF CHARACTERISTICS <i>Math. from Ed. Times.</i> Vol. V. 40, 50. [1866] | 415 |
| I. | INSTRUMENTS USED IN MEASUREMENT <i>Handbook...Scientific Apparatus</i> , 55--9. [1876] | 410 |
| II. | INSTRUMENTS ILLUSTRATING KINEMATICS, STATICS, AND DYNAMICS <i>Handbook...Scientific Apparatus</i> , 60--77. [1876] | 424 |

APPENDIX.

| | |
|--|-----|
| ALGEBRAIC INTRODUCTION TO ELLIPTIC FUNCTIONS. [1877] | 448 |
| ON ELLIPTIC FUNCTIONS | 474 |
| NOTES OF LECTURES ON QUATERNIONS. [1877] | 478 |
| SYLLABUS OF LECTURES ON MOTION. [1878] | 516 |
| LECTURE NOTES. [1870] | 524 |
| ANALYSIS OF LONATSCHEWSKY | 531 |
| THE POLAR THEORY OF CUBICS | 532 |
| ON TRAFEGANS | 535 |
| ANALYSIS OF CREMONA'S TRANSFORMATIONS. [1869] | 538 |
| BITANGENT CIRCLES OF A CONIC. [1866] | 543 |
| OF POWER-COORDINATES IN GENERAL. [1866] | 546 |
| THEORY OF POWERS | 555 |
| REVIEWS. | |
| I. DE MORGAN'S BUDGET OF PARADOXES. [1873] | 559 |
| II. DR BOOTH'S NEW GEOMETRICAL METHODS. [1877] | 562 |
| PROBLEMS AND SOLUTIONS from the <i>Educational Times</i> | 565 |
| SYLLABUS OF TEN LECTURES TO LADIES ON GEOMETRY DELIVERED AT S. KENNINGTON. [1869] | 568 |
| SYLLABUS OF LECTURES ON SYNTHETIC GEOMETRY AND GRAPHICAL STATICS | 567 |
| NOTES. | 540 |
| INDEX | 558 |

PREFACE:

I. BIOGRAPHICAL.

II. BIBLIOGRAPHICAL.

BY

ROBERT TUCKER, M.A.,

HON. SEC. TO LONDON MATHEMATICAL SOCIETY,
SOMETIME SCHOLAR OF ST JOHN'S COLLEGE, CAMBRIDGE.

WILLIAM KINCHON CLIFFORD,

Born at Exeter, May 4th, 1845. Died at Madeira, March 3rd, 1879.

AFTER the touching Biography prefixed to the *Lectures and Essays*, written with consummate literary skill by one who knew and loved Clifford well — "as his own soul," — I feel it is unnecessary for me to say more than a few words as a preface to the present collection of his Mathematical Papers, and in the few following lines I shall confine myself to the simple statement of facts bearing upon his life and work as a Mathematician.

Clifford received his early education at Mr Templeton's school in Exeter, where in 1858 and 1859 he gained numerous distinctions in a very extended range of subjects at the University Local Examinations, both of Oxford and Cambridge. From this school he proceeded, in 1859, to King's College, London, where in like manner his efforts were crowned with success in more than one branch of study. Having in 1863 obtained a minor scholarship, he entered at Trinity College, Cambridge; at the end of his course in 1867 he graduated Second Wrangler and gained the Second Smith's Prize. In 1868, he was elected a Fellow of his College. He proceeded to M.A. in 1870; joined the Eclipse Expedition to Sicily of the same year, and was in 1871 elected Professor of Applied Mathematics and

Mechanics in University College, London, a post he occupied until the time of his death.

In June, 1874, Clifford was elected a Fellow of the Royal Society. He became a Member of the London Mathematical Society, June 18th, 1866, and served on its Council for every session from 1868—9 to 1876—7. Though he then ceased to be a member of the Council, he continued to take a warm interest in the proceedings of the Society, and on many a night he was present at the meetings when he was far from well and ought to have remained at home. I go back in imagination to the first meeting he attended—his was ever a welcome face in our room at University College and subsequently at Burlington House and Albemarle Street—it was on the evening of January 23rd, 1868, when Dr Hirst, our Treasurer, occupied the chair in the absence of our then President, Prof. Sylvester. Of the little band then present four have since been removed by death, viz., Clifford, Clerk Maxwell, Archibald Smith, and Thomas Cotterill. Clerk Maxwell made two communications: i. "on the construction of stereograms of surfaces," ii. "on the doctrine of Reciprocal Diagrams of Forces with the extension of Airy's Function of Stress from two dimensions to three." The former was made interesting by the fact that the author brought with him a real Image Stereo-

condition, which is easily obtained for a cubic but is more difficult to find in the case of equations of higher degrees. Clifford at once said that we obtain the condition required by forming an equation whose roots are the sums of the roots of the original equation taken in pairs and determining the condition that the real roots of this equation should be negative. This may serve to give an idea of his readiness and helpfulness at our meetings. He rarely rose to speak of his own accord, but when a direct appeal was made to him he was ever ready to contribute a few pertinent remarks of an eminently suggestive kind, often showing in the course of the discussion that he had carefully considered the subject for himself. In this way he often threw out hints which enabled others to advance still further the researches which they had hitherto almost looked upon as their own. His readiness is exemplified in the following extract from a letter I have received from his private tutor, the Rev. Percival Frost: "We were capital friends, yet I was so much engaged with a large number of pupils that I did not see very much of him except in a professional way. Even when he came to see me out of his working hours we used to get upon some mathematical curiosity, and both being fond of mathematics for their own sakes, we have often pursued our amusement into the small hours—once between 2 and 3—for which his tutor called him to account, good-naturedly excusing him when he heard of how he had been occupied. He often used to amuse me by solving in his head difficult problems, when some conversation like the following would take place. *Fr.* The men in the next room tell me this problem won't come out: there must be a mistake: just read it over and tell me where the setter has blundered. *Cl.* (reads it over and thinks a few minutes) I see how it is, he has, &c., &c."

Few anecdotes of Clifford's young days which show any foreshadowing of his future mathematical power have reached

¹ See also *Nature* (March 18th, 1870).

me. The following illustration, though trifling, may find a place here. His aunt (Mrs McLeod) writes: "In the year of the first Exhibition, 1851, his parents came to town, and I had the charge of him with my own child. When putting him to bed one night, I saw dear Willie looking very thoughtful. When I asked him what it was all about, he looked up in his smiling, loving way, saying, 'Aunt Annie, I don't think you would know.' But on my asking again, he said he was calculating 'how many edges (*sharp*) of a penknife it would take to go round the wheel of a coach.'

"We had a little talk about it, for it seemed impossible to me he could arrive at any conclusion. He then gave the figures and I daresay the size of the wheel; but all that I have long forgotten." The question and answer were submitted to an uncle, the late Mr Frank Kingdon, who was no mean mathematician, and he said the result was correct within a few figures.

A passage in Mr Pollock's Introduction to the *Lectures and Essays* (p. 41) would lead one to infer that Clifford turned his attention to kites at a somewhat late period of his life: it is however clear from a letter he himself wrote to Mr Miller that he had practised the flying of kites when a boy, for in 1863 he says "I have had in my mind, almost from the time I began to fly kites (I have not yet left off), the problem of finding the form of a kite-string under the action of the wind. On a rough trial, the other day, the intrinsic equation seemed not very difficult to obtain; if I get any result, I will send it to you hereafter".

One more anecdote which will give an idea of the quick-

¹ The question figures as 6009 in the *Educational Times*, July 1870, May 1880 (Reprint, vol. xxxlii, p. 59). In the same letter he adds, "I have been trying to construct a second interpretation of mechanical equations similar to that of tangential co-ordinates, but have failed hitherto. Being a firm believer in the duality of symbols, I should look upon complete failure as a proof that our symbolical system is wrong." Cf. *Lectures and Essays*, p. 41, "his thoughts often ran upon mechanical inventions."

ness and clearness of his perception of complicated relations in space, must close this brief sketch. Mr Frost, in his previously cited letter, goes on to say, "My brother A. H. Frost, who was in England for a short holiday from his missionary work in India, brought with him a complicated puzzle which was to be taken to pieces. It was not a two-dimension one, like many, but solid: about as big as a good snowball. My brother said, 'I have heard you talk of the wonderful capacity of Clifford, prove it to me by asking him to tea, and I will believe you if he can take my puzzle to pieces.' I accordingly asked him, and on my brother's giving him the thing, he, without fingering it but simply looking it over for a few minutes, put his head in his hands for some ten minutes and then took hold of the puzzle, and at once, to my brother's astonishment, dislocated it, and my brother believed in him ever afterwards."

These few personal details of one who stood in the front rank of the mathematicians of our time, though trivial, will not, I think, be deemed out of place in such a volume as this. Ever kindly and unselfish, Clifford maintained in his relations with his brother mathematicians the same amiable bearing which endeared him to a large circle of private friends.

If it were possible to ascertain the chronological order of creation of Clifford's papers, that order would have been followed in this work; but lacking the guidance of this "sacred hand," I soon found that I could not attempt a chronological arrangement and resorted to the following plan. In view of the publication of them in almost any order, I decided, for this plan I received the approval of all the members of the group whom I consulted upon the matter. In the following alphabetical sketch, I have endeavored to convey, in the best of this arrangement by assigning to all cases the idea that, by my ability, the lower limit of publications was the date of posthumous papers, of composition. It is possible that some readers may be able to furnish data for correcting my statements; such information will be gratefully accepted.

When the number of a "Reprint" problem is given in Roman type, this shows that the solution is a *reprint*, that is, a *problem**. The titles of unpublished papers are printed in italics.

The MSS even the reprints, are all very neatly written and by the style of penmanship indicate four or five different epochs; it is by attention to this fact that I have been able to assign the respective dates to the posthumous papers.

* "Reprint" stands for "Mathematical Questions with their Solutions from the Educational Times," edited by W. J. G. Clifford, 1884, and for "Lectures and Essays," edited by Leslie Hogben, 1884.

A lithographed specimen of the smallest handwriting faces the title-page*.

1863.

"On Jacobians and Polar Opposites," vi. pp. 23—33.

"Analogues of Pascal's Theorem," Nov. 20th, x. pp. 72—79.

Reprint, 1362, 1373, 1378, 1379, 1387, 1389, 1393, 1399, 1409,
4097 (1415)†, 1418, 1423, 1448, 4143 (1459).

1864.

"Analytical Metrics" (see footnote, p. 80) xi. pp. 80—109.

"Geometrical Theorem," xlvii. pp. 410, 411.

Reprint: 1319, 1394, 1416, 1421, 1442, 1443, 1468, 1479, 1486,
1505, 1507, 1514, 1517, 1519, 2108 (1526), 1585, 1605.

1865.

"On Triangular Symmetry," xlviii. pp. 412—414.

Reprint: 1497, 1638, 1652, 1675, 1679, 1680, 1691, 1724, 1732,
1733, 1748, 1750, 1775, 1795, 1823.

1866.

"On some extensions of the fundamental proposition in M. Chasles's
Theory of Characteristics," March, xlix. pp. 415—418.

"On the General Theory of Anharmonics," Nov. 22. xii. pp. 110—
114.

"Bitangent Circles of a Conic," pp. 543—545, and

"Of Power-coordinates in General," pp. 546—555, may belong to
this year.

Reprint: 1878, 1888, 4199 (1907), 1918, 1929, 1962, 1996, 2220,
2229, 2253, 4696 (2281), 2301.

* Professor Henrioi tells me that Clifford proposed to him that they should write in conjunction a series of books on Mathematics, beginning at the very commencement and carrying the subject in each case rapidly to the most advanced stages. The question was often discussed during the years in which Clifford was in full health, but this sheet is all that remains. Professor Henrioi's occupation as Examiner to the London University prevented his actively commencing his part of the work. The short abstract xi. p. 650, is suggestive as to the place which some of the papers published in this volume were intended to take in a more extended scheme.

† The number in the bracket is that belonging to the first proposal of the question, the other number is that of the re-proposed question to which the solution was given.

"*On the Contact of Surfaces of the Second Order with other Surfaces.*" Title only in British Association Report of this year.

An *Athenæum* correspondent says (Aug. 24), "Prof. Clifford explained that the radial polarization settles the fact of there being floating clouds of solid or liquid matter in the corona." No. 2339.

"On a Theorem relating to Polyhedra, analogous to Mr Cotterill's Theorem on Plane Polygons." Nov. 14. xviii. pp. 168—176.

"*The Dawn of the Sciences in Europe.*" Nov. 17. L. and E., Vol. i. p. 68.

"Geometry on an Ellipsoid." Dec. 12. xix. pp. 177—180.

1873.

"The Philosophy of the Pure Sciences." March 1, 8, 15. L. and E., Vol. i. pp. 264—340.

"On the Hypotheses which lie at the bases of Geometry." May 1, 8. ix. pp. 55—71.

"*The Relations between Science and some Modern Poetry.*" May 4. Recast as "Cosmic Emotion." L. and E., Vol. i. p. 68.

"Preliminary Sketch of Biquaternions." June 12. xx. pp. 181—200.

"On Mr Spottiswoode's Contact Problems." June 19. xxxii. pp. 287—304.

"*On some Curves of the Fifth Class,*" and "*On a Surface of Zero Curvature and Finite Extent.*" Titles only in British Association Report for this year.

"Review of De Morgan's 'Budget of Paradoxes.'" August 15. pp. 559—561.

"Graphic representation of the Harmonic Components of a Periodic Motion." Dec. 11. xxi. pp. 201—204.

"Syllabus of Lectures on Motion." (?) pp. 516—523.

Reprint: 3876, 3961, 3980, 4010, 4069.

1874.

"Review of Vol. I. of G. H. Lewes' 'Problems of Life and Mind.'" Feb. 7. L. and E., Vol. i. p. 68.

"The First and Last Catastrophe." April 12. L. and E., Vol. i. pp. 191—227.

"On the Education of the People." May 22. L. and E., Vol. I. p. 68.

"On a Message from Prof. Sylvester," and "On the General Equations of Chemical Decomposition." Titles only given in the British Association Report for this year.

"Prof. Clifford exhibited a jointed frame in illustration of some beautiful discoveries recently made in connexion with what machinists call parallel motion." *Athenaeum*, Sept. 5. No. 2445 (alluding to the discoveries of Pouucellier, Hart, and Sylvester).

"This paper (*Chemical Decomposition*) was read before Section A. The Author thinks that Chemical Equations may be brought under a general formula. Thus, $H_2 + Cl_2 = 2HCl$. If we assume that there is a structure common to the hydrogen and the chlorine atoms, also a structure confined to the hydrogen, and likewise a structure confined to the chlorine atoms, we may represent the equation thus: $XYX + XZZ = 2 XYZ$, when X represents the common structure, and Y and Z the structures which are confined to hydrogen and chlorine respectively. So $2H_2 + O_2 = 2H_2O$ may be represented thus: $2XY + XXZ = 2XXY$. These equations involve no hypotheses, because the fundamental facts of the molecular theory are now firmly established. Reasoning from these and similar equations, the author deduces the result that the ordinary equations of chemistry, such as those just stated, are expressive of facts, and that the hydrogen molecule really consists of two equal atoms." *Nature*, Sept. 24, No. 256.

"Body and Mind." Nov. 1. L. and E., Vol. II. pp. 31—70.

"On the Nature of Things in themselves." L. and E., Vol. II. pp. 71—88.

"Seeing and Thinking." December. Published in 1879 (*Nature Series*).

"Motion of a Solid in Elliptic Space." *XLI. pp. 378—384. (See note, p. 378, but I do not think the paper was written until 1876).

1875.

"The general features of the History of Science." Feb. 27, March 6, 13, 20. L. and E., Vol. I. p. 69.

"Ultramontaniam." April 28. L. and E., Vol. I. p. 68.

"The Unseen Universe." June. L. and E., Vol. I. pp. 228—253.

"On the Scientific Basis of Morals." L. and E., Vol. II. pp. 106—123.

"On the Theory of Linear Transformations: (i) the Graphical representation of Invariants; (ii) the Expansion of Unsymmetrical

Functions in Symmetrical Functions and Determinants ; (iii) the Notation of Matrices." Title only in British Association Report of this year.

"Prof. Clifford astonished the section by some remarkable applications of Grassmann's " polar multiplication " defined by the law that ba is minus ab . He applied it to the graphical representation of invariants, to the expansion of unsymmetrical functions, and to the notation of matrices, illustrating his remarks by drawings representing atoms hung together in various ways."—*Athenæum*. Sept. 11th. No. 2498.

"A Fragment on Matrices," *xxxv. pp. 337—341, perhaps belongs to this period.

"Right and Wrong: the scientific ground of their distinction." Nov. 7th. L. and E., Vol. II. pp. 124—176.

"On the Transformation of Elliptic Functions." Dec. 9th. xxii. pp. 205—217.

Reprint: 4641, 4819, 4843.

1876.

"On the Free Motion under no forces of a Rigid System in an n -fold homaloid." (Provisional Notice.) Jan. 13th. xxvi. pp. 236—240.

"Sight and what it tells us." Feb. 24. L. and E., Vol. I. p. 69.

"Instruments used in Measurement." L. pp. 419—423; L. and E., Vol. II. p. 3—8.

"Instruments illustrating Kinematics, Statics and Dynamics." L. pp. 424—440; L. and E., Vol. II. pp. 9—30.

"The Ethics of Belief." L. and E., Vol. II. pp. 177—211.

"Notes on the communication entitled 'On the Transformation of Elliptic Functions.'" xxii. pp. 218—228.

"On the Classification of Geometric Algebras," *XLIII. pp. 397—401.

"On In-and-circumscribed Polyhedra" (?) *xxiv. pp. 229—233.

"On the Theory of Screws in a space of constant positive curvature" (?) *XLIV. pp. 402—405.

"On Tricircular Sextics" (?) *xxxvi. pp. 342—345.

"On the Double Theta-functions" (?) *XL. pp. 368—377.

"Further Note on Biquaternions" (?) *XLII. pp. 385—390.

Reprint: 4871, 4897, 4925, 4950, 4972, 4996.

1877.

"On the Types of Compound Statement involving four classes."

Jan. 9th: i. pp. 1—13. L. and E., Vol. ii. pp. 89—106.

"The Ethics of Religion." March 4. L. and E., Vol. ii. pp. 212—243.

"The Influence upon Morality of a Decline in Religious Belief."

April: L. and E., Vol. ii. pp. 244—252.

Review of Dr Booth's "New Geometrical Methods." June: pp. 562—564.

"On the Canonical Form and Dissection of a Riemann's Surface."

June 14th: xxvii. pp. 241—254.

"Cosmic Emotion." October: L. and E., Vol. ii. pp. 253—285.

"Notes on Vortex-Motion, on the Triple-generation of Three-bar Curves, and on the Mass-centre of an Octahedron." November 8th: XLVI. pp. 407—409.

"Enumeration of the Types of Compound Statements." (?) * ii. pp. 14—16.

In the Michaelmas Term of this year (cf. L. and E., Vol. i. p. 27)

Prof. Clifford delivered a course of ten lectures on Quaternions with a view to their physical applications, for students of Physics, "who are not able or willing to read very high Mathematics."

For an admirable collection of Notes of these Lectures (see *infra*, pp. 478—515), I am indebted to the kindness of Miss Ellen Watson*, who also placed at my disposal other Lecture Notes which I have not made use of in the present work. Mr G. Griffith, of Harrow, who also attended the Course, lent me his skeleton Notes which enabled me to clear up one or two minor points.

Two other courses were announced. In the Lent Term of 1878,

"On Elliptic Functions and some of their Physical Applications treated on a basis of Elementary Algebra, but assuming a Knowledge of the Elements of the Differential Calculus†."

* Miss Watson was the first woman to enter the Classes of Mathematics at University College, London. Of her, Clifford said, "that her proficiency would have been very rare in a man," and that "he was totally unprepared to find it in a woman."—She was obliged to leave England on account of failing health, and died in December 1880, at Grahamstown, South Africa.

† One or two Lectures of this course were delivered, but the few Notes of them in Miss Watson's MSS. are too fragmentary to be of service.

These were to be followed by a third Course in the Midsummer Term, "On Spherical Harmonics and other functions rising out of the Theory of the Potential and Allied Theories treated by means of Partial Differential Equations and series with special attention to problems of Electricity."

"Algebraic Introduction to Elliptic Functions" (?) commenced and added to at different times), pp. 443—477.

"On Groups of Periodic Functions" (?) * xxxviii. pp. 350—355.
Reprint: 5304, 5457.

1878.

"Remarks on the Chemico-Algebraical Theory." xxviii. pp. 255—257.

"Virchow on the Teaching of Science." April: L. and E., Vol. II. pp. 286—321.

"On the Classification of Loci." April 8: xxxiii. pp. 305—329.

"Childhood and Ignorance." May: L. and E., Vol. I. p. 70.

"Applications of Grassmann's Extensive Algebra." xxx. pp. 266—276.

"Elements of Dynamic: an Introduction to the Study of Motion and Rest in Solid and Fluid Bodies." Part I. Kinematic.

It may not be out of place to give here an analysis of what Clifford intended to give in Vol. II. of his book, the manuscript of which is now in my possession.

BOOK IV. FORCES.

Cap. I. The Laws of Motion.

Cap. II. The Conditions of Equilibrium of a Rigid Body.

Cap. III. The Composition of Forces.

§ 1. The Link-polygon, Reciprocal diagrams, &c.

§ 2. Centres of Inertia, Second Moment, Cores.

§ 3. Attractions, Potential and Level Surfaces, Sources and Sinks. Theorems of Stokes and Chasles. Electric Images. Centrobatic Bodies.

Cap. IV. Motion of a Rigid Body.

BOOK V. STRESSES.

Cap. I. Solids.

"Notes on Quantics of Alternate Numbers, used as a means for determining the Invariants and Covariants of Quantics in general." (?) *xxix. pp. 258—265.

"Binary Forms of Alternate Variables." (?) *xxxi. pp. 277—286.

"On Bessel's Functions." *xxxvii. pp. 346—349.

"Theory of Marks of Multiple Theta-functions." (?) *xxxix. pp. 356—367.

The manuscript of "The common sense of the Exact Sciences" is in Prof. Rowe's hands, and is almost ready for publication.

All introduced matter is, with one or two exceptions, included in []: this has necessitated the adoption of another form of *bracket* when [] occurs in Clifford's papers. I am in the main responsible for these additions, though I have in all cases submitted the suggestions to one or more of the mathematicians mentioned in my dedicatory letter: their own additions are suitably initialled.

INTRODUCTION

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INTRODUCTION.

It will be generally admitted that the publication in a collected form of the works of the eminent men, who have moulded the mathematical sciences into their present form, has become little less than a necessity to those who desire to follow in their footsteps, and to advance, if possible, beyond the limits attained by them. And mathematicians will gratefully acknowledge that no inconsiderable progress has already been made towards satisfying this requirement. To the Academy of Sciences at Paris we are indebted for magnificent editions of the complete works of Laplace and of Lagrange; the Government of Norway has given to the world the collected memoirs of Abel; the Academy of Goettingen has fulfilled the same duty toward the great names of Gauss and Riemann; and the Academy of Berlin has followed the example by undertaking editions of the works of Steiner and Jacobi. In our own country we have the collected works of Green, of Mac Culagh, of Gregory, of Leslie Ellis, and of Macquorn Rankine; not to mention the volumes of reprinted memoirs which we owe to living writers; for example, to Sir William Thomson and to Professor Stokes. Such collections, we may hope, will form an increasing portion of every scientific library. At the present time the results of mathematical research almost always appear in the Transactions of Societies, or in periodicals specially devoted to mathematical writings: the contents even of the most original treatises being generally anticipated by their authors in memoirs, which are often not wholly superseded

by the works themselves. But the number of the periodical repositories of mathematical literature has become so great, that papers consigned to them, although preserved, as we may hope, for all time, are in imminent danger of passing out of sight within a few years after their first appearance. They are preserved from destruction, but not from oblivion; they share the fate of manuscripts hidden in the archives of some great library from which it is in itself a work of research to disinter them. This "mislaying," if it may be so termed, of important memoirs is not only a loss to the history of science, but interferes seriously with the discovery of new knowledge. For notwithstanding the ardour with which mathematical investigation is at present pursued in every direction, a much longer time than is perhaps sometimes supposed elapses before a mathematical work of genuine originality, be it a brief note, or an elaborate treatise, becomes antiquated. It would be out of place in this connexion to mention the *Principia* of Newton, which stands apart by itself, and of which not only the methods and results, but even the very words have become the common property of all men of science. Nor need we even refer to works which have marked the beginning of a new epoch in their respective departments, such as the *Mécanique Céleste*, the *Mécanique Analytique*, the *Disquisitiones Arithmeticae*, the *Traité Analytique de la Chaleur*, the *Fundamenta Nova*, or the *Systematische Entwicklung der Geometrischen Gestalten*—the freshness of which time has hardly impaired, while the superstructures which have been based upon them have added incalculably to their importance. But leaving out of count these and other great classics of the science, the trains of thought hidden in the opuscula of Euler, of Lagrange, of Gauss, of Poisson, of Cauchy, of Abel, and Jacobi, are still unexhausted; and, far from having lost their value by the lapse of time, have in many cases acquired an increased suggestiveness from the light which the more extended knowledge of recent times has thrown upon them.

Such a prospect of future and long-continued usefulness, we may venture to hope, awaits many of Clifford's memoirs; and, even more than the immediate interest attaching to them, justifies, if any justification be needed, their appearance in their present collected form.

It might be interesting to enquire why it is that mathematical writings retain a scientific (as opposed to a merely historical) value for a longer time than memoirs recording researches in the sciences of experiment and observation. Among many partial answers which might be given to this question, one is suggested by the character of many of Clifford's papers, and has its foundation in the nature of the subjects with which they deal. Speculation in pure mathematics resembles metaphysical speculation in this, that the whole universe of thought to which it refers is so closely inter-dependent, that a clear-sighted and powerful thinker cannot fix his mental vision (however keen his effort after concentration may be) on any one region in it, without catching glimpses of something that lies beyond, and without discerning, more or less dimly, new relations to be examined, and new lines of research, which may perhaps have no immediate relevancy to the particular enquiry in which he is engaged. And these glimpses, if recorded, or even if only half unconsciously indicated, in the account which he afterwards gives of his work, are not unlikely to suggest a wholly new departure to some kindred spirit in a future time. On this ground, more strongly perhaps than on any other, we may venture to commend the present volume to the rising generation of English mathematicians. The collection includes papers—some of them youthful efforts—some suggested, one might say casually, by the researches of scientific friends—which relate to special problems, and which nevertheless would be sufficient of themselves to establish a considerable mathematical reputation. There are others, again, the work of a maturer time, and planned with a wider scope, which are models of artistic perfection, in respect both of

reference to geometry, are few and far between. For even the Elliptic and Abelian functions were approached by Clifford from the side of geometry. His early note "On some Porismatic Problems," relating to the theorem of Poncelet, which asserts that "given two conics, a polygon of a given number of sides, all whose vertices shall lie on one of them, and all whose sides shall touch the other, can either not be drawn at all, or else can be drawn in an infinite number of different ways," led him to the study of the connexion, established by Professor Cayley, between this theorem and the addition of elliptic functions of the first species; and thus to the discovery of a geometrical theory of the transformation of elliptic functions, which forms the subject of one of his most brilliant investigations (XXII.). But in his further prosecution of the subject the elliptic functions again disappear from view, and he returns to the geometrical doctrine of correspondences, and to the theory of the polyhedra, of which the faces osculate a skew cubic curve.

In like manner it would appear that he was attracted to the consideration of the Abelian Integrals by their relation to another and widely different part of geometry, the Geometry of Situation, as it has been termed. His memoir on the "Canonical Dissection of a Riemann Surface," founded on the researches of Clebsch and Lüroth, contains the simplest account which has yet been given of this important chapter of a great theory; and the reduction of a Riemann surface to the surface of a solid having a certain number of holes through it, presents to the mind what is perhaps the clearest image which it is possible to obtain of the space of two dimensions upon which a many-valued algebraical function can be mapped with the same distinctness with which a one-valued function can be mapped on a plane. But the study of the Abelian Integrals—however geometrical may have been the form in which he first envisaged this theory, led him by an inevitable sequence of ideas to the Theta functions, which form the indispensable

basis for a study of the relations subsisting between the upper limits of a system of algebraical integrals, and the values of the integrals themselves. His posthumous and unfinished memoirs on the Double and Multiple Theta functions (XXXVIII., XXXIX. and XL.) form a not unimportant contribution to pure analysis; and, though now published long after their proper date, are, it may be hoped, neither too late, nor too incomplete, to exercise some influence on the development of this rapidly growing theory. The "Algebraical Introduction to Elliptic functions," which is in fact a treatise on the single Theta functions, probably took its rise in connexion with these ulterior researches; Clifford desiring to obtain a complete command over the manipulation of these series in the simplest cases, before proceeding to apply them to the general problem of the inversion of algebraical integrals.

Enough has been said to show that Clifford's predilection for geometry lay deep. But to this his favourite science he attributed the widest imaginable scope, and at times regarded it as co-extensive with the whole domain of nature. He was a metaphysician (though he would only have accepted the name subject to an interpretation) as well as a mathematician; and geometry was to him an important factor in the problem of "solving the universe"¹. Thus he was a geometer of a type peculiarly his own; and his dealings with the science were characterized by an amount of scepticism and an amount of faith which one would hardly expect to find combined in a mathematician. He had early read and translated Riemann's celebrated discourse (ix.) "On the Hypotheses which lie at the Basis of Geometry," and had imbibed the views set forth in it as a part of his intellectual nature. Some men who have an ardent love for new knowledge find it

¹ "At the Savile I met C.... and solved the universe with great delight until A. came in and wanted to take him off... Of course I would not let him go... We walked about in the New Road solving more universe," *Lectures and Essays*, Introduction, p. 80.

difficult to maintain an unflagging interest in geometry, because they regard it as a purely deductive science, of which the first principles (axioms, postulates and definitions), whether derived from experience or not, are unquestionable, and contain implicitly in themselves all possible propositions concerning space. Thus the unknown, or at least the unforeseen, seems to be excluded from geometry, because whatever may be found out hereafter must be latent in what is already known. But upon the view put forward by Riemann and adopted by Clifford, the essential properties of space have to be regarded as things still unknown, which we may one day hope to find out by closer observation and more patient reflection, and not as axioms to be accepted on the authority of universal experience, or of the inner consciousness.

These speculations had so much influence on a great part of Clifford's work that it may not be out of place to pursue the subject a little further. In his lecture "On the Postulates of the Science of Space"¹, he has stated his own views on the question with singular clearness and brilliancy; and the pages in which he has expressed them are likely to be remembered, as marking an important moment in the controversy concerning the nature of space and the origin of our knowledge of it, which is likely to last as long as metaphysical enquiries have any interest for mankind. In this lecture he enumerates four fundamental postulates on which the ordinary conception of space is founded, (1) its continuity, (2) its flatness in its smallest parts, (3) its similarity to itself at every point, or, which is the same thing, the possibility of the existence of the same figure in any two different places, (4) the possibility of the existence of figures similar to one another, but on different scales of magnitude. The second of these postulates requires some comment to make it intelligible. Perhaps the simplest account that can be given of a space which is flat in its smallest parts is, that if anywhere

¹ *Lectures and Essays*, Vol. I. pp. 295—325.

in it we take three points very near to one another and join them by the shortest lines that can possibly be drawn, the triangular figure so formed will lie very nearly in a plane; the mathematical equivalent of this statement being that the square of the distance between any given point and any other infinitely near to it can always be expressed as a homogeneous quadratic form, in which the indeterminates are the infinitesimal differences between the co-ordinates of the two points, and the coefficients are functions of the co-ordinates of the given point. If we go further and join one of the three infinitely near points by the shortest lines possible to every point on the line already joining the other two, the assemblage of these lines will form a triangular surface-element: if this surface-element is absolutely plane, whatever be the three infinitely near points which we have taken, the space is flat; if the surface-element has a finite curvature, the space, while retaining the property of elementary flatness, is said to have curvature; and this curvature is measured, for the surface direction determined by the three points, by the curvature of the surface-element which we have constructed.

As to the first postulate, Clifford indicates his readiness to adopt either of the two opposite hypotheses that space is continuous or that it is discontinuous, while admitting fully that no phenomena have yet been observed which point to its discontinuity.

Of the second postulate, in this respect following Riemann, he speaks in the same general terms; we must not shrink from rejecting it, if its rejection should be found to assist us in the explanation of natural phenomena. The postulate is not inconsistent with a hypothesis which at one time was a great favourite with him, and which he has described in a remarkable communication (v.), presented to the Cambridge Philosophical Society, in 1870. In this brief note, comprised within a single page, he appears to adopt the hypothesis (for his language on the point is not quite free from ambiguity), that space

has everywhere a finite curvature, but that this curvature continually changing, and that all the phenomena of the universe may possibly consist in changes of the curvature of space. A finite curvature, it will be remembered, is consistent and indeed implies elementary flatness. Unfortunately Clifford though in earlier days he was fond of discussing this theory, doubt as one possible mode of "solving the universe," has no memoranda relating to it, perhaps because the efforts he made to work it out in detail, led him to no satisfactory conclusion. In the note of 1870 he speaks of it with confidence which must not be taken too literally. He would probably have allowed that Lord Bacon's criticism on Galileo "*postquam in contemplationibus magnetis se laboriosis exercuisset, confinxit statim philosophiam consentaneam apud ipsum præpollenti,*" admitted of an application, *mutandis*, to his own effort to resolve all philosophy into geometry; though he would no doubt have maintained with the utmost depth of conviction that, for aught we know to the contrary, the properties of space may change with time.

But whatever importance he may have temporarily attached to the opinion that space may not be independent of time, this idea has left no other perceptible traces in his mathematical writings. Very different is the case with another hypothesis as to the nature of space, which is somewhat widely divergent from ordinary conceptions, and to which Clifford appears at all times to have turned with peculiar favour.

This hypothesis admits the first three of the postulates enumerated above, as expressing true properties of space, but rejects the fourth, substituting for it the new postulate that space has a finite but very small curvature, which is approximately the same for any two points, and for any two surface directions at the same point. Admitting this postulate, we find ourselves in the presence of two alternatives, between which we have to choose. For we may imagine either the

the curvature of all surface-elements, constructed in the manner above described, has the same positive value, or that it has the same negative value; understanding by a positive curvature a curvature such as that of the outer portion of the surface of an anchor-ring, where the tangent plane at any point just meets the surface and does not cut it; and by a negative curvature a curvature such as that of the inner portion of the same surface, where the tangent plane cuts the surface at the point of contact. To the hypothesis that space has a constant negative curvature considerable historical interest attaches. For this hypothesis was first arrived at, not by following out such general views as those indicated by Riemann, but in a much more elementary manner. The celebrated twelfth axiom, as is well known, is the basis of Euclid's theory of parallel lines; and the assertion made in it is in fact equivalent to an assumption of the fundamental proposition of plane geometry, that the three angles of a triangle are equal to two right angles. It is now universally allowed that all efforts to demonstrate Euclid's axiom have failed; but the Russian mathematician, Lobatchewsky, appears to have been the first person to whom the idea occurred of dispensing with the axiom altogether, and trying to see what would become of geometry without it. The idea was obvious, but it was also profound; and Lobatchewsky was rewarded by the discovery that it is possible to construct a consistent and complete system of geometry upon the hypothesis that the three angles of a triangle are less than two right angles. Till the discovery of Lobatchewsky, the only substantial addition that had been made to Euclid's theory of parallel lines was a demonstration by Legendre, that the angles of a triangle cannot be greater than two right angles. As a matter of fact the demonstration of Legendre depends on the assumption that space is infinite; an assumption which, from the point of view taken by Riemann, cannot be regarded as justified by experience: but the considerations upon which the demonstration rests decided Lobat-

chewsky, as between the two alternative hypotheses that the angles of a triangle are less, and that they are greater, than two right angles, to adopt the former. This was in effect to adopt the hypothesis (though it does not appear to have occurred to Lobatchewsky in that light) that a plane has negative curvature. It was reserved for an Italian mathematician, Beltrami, to show that the plane geometry of Lobatchewsky is identical with the geometry of a pseudo-spherical surface, *i.e.* of a surface of constant negative curvature.

What Clifford thought of the philosophical importance of the work of Lobatchewsky the following quotation may serve to show :

"Each of them [Copernicus and Lobatchewsky] has brought about a revolution in scientific ideas so great that it can only be compared with that wrought by the other. And the reason of the transcendent importance of these two changes is that they are changes in the conception of the Cosmos. Before the time of Copernicus men knew all about the universe. They would tell you in the schools, pat off by heart, all that it was, and what it had been, and what it would be.... In any case the universe was a known thing. Now the enormous effect of the Copernican system, and of the astronomical discoveries that have followed it, is that, in place of this knowledge of a little, which was called knowledge of the universe, of Eternity and Imminence, we have now got knowledge of a great deal more ; but we only call it the knowledge of Here and Now.... This then was the change effected by Copernicus in the idea of the universe. But there was left another to be made. For the laws of space and motion implied an infinite space and an infinite duration, about whose properties as space and time everything was accurately known. The very constitution of those parts of it which are at an infinite distance from us, 'geometry upon the plane at infinity,' is just as well known, if the Euclidean assumptions are true, as the geometry of any portion of this room.... So that here we have real knowledge of something at least that concerns

the Cosmos; something that is true of the Immensities and the Eternities. That something Lobatchewsky and his successors have taken away. The geometer of to-day knows nothing about the nature of actually existing space at an infinite distance; he knows nothing about the properties of this present space in a past or a future eternity. He knows, indeed, that the laws assumed by Euclid are true with an accuracy that no direct experiment can approach...; but he knows this as of Here and Now; beyond his range is a 'There and Then, of which he knows nothing at present but may ultimately come to know more. So, you see, there is a real parallel between the work of Copernicus and his successors on the one hand, and the work of Lobatchewsky and his successors on the other."—*Lectures and Essays*, Vol. I. pp. 298—300.

But in spite of this eulogium, the conception of space which has left the deepest traces in Clifford's writings, is not that of Lobatchewsky, but that founded on the alternative hypothesis (rejected by the Russian geometer) of a constant positive curvature. This conception lies at the bottom of Clifford's theory of biquaternions, to which he devoted much continuous thought, and which was the origin of his researches into the classification of geometric algebras. A space of constant positive curvature is most easily represented to the mathematician (in the absence of any possibility of imaging it to the mind) as the locus of an equation of the form

$$x^2 + y^2 + z^2 + w^2 = \text{constant}$$

in a flat space of four dimensions in which $wyzw$ are rectangular co-ordinates. It is related to the two dimensional surface of a sphere, just as in ordinary geometry space of three dimensions is related to a plane surface. The following description of a space of this kind is taken from the Lecture "On the Postulates of the science of Space." It can hardly be necessary to point out, that in the last sentence Clifford is half laughing at himself.

"I cannot perhaps do better than conclude by leaving you as well as I can what is the nature of things in position that the curvature of all space is uniform and positive."

"In this case the universe, as known, becomes a given number of miles. And this comes about in a curious way. If you start in any direction whatever and move in that direction a perfect straight line according to the definition of a straight line after travelling a most prodigious distance, to which I have attached the metric unit 200,000 times the diameter of the earth, or thereabouts, you would arrive at the place whence you started. This supposition of a positive curvature the whole of geometry becomes far more complete and interesting: the principles of geometry instead of half breaking down over metrical relations, stand firm as all propositions without exception. In fact, I do not confess that I personally have often found relief from the infinity of homotoidal space in the concealing hope that all this other may be the true state of things" — *Mathematical and Essays*, Vol. I, pp. 322–3.

A third line of thought, different from those of Lobatchewsky and of Riemann, had no doubt a large part in determining Clifford to regard the hypothesis of constant positive curvature with special favour. One of the earliest and most fruitful enquiries of wide scope which interested him was the connexion between the descriptive and metrical properties of figures. In two unfinished memoirs "On Analytical Methods of Geometry" and "On the Foundations of Geometry,"

For these two imaginary points, in the geometry of a spherical surface, we have to substitute the imaginary circle in which the sphere is cut by the plane at an infinite distance. In ordinary space of three dimensions the Absolute is the same imaginary circle and the plane at an infinite distance in which it lies. Professor Cayley in his celebrated "Sixth Memoir on Quantics", generalized this conception by substituting any quadric whatever for the imaginary circle at an infinite distance (which may be regarded as a quadric surface of which one dimension has vanished). The effect of this substitution is to change the metric properties of space, the nature of the change depending on the nature of the quadric chosen as the Absolute. If we wish the space which we thus bring under contemplation to possess one of the most obvious properties which we know by experience to characterize the space in which we live, (viz. that the rotation round a fixed axis, which brings a body from any given position back into the same position again, is a finite and not an infinite operation), our choice of the form of the Absolute is limited to three hypotheses, (1) the Absolute is an imaginary quadric, (2) the Absolute is a real umbilical quadric (*i.e.* a quadric not having real right lines on it) and the space considered is internal to the quadric, (3) the Absolute is an imaginary quadric which has degenerated into a conic section by losing one of its dimensions. Of these three hypotheses the last corresponds to the ordinary conception of space: the spaces characterized by the suppositions (1) and (2) have been termed elliptic and hyperbolic respectively by Professor Klein, who succeeded in showing that in each of them the curvature is constant, being positive in the elliptic, and negative in the hyperbolic space. Thus the geometry of Lobatchewsky is the geometry of hyperbolic space: and Professor Klein's discovery of the identity of the two has thrown a wholly new light upon the researches of the former geometer. Of Clifford's study of the details of the system of Lobatchewsky only one brief note

is preserved (Appendix, p. 531). Indeed he seems to have quickly abandoned hyperbolic for elliptic geometry, influenced no doubt by the reason indicated in the passage which we have quoted—the perfect duality of the properties of elliptic space.

In the geometry of Lobatchewsky every straight line has two real points on it at an infinite distance, viz. the two real points in which it intersects the Absolute. Again, among the planes which pass through a given straight line there are two which belong to the Absolute, and which therefore are to be regarded as planes at an infinite distance. But these two planes are imaginary, being the two planes which can be drawn through the given line to touch the Absolute. Thus in the hyperbolic geometry there is no perfect duality, because when we compare the points which lie along a line, and the planes which pass through it, the absolute elements are real in the one case, and imaginary in the other: in fact, the space which is the dual correlative of an hyperbolic space is not itself a similar space, but is analogous to the space *outside* the Absolute of the hyperbolic space. On the other hand, in elliptic geometry all the elements of the Absolute, whether points or planes, are imaginary, and the duality is as perfect as it is on the surface of a sphere. It follows at the same time that all distances as well as all rotations are finite, and that a point moving on a straight line (or more properly on a shortest line) will come round after a finite journey to the point from which it set out, just as a plane revolving round a straight line returns after describing a finite angle of 360° to its original position.

We proceed to give an enumeration of the memoirs contained in this volume, grouped according to their subjects. Perfect accuracy is not important, nor indeed attainable, in such a classification, of which the only object is to convey a general impression of what Clifford has done in each department

of mathematical science. The grouping, however rough, will of itself serve to distinguish the problems with which he occupied himself habitually, and by deliberate preference, from those which had only a temporary interest for him, and were suggested by some accidental circumstance. The enumeration is followed by a few remarks on the methods or results of some of the more important papers.

A. ANALYSIS.

(a) *Mathematical Logic.*

- (1) On the types of compound statement involving four classes, (I.).
- (2) Enumeration of the types of compound statement, (II.).

(b) *Theory of Equations and of Elimination.*

- (1) Proof that every equation has a root, (IV.).
- (2) On a case of Evaporation in the order of a Resultant, (XVII.).

(c) *Abelian Integrals and Theta Functions.*

- (1) On the Canonical form and Dissection of a Riemann's surface, (XXVII.).
- (2) On groups of Periodic Functions, (XXXVIII.).
- (3) Theory of Marks of Multiple Theta Functions, (XXXIX.).
- (4) Double Theta Functions, (XI.).

(d) *Invariants and Covariants.*

- (1) Remarks on the Chemico-Algebraical Theory, (XXVIII.).
- (2) Notes on Quantics of Alternate Numbers, (XXIX.).
- (3) Binary forms of Alternate Variables, (XXXI.).

(e) *Miscellaneous.*

- (1) Remarks on a Theory of the Exponential Function, (xlv.).
- (2) On Bessel's Functions, (xxxvii.).
- (3) On a Canonical form of Spherical Harmonics, (xxv.).
- (4) A Fragment on Matrices, (xxxv.).

B. GEOMETRY.

(a) *Projective and Synthetic Geometry.*

- (1) On some Porismatic Problems, (iii.).
- (2) A Synthetic proof of Miquel's Theorem, (viii.).
- (3) Analogues of Pascal's Theorem, (x.).
- (4) Analytical Metrics, (xi.).
- (5) On the Theory of Distances, (xvi.).
- (6) Geometry on an Ellipsoid, (xix.).
- (7) On the general Theory of Anharmonics, (xii.).
- (8) A Geometrical Theorem, (xlvii.). [An early note on the Properties of the Quadrilateral].
- (9) Triangular Symmetry, (xlviii.).
- (10) On the Powers of Spheres, (xxxiv.).
- (11) Tricircular Sextics, (xxxvi.).
- (12) On the Triple Generation of Three Bar Curves, (xlvi. ii.).
- (13) On the Mass-centre of an Octahedron, (xlvi. iii.).
- (14) On some extensions of the Fundamental Proposition in M. Chasles's Theory of Characteristic, (xlix.).

(b) *Applications of the Higher Algebra to Geometry.*

- (1) On Jacobians and Polar Opposites, (vi.).
- (2) On a Generalization of the Theory of Polars (xiii.).
- (3) On Syzygetic relations among the powers of Linear Quantics, (xiv.).
- (4) On Syzygetic relations connecting the powers of Linear Quantics, (xv.).

(e) *Miscellaneous.*

- (1) Remarks on a Theory of the Exponential Function, (xlv.).
- (2) On Bessel's Functions, (xxxvii.).
- (3) On a Canonical form of Spherical Harmonics, (xxv.).
- (4) A Fragment on Matrices, (xxxv.).

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- (11) Tricircular Sextics, (xxxvi.).
- (12) On the Triple Generation of Three Bar Curves, (xlvi. ii.).
- (13) On the Mass-centre of an Octahedron, (xli. vi. iii.).
- (14) On some extensions of the Fundamental Proposition in M. Chasles's Theory of Characteristics, (xlix.).

(b) *Applications of the Higher Algebra to Geometry.*

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- (2) On a Generalization of the Theory of Polars (xiii.).
- (3) On Syzygetic relations among the powers of Linear Quantities, (xiv.).
- (4) On Syzygetic relations connecting the powers of Linear Quantities, (xv.).

transformed without tearing into the surface of a body with $\frac{1}{2}w - n + 1$ holes through it. For example, an anchor ring is a body with one hole through it; and into the surface of an anchor ring a Riemann's surface with two sheets and four spiral points can be transformed. Again, in chains of a not uncommon pattern, each link is of the form of a flattened ellipsoid with two circular holes through it; the surface of such a body would represent the equation $y^2 = f(x)$, where $f(x)$ is a rational and integral function of the sixth order, and might therefore be used as the geometrical scaffolding for the theory of hyper-elliptic integrals. A closed box with p tubes, open at each end, carried through it from its lid to its bottom, is a simple example of a body with p holes through it. Such a surface would serve to represent (in a certain sense) all the real and imaginary points of a curve of deficiency p ; or, which is the same thing, all the pairs of values of two variables which satisfy an algebraical equation of that deficiency. What however is represented in this way is not the quantitative relation between the two variables, but the *connexion* between the different values of one of them corresponding to one and the same given value of the other; viz, if z is the independent and s the dependent variable in an algebraical equation between z and s , and if s_0, s'_0 are two different values of s corresponding to the same value z_0 of z , these two values are *connected* in the sense that it is always possible, by making z pass through a continuous cycle of values beginning and ending with z_0 , to cause s , though it may have set out with the value s_0 , to end with the value s'_0 . Each point of any Riemann's surface representing the equation corresponds to a pair of values of z and s ; and the different tracks on the surface, irreducible to one another by any continuous change, along which we can pass from the point (z_0, s_0) to the point (z_0, s'_0) , answer precisely to the different courses of values by which we can pass with analytical continuity from the pair of values (s_0, z_0) to the pair of values (s'_0, z_0) . Remodelled as it has been by Clifford, the Riemann's

transformed without tearing into the surface of a body with $\frac{1}{2}w - n + 1$ holes through it. For example, an anchor ring is a body with one hole through it; and into the surface of an anchor ring a Riemann's surface with two sheets and four spiral points can be transformed. Again, in chains of a not uncommon pattern, each link is of the form of a flattened ellipsoid with two circular holes through it; the surface of such a body would represent the equation $y^2 = f(x)$, where $f(x)$ is a rational and integral function of the sixth order, and might therefore be used as the geometrical scaffolding for the theory of hyper-elliptic integrals. A closed box with p tubes, open at each end, carried through it from its lid to its bottom, is a simple example of a body with p holes through it. Such a surface would serve to represent (in a certain sense) all the real and imaginary points of a curve of deficiency p ; or, which is the same thing, all the pairs of values of two variables which satisfy an algebraical equation of that deficiency. What however is represented in this way is not the quantitative relation between the two variables, but the *connexion* between the different values of one of them corresponding to one and the same given value of the other; viz. if z is the independent and s the dependent variable in an algebraical equation between z and s , and if s_0, s_0' are two different values of s corresponding to the same value z_0 of z , these two values are *connected* in the sense that it is always possible, by making z pass through a continuous cycle of values beginning and ending with z_0 , to cause s , though it may have set out with the value s_0 , to end with the value s_0' . Each point of any Riemann's surface representing the equation corresponds to a pair of values of z and s ; and the different tracks on the surface, irreducible to one another by any continuous change, along which we can pass from the point (z_0, s_0) to the point (z_0, s_0') , answer precisely to the different courses of values by which we can pass with analytical continuity from the pair of values (s_0, z_0) to the pair of values (s_0', z_0) . Remodelled as it has been by Clifford, the Riemann's

A (a). The paper (I.) on the types of compound statement involving four classes, and the note (II.) upon the general problem of compound statements involving any number of classes, belong to the theory of combinations, which at no time appears to have engaged Clifford's attention continuously. Indeed the only other problems of combinatorial analysis which he has treated are those relating to the marks or systems of indices of the multiple Theta functions.

He appears to have undertaken a discussion of the problem of compound statements rather from an interest casually awakened by its acknowledged difficulty (see the Editor's note, p. 16), than from any predilection for the mathematical theory of Logic, to which (so far as can be gathered from his remaining papers) he never returned again.

A (b). The proof given in (IV.) of the fundamental theorem that every equation has a real quadratic factor, is very remarkable because it depends solely on the theory of elimination. A proof substantially the same was afterwards obtained independently by Mr J. C. Malet, of Trinity College, Dublin, and is published in the *Transactions of the Royal Irish Academy* for 1878. The principle of the demonstration is so simple and natural that, when once it has been pointed out, the only wonder is it should not have occurred to any one before. It turns on what Clifford calls an obvious remark that the order of a resultant is reduced when the weight of the coefficients, in the two equations from which the elimination takes place, falls, not by successive units, but by some other constant integer. The development of this remark forms the subject of XVII.

A (c). We have already referred to the canonical form of a Riemann's surface imagined by Clifford. His theorem in effect is that a Riemann's surface having n sheets or leaves, and w spiral or branch points, by winding around which the different sheets are connected with and pass into one another, may, if the material of which it is made is sufficiently extensible, be

surface enables us to form a distinct conception of these different tracks, as well as of the systems of curves or "cross-cuts" which have to be drawn in order to reduce the surface to one simply connected; i.e. to a surface in which but one set of irreducible tracks can be drawn from any one given point to another. The advantage resulting from Clifford's simplification will be acknowledged by students approaching the theory for the first time, who generally find considerable difficulty in eliciting from Riemann's description a clear image either of the surface itself or of the complicated systems of curves which he directs to be drawn upon it.

The unfinished memoir on the Double Theta Functions (XL) relates to the problem of the inversion of hyper-elliptic integrals, solved, so far as its main outlines are concerned, by Rosenhain and Goepel. Except in a few matters of detail Clifford does not seem to have added anything to the results of his predecessors, and the papers may be regarded as a continuation of the "Algebraical Introduction to Elliptic Functions" (Appendix, p. 443). The memoir on groups of periodic functions (xxxviii.) deals with the multiple Theta functions. In it Clifford considers the different Theta functions of the same arguments as multiples by an exponential of any one of them, in which the arguments are increased by *quadrants* (i.e. by multiples of the halves of the periods and of the quasi-periods): and he determines the number of even and uneven Theta functions respectively. Leaving these known results, he proceeds to investigate the differential coefficient of the quotient of two Theta functions (p. 354). The method is the same as that employed in the "Algebraical Introduction;" viz. two Theta series are multiplied together, and the sums of two squares which appear as exponents in the product are transformed by the formula $(m-n)^2 + (m+n)^2 = 2(m^2 + n^2)$. But the research is left unfinished, and it does not appear whether Clifford had succeeded in completing it. In the memoir "on the Marks of Multiple Theta Functions" the point of view is somewhat

of the method are perhaps even more apparent upon an examination of the lithographed fragments to which reference is made in the note on xxxi. The interpretation of each of the graphs recorded in these fragments is not difficult to decipher. But the method does not afford any answer to the question whether the corresponding invariant or covariant is really existent, or is identically evanescent and therefore nugatory. Thus it will be noticed in Dr Sylvester's note on xxviii., that there is no difficulty in finding the 'algebraical content' of the graph which represents the discriminant of a cubic; but to establish the non-evanescence of this algebraical content, considerations are required which Dr Sylvester supplies from another source.

A (e). Of the notes included under this head perhaps one only—that on Bessel's functions—is purely analytical. To these functions Clifford does not appear to have devoted any continuous attention, though he has again considered them from the same point of view in the paper 'on the multiplication of two infinite series.' (Appendix, p. 474). The note on the exponential function is intended to shew that an exponential operator applied to vectors in one line, in a plane, and in space, leads successively to the conceptions of a ratio having a sign, of a complex ratio, and of a quaternion. The note on spherical harmonics points out that Laplace's equation may be interpreted as signifying that the curve represented by any spherical harmonic equated to zero stands in a certain covariant relation to the absolute imaginary circle upon the sphere. Lastly, the fragment on matrices contains (1) a geometrical investigation of the effect of transformation by a matrix of which the determinant vanishes, (2) a geometrical investigation of the condition that two matrices should be commutative in multiplication.

B (a). The papers viii., xlviii., xi., xvi., xix. may perhaps be classed together as all turning on the derivation of the metrical from the descriptive properties of figures by the intro-

In this singularly beautiful series of conceptions it is difficult to say how much belongs to Clifford and how much to Dr Sylvester: the more so because each of them was ready to attribute to the other the larger share. (See the letter to Dr Sylvester entitled "Remarks on the Chemico-algebraical Theory," and the note of Dr Sylvester at p. 257).

A point of considerable interest in itself appears to have had a special attraction for Clifford. "The part," he writes to Dr Sylvester, "of the theory which astonished me most is its application to intergradient variables when the number in a set is greater than three, such as the six coordinates of a line in the case of quaternary forms. When the original variables are regarded as alternate numbers, then intergradients are simply their binary products. Thus by simply multiplying the linear forms representing two planes we get one intergradient form representing their line of intersection, etc."

The theory of the *intervariants* appertaining to intergradient variables has been so little developed in detail that it is a matter of regret that no other record remains of Clifford's meditations on the subject.

It may perhaps be right to point out that simple and beautiful as are the methods of the formation of invariants by polar multiplication, and of their representation by graphs, the application of these methods to individual cases is not exempt from difficulties. One source of these difficulties Clifford indicates when, in the letter just quoted, he says "of course the main thing is to pass from this system of separate variables [in the multipartite binary forms] to that in which the same variables occur to higher orders in the same form, or back again—what you call unravelment."

Thus in the papers xxix. and xxxi. in which alone we have any detailed applications of the method to the actual determination of invariants and covariants, a large amount of space has to be devoted to the consideration of the distinction between symmetric and unsymmetric forms. The difficulties

the domain of symbolic geometry the theory of screws invented by Dr Ball. And it must be allowed that Clifford has succeeded in combining these, in form if not in substance, somewhat heterogeneous elements into a theory of great beauty and wide scope, which is certainly his own. He appears to have shared the conviction of Grassmann* that Sir William Hamilton's quaternions lie entirely within the four corners of the *Ausdehnungslehre*. Some exception may perhaps be taken to this view. In the first place Clifford and Grassmann do not altogether agree as to the way in which the symbols of Grassmann are to be identified with those of Hamilton; and each of them has to assume a law of multiplication for the fundamental units, which is at variance with the law actually adopted in the *Ausdehnungslehre*, though included as a special case under one of the general types of multiplication described by Grassmann in his Memoir "Sur les différents genres de Multiplication" (Crelle's *Journal*, Vol. XLIX. p. 123). In fact, throughout the *Ausdehnungslehre* of 1862 only one species of multiplication (beside that of ordinary algebra) is employed: viz. the polar multiplication (see above, p. lvi), characterized by the equations $[ab] = -[ba]$, $[a^2] = [b^2] = \dots = 0$, where a, b, \dots are any extensive magnitudes whatever, and the square brackets are employed to signify that the multiplication is polar and not algebraical. It is true that Grassmann uses a great number of different terms to describe what he considers to be different kinds of multiplication (outer, inner, progressive, regressive, planimetric, stereometric); but these terms really define the interpretation to be given to the symbols in certain combinations, and not the law of the multiplication itself; for example, the *inner* product of an unit a by another unit b is the polar (or, to use Grassmann's expression, the *combinatorial*) product of a , not by b , but by a certain extensive magnitude (viz. the product taken with a definite sign of all the fundamental units

* See his memoir "Der Ort der Hamilton'schen Quaternionen in der Ausdehnungslehre," *Mathematische Annalen*, Vol. XII. p. 375.

nungslehre. How far, at the time when he thus expressed himself, geometers were from having kept pace with the rapid development of algebra, may be inferred from the fact, to which Clebsch often adverted in conversation, that the geometric interpretation of the simplest known linear covariants—those of the binary quintic—has not yet been given. In the two papers “on syzygetic relations” (which do not imply any reference to the methods of Grassmann), Clifford generalizes in various directions the methods employed and the results obtained by M. Paul Serret in his *Géométrie de Direction*.

Among researches of a somewhat more special kind, we must especially mention (1) “The Theorem relating to Polyhedra” (XVIII.) enunciated at p. 169, which is the analogue of an equally beautiful property of polygons discovered by the late T. Cotterill; and (2) the investigations relating to Mr Spottiswoode’s problem of “the contact of conics and of quadrics with surfaces.”

B (e). Passing over the important papers on Elliptic Transformation, the general character of which we have already indicated, and the few notes on kinematical subjects (which are of less importance as Clifford has left a complete work on dynamical science), we come to the class of researches in which he seems to have found the fullest scope for his mathematical imagination.

To the translation of Riemann’s discourse we have already referred; and, as Clifford has added no notes of his own to it, we may pass it over here, only observing that the Lecture on the Postulates of the Science of Space is not only a popular account of the theory, but a commentary, which any young mathematician reading Riemann’s discourse for the first time will find invaluable. Of the other papers enumerated under this head, all except the last agree in the employment of symbolic methods, founded on those of Grassmann and Sir William Hamilton, but modified and enlarged so as to include within

which are not factors of b), termed by Grassmann the *complement* of b . To bring quaternions within the methods of the *Ausdehnungslehre*, Grassmann has to suppose a law of multiplication, such that the product of two factors is equal to the complement of their outer or combinatorial product, diminished by their inner product*; and Clifford has to suppose a system of three fundamental units, of which the squares are -1 instead of zero. Each of these suppositions seems to lie wholly outside the particular system of symbolic geometry developed in the *Ausdehnungslehre*; though it is not contended that either of them lies outside the general principles of the memoir "Sur les différents genres de Multiplication." So far as Clifford's own work is concerned, the question of the true relation between the quaternion algebra and that of extensive magnitudes is not perhaps very important; it is certain however that a comparison of the points of difference and of agreement between the two theories exercised a marked influence on the course of his own speculations.

The conception of a vector in space, and of the addition and subtraction of vectors, is common to both systems, and was no doubt familiar to geometers before the invention of either. The idea of a quotient of two vectors is peculiar to the calculus of quaternions; and the expression of such a quotient as the sum of a scalar and a vector is the central point in the whole theory. There is nothing properly corresponding to this in the *Ausdehnungslehre*. We do indeed meet in the work of Grassmann with quotients of vectors†; but these quotients are operators, converting simultaneously three given vectors into three other given vectors; and thus affording a symbolic representation of the homographic transformation of space. Nor again are the quotients of Grassmann themselves extensive magnitudes; thus they do not possess the characteristic property of the vector-quotients

* *Mathematische Annalen*, Vol. vii. p. 374.

† *Ausdehnungslehre*, Arts. 377-380. See also *Math. Ann.* Vol. xii. p. 381-383.

Of the papers entitled "Applications of Grassmann's extensive Algebra" and "On the classification of Geometric Algebras," no clearer account can be given than that contained in Clifford's letter to Dr Sylvester: "I had designed for you a series of papers on the application of Grassmann's methods, but there is only one of them fit for printing yet. It is an *explanation* of the laws of quaternions and of my biquaternions by resolving the units into factors having simpler laws of combination; a determination of the compounding systems for space of any number of dimensions; and a proof that the resulting algebra is a compound (in Peirce's sense) of quaternion algebras. It thus appears that quaternions are the last word of geometry in regard to complex algebras."

The fundamental units $i_1, i_2 \dots i_n$ in these papers, in accordance with the *Ausdehnungslehre*, are points or vectors; vectors being equivalent to points of weight zero at an infinite distance. But these units differ essentially from those of Grassmann, inasmuch as $i_r^2 = \pm 1$, instead of $i_r^2 = 0$, although $i_r i_s = -i_s i_r$. It must be admitted that in the combination of these conditions there is something paradoxical, for the equation $i_r i_s = -i_s i_r$, if we suppose it to hold when $r = s$, gives necessarily $i_r^2 = 0$: and Clifford has nowhere clearly explained why the square of a symbol denoting a point should be a positive or negative unit in the same symbolical system in which the product of two points represents a distance.

The only application of this geometrical algebra which he has left consists in his solution of the problem of the motion of a rigid body round a fixed point in a flat space of n dimensions. The paper (XXVI.) containing this solution is further remarkable as sketching the way in which Theta functions of $n-r$ arguments might be employed to integrate the equations of motion in the case in which no forces are acting; just as Jacobi has expressed as Theta functions of the time the nine direction cosines of a rigid body, rotating under the action of no forces round a fixed point in space of three dimensions.

had to be invented, that of the Biquaternion*. It is possible that Clifford might not have been guided to this conception if he had not considered the problem with reference to elliptic instead of parabolic space. In an elliptic space a motor can in general be represented in an infinite number of ways as the sum of two rotors, and in one way, and one way only, as the sum of two rotors reciprocal to one another with regard to the absolute. If we represent by ω a rectangular twist of unit pitch, Clifford shews (1) that all such twists are equivalent to one another; (2) that if α be any rotor, $\omega\alpha$ is the rotor reciprocal, and equal to it; (3) that every motor can be expressed in the form $\alpha + \omega\beta$, where α and β are rotors passing through the origin; and (4) that the ratio of any two motors can be exhibited in the form $s + \omega t$ where s and t are quaternions. The symmetry of this system is very striking; in parabolic space the definition of the symbol ω is far less simple and natural (see p. 186)†.

These ideas are developed (though only too briefly) in the "Preliminary Sketch of Biquaternions," the "Further Note on Biquaternions," and the note "On the theory of Screws in a space of Constant Curvature." The paper "On the motion of a body in elliptic space" contains a singularly beautiful application of the whole theory.

* The word Biquaternion had already been employed by Sir W. R. Hamilton. But his biquaternions are entirely different from those of Clifford, and are simply quaternions of which the coefficients are complex quantities containing the imaginary unit of ordinary algebra.

† Dr Ball, who kindly allowed me to submit to him a proof of the above very imperfect account of Clifford's theory of biquaternions, writes to me as follows: "I think it might be well to add a line or two with a view of prominently bringing out Clifford's conception of the 'vector' in elliptic space. A 'right vector' is the result of two equal rotations about a pair of conjugate polars with regard to the absolute; a 'left vector' is the result of two equal and opposite rotations about a pair of conjugate polars. Clifford shows beautifully that this is the legitimate generalization of the Hamiltonian vector, and he enunciates the splendid theorem that *any* motor must be *in one way* the sum of a right vector and a left vector."

The fundamental principle of Clifford's investigation has been so clearly explained by Professor Henrici in his note on p. 307, that we need not dwell on it here. It may however be worth while to observe that Grassmann (whose footsteps Clifford is certainly not following in this paper) had already proposed to regard the coefficients in the Cartesian equation of a curve as coordinates; and as an example of this method had worked out the geometry of a system of circles in a plane, shewing its analogy to the geometry of the points of space of three dimensions: viz. if $XYZW$ are the equations of four circles, the equation of any given circle may be expressed in the form $lX + mY + nZ + rW$, and l, m, n, r may be regarded as the homogeneous coordinates of this circle. These coordinates of Grassmann's are closely related to the 'power coordinates' of a circle and of a sphere employed in xxxiv. and in the Appendix, p. 546; Clifford's power coordinates of a circle being in fact linear functions of Grassmann's. It is remarkable, however, that Grassmann, at least in his general definition, contemplates only a single equation between his generalized coordinates. Thus the circles of which the coordinates satisfy the homogeneous relation $f(l, m, n, r) = 0$, form the only "Kurvegebilde" of circles which he defines. This according to Clifford would be a "two spread" of circles, and a "one spread" or "one way locus" would be formed by the circles of which the coordinates satisfy two equations. All that remains to us of Clifford's work on the subject relates to "one-way loci."

The application of Abelian functions to this new aspect of geometry awakened all Clifford's enthusiasm. He spoke to me of this part of his theory as opening a boundless field for new researches—as "altogether too big a thing" for one man to manage; and, with the instinct for companionship so characteristic of his nature, he expressed an earnest desire to get others to join him in the work. All that remains to indicate the nature of the discoveries which he had made in this direction is (1) a brief but masterly summary of the known results of the

The last paper under this head, that "on the Classification of Loci," is of a different character from those which have preceded; and is perhaps, even in the unfinished state in which we have it, the most profoundly interesting of all Clifford's mathematical writings. No use is made in it of any special symbolical methods: the only apparatus employed is that of ordinary algebra. And, if the language of the geometry of many dimensions is adopted throughout, yet the results are capable of immediate translation into the language of algebra. Thus the Theorem A, p. 307, "Every proper curve of the n^{th} order is in a flat space of n dimensions or less," is equivalent to the following algebraical proposition:

"If there be a system of $n+m+1$ quantities x connected by $n+m-1$ homogeneous equations; and if this system be such that, upon the addition to it of one equation more, linear and homogeneous in the quantities x and having arbitrary coefficients, it gives n sets of values for the ratios of the quantities x , these $n+m+1$ quantities can always be expressed as linear and homogeneous functions of $n+1$ quantities."

The value of such an algebraical proposition cannot be questioned, because the study of the properties of systems of algebraical equations is of importance for every part of analysis. But the advantage of the geometrical statement in point of clearness and precision is unmistakeable. And our sense of this advantage would be yet further quickened if we were to attempt to render into pure algebra the proof, in three lines only, which Clifford has given of the Theorem. As an example, having no immediate connexion with Clifford's memoir, of the use which has been made by other mathematicians of a similar extension of geometrical language, we may refer to the researches on elimination contained in Lesson XVIII. of the Higher Algebra of Dr Salmon, who cannot be justly accused of having shewn any undue partiality for space of more than three dimensions.

MATHEMATICAL PAPERS.

theory of Abelian functions, stated in a form suitable to the applications which he contemplated; (2) a generalization, to which we have already referred, of an important result obtained by Riemann; and (3) a remarkable theorem limiting, in certain cases, the number of dimensions requisite for the existence of a curve of given order and deficiency. How much may have perished unrecorded we cannot tell. But, however this may be, no geometer will look for a more splendid monument of Clifford's genius, or for a more touching memorial of his early death, than is to be found in the unfinished pages 'On the Classification of Loci' which embody the last and perhaps the greatest effort of his inventive powers.

I.

ON THE TYPES OF COMPOUND STATEMENT INVOLVING FOUR CLASSES*.

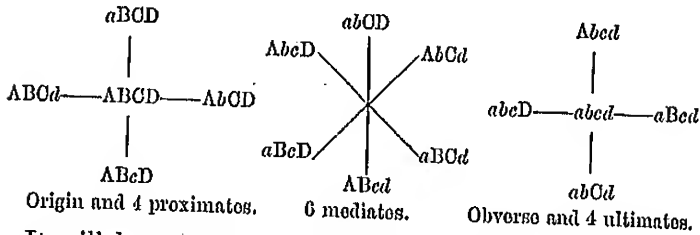
PROFESSOR STANLEY JEVONS has enumerated† the types of compound statement involving three classes, among which the premises of a syllogism appear as a type of four-fold statement. He propounded at the same time the corresponding problem of enumeration for four classes, which is solved in the present communication. The reader is referred to the paper or the book just mentioned for further explanation of the nature and purpose of the problem than is to be found in Art. 1. It may, however, be premised that the letters A, B, C, D, denote four *classes* or *terms* (for example, hard, wet, black, nice), and that, according to a convenient notation of De Morgan's, the small letters, *a*, *b*, *c*, *d* denote the complementary classes or contrary terms (not hard, not wet, not black, not nice). A *simple* statement is of the form $ABCD = 0$ (no hard, wet, black, nice things exist, or, which is the same thing, all hard, wet, black things are nasty). The statement $ABC = 0$ (no hard, wet, black things exist, or all hard, black things are dry) is to be regarded as made of these two, $ABCD = 0$, $ABCd = 0$ (no hard, wet, black, nice things exist, and no hard, wet, black, nasty things exist) and so is called a *compound* (in this case a *two-fold*) statement. The notion of *types* is defined in Art. 1.

1. Four classes, or terms, A, B, C, D, give rise to sixteen cross-divisions or *marks*, such as $AbCd$. A denial of the existence of one of these cross-divisions, or of anything having its mark (such as $AbCd = 0$), is called a simple statement. A

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† *Proceedings of the Manchester Philosophical Society*, vol. vi. pp. 65—68, and *Memoirs*, Third Series, vol. v. pp. 119—130. *The Principles of Science*, vol. i. pp. 154—164. [New Edition, pp. 134—143.]

four at distance *three*, and may be called *ultimates*. Finally, the obverse is at distance *four*.



It will be seen from the above table that the four proximates are respectively obverse to the four ultimates, and that the mediate form three pairs of obverses. Every proximate or ultimate is distant 1 and 3 respectively from such a pair of mediate. Thus each proximate or ultimate divides the mediate into two classes; three of them are at distance 1 from it, and three at distance 3. Two mediate which are not obverse are at distance 2. Two proximates, or two ultimates, or an ultimate and a proximate which are not obverse, are also at distance 2.

This view of the mutual relations of the marks is the basis of the following enumeration of types.

4. There is clearly only one type of simple statement. But of two-fold statements there are four types; viz. the distance may be 1, 2, 3, or 4; and so, in general, with n classes there are n types of two-fold statement.

5. A compound statement containing no pair of obverses is called *pure*. In a three-fold statement there are three distances; one of these must be not less than either of the others. If this be 2, the remaining mark must be at odd distance from both of these or at even distance from both; thus we get the types 1, 1, 2, and 2, 2, 2. If the not-less distance be 3, the remaining distances must be one even and the other odd; the even distance must be 2, the odd one either 1 or 3; and the types are 1, 2, 3; 2, 3, 3. Thus there are 4 pure three-fold types. With a pair of obverses, the remaining mark must be at odd or even distance from them; 1, 3, 4; 2, 2, 4. In all six three-fold types observe that there is necessarily one even distance.

denial of two or more even statements are called *compound statements*, and, moreover, two-fold, three-fold, etc., statements are called the number denoted.

When two compound statements can be transformed into each other by interchange of the letters A, B, C, etc., with their other or with their complementary statements, then they are called *similar*; and all similar statements are said to belong to the same *type*. The problem before us is to determine all the types of compound statements that can be made by the compounding

2. Two statements are called *complementary* when they deny between them all the statements that can be made by putting any mark; or, which is the same thing, when between them put those marks which the other denies, and vice versa. It is obvious that when two statements are similar, then any two of any statements will also be similar, and, consequently, that every type of *n*-fold statement has a unique complementary type of $(16-n)$ -fold statement. It follows that we need only consider the types as far as the eighth order, for the higher and more than eight fold statements will already have been given as complementary to types of lower orders. Thus, a eight-fold statement is complementary to one eight-fold statement, and, but there are not necessarily of the same type.

3. One mark ABCD may be changed into any other kind by interchanging one or more of the letters A, B, C, D with the complementary state. The number of such changes is called the *distance* of the four marks. Thus in the example given the distance is 4. In two similar compound statements the distance of the marks denoted must be the same, but it does not follow that when all the distances are the same the two statements are similar. Moreover, it would be well to give one example of two dissimilar statements having the same distance. When the distance is 4 the two marks are said to be *obverse* to one another, and the statements denoted are called *obverse statements* as ABCD and abcd, or, again, ABCd and aBd. When any one mark is given to itself and the others may be provided in sequence and all the results are listed as follows: Four are at distance one from each other, and all are called *proximate*; six at distance two, and all are called *second-order*

$13^2, 13$; $13^3, 31$; $22^2, 22$; $13^1, 22$; $13^3, 22$. With *two* pairs of obverses they must be either at odd or even distances from one another; two types. Altogether $12 + 5 + 2 = 19$ four-fold types.

8. In a *pure* five-fold statement there is always a triad of marks at distance 2 from one another. For there is a pair evenly distant; if there is not another mark evenly distant from these, the remaining three are all oddly distant, and therefore evenly distant from one another. First, then, let the remaining two marks be both oddly distant from the triad. In regard to the origin of which these are proximates, the two to be added must be either two mediates, like (of two kinds) or unlike, or a mediate of either kind with the origin or the obverse; 7 types. Next, if one of the two marks be evenly distant from the triad, it must form with the triad either a proper or an improper group of four. To a proper group we may add the origin, the obverse, or a mediate; to an improper group, the origin or the obverse (the mediates give no new type), 5 types; or in all 12 pure five-fold types.

9. In a five-fold statement with one pair of obverses there must be another pair of marks at distance 2. We have therefore to add one mark to each of the following three types of fourfold statement,—a pair of obverses together with (1) two proximates, (2) a proximate and an ultimate, (3) two mediates. To the first we may add another proximate, an ultimate, or a mediate of three kinds, viz. at distances 11, 13, 33 from the two proximates; 5 types. To the second if we add a proximate or an ultimate, we fall back on one of the previous cases; but there are again three kinds of mediates, at distances 11, 33, 13 from the proximate and ultimate; 3 types. To the third we may add another mediate, whereby the type becomes a proper group together with the obverse of one of its marks, which is the same thing as an improper group together with the obverse of one of its marks—or a proximate or ultimate which are of three kinds, at distances 11, 13, 33 from the two mediates; 4 types. Thus there are 12 five-fold types with one pair of obverses. With two pairs of obverses at odd distances, there is only one type, all the remaining marks

11. If a six-fold statement contains one pair of obverses, the remaining four marks cannot all be evenly distant from this pair. For in that case they would constitute a group; and it is easy to see that the marks evenly distant from a group, whether proper or improper, do not contain a pair of obverses. We have therefore only these four cases to consider:—

- (1) The four marks are all oddly distant from the obverses.
- (2) One is evenly distant and three oddly distant.
- (3) Two are evenly distant and two oddly.
- (4) Three are evenly distant and one oddly.

In the first case the four marks form a group. If this is a proper group, the pair of obverses must be either the origin and obverse of the group, or a pair of mediates; 2 types. If the group is improper, the pair must be an origin and an obverse; 1 type. In the second case, we have an origin, an obverse, and a mediate, to which we must add three marks taken out of the proximates and ultimates. We may add 3 proximates distant respectively 113 or 133 from the mediates (2 types),—or 2 proximates distant respectively 11, 13, 33 from the mediate, and with each of these combinations an ultimate distant either 1 or 3 (6 types). To interchange proximates with ultimates clearly makes no difference; so that in reckoning the cases of 1 proximate and 2 ultimates or 3 ultimates, we should find no new types. In the third case we have an origin, an obverse, and two mediates distant 2 from each other; and to these we have to add either two proximates or a proximate and an ultimate. The two proximates may be distant from the two mediates 11, 13, or 11, 33, or 13, 13, or 13, 33; 4 types. The proximate or ultimate must not be respectively distant 11, 33, or 33, 11; for then they would form a pair of obverses; there remain the cases 11 with 11 or 13, 13 with 13, and 33 with 13 or 33; 5 types. In the fourth case we have an origin, obverse, and three mediates distant 2 from one another; the remaining mark must be distant either 1 or 3 from these mediates; 2 types. This makes twenty-two types of six-fold statement with one pair of obverses.

being similarly related to them; at even distance the remaining mark may be evenly or oddly distant from them; 2 types. On the whole we have $12 + 12 + 3 = 27$ types of five-fold statement.

It is to be remarked that there is no pure five-fold statement in which all the distances are even, and that, if there is only one pair of obverses with all the distances even, the type is a proper group together with the obverse of one of its marks.

10. We may now prove, as a consequence of the last remark, that a pure six-fold statement either contains a group of four with a pair oddly distant from it or consists of two triads oddly distant from one another. For there must be a pair at distance 2; if the other four are all oddly distant from these, they form a group; if one is evenly distant and three oddly distant, we have the case of the two triads; if two are evenly distant, we again have a group. We must add, then, first to a proper group, and then to an improper group, a pair oddly distant from it. To a proper group consisting of the proximates to a certain origin we may add the origin or its obverse with a mediate, or two mediates; 3 types. An improper group is symmetrical; that is to say, if we substitute for any one of its marks the obverse of that mark, we shall obtain a proper group. In this way we shall get four origins distant 1113 from the group, and four obverses distant 1333; if we add to these the obverses of the marks in the group itself, we have described the relation of the twelve remaining marks to the group. To form, therefore, a pure six-fold statement we may add either two origins or two obverses or an origin and an obverse; 3 types.

In the case of the two triads, since they are oddly distant from one another their origins must be oddly distant; that is, they must be distant either 1 or 3. If they are distant 1, neither, both, or one of the origins may appear in the statement; if they are distant 3, neither, both, or one of the obverses; 6 types. Thus we obtain 12 types of purely six-fold statement.

and three oddly distant from them; or of five marks evenly distant from one another and two oddly distant from them. In the former case the pair of obverses may be in the four or in the three. If they are in the four, the three form a triad which are proximates to one origin; and then the pair may be the origin and obverse or a pair of mediates. If the pair are origin and obverse, the other two (at distance 2) are mediates distant 11, 13 or 33 from the proximate which is not in the triad; if the pair are mediates, the two may be the origin or obverse with a mediate distant 1 or 3 from that proximate (4 types) or two mediates distant 11, 13, 33 from it (3 types). If the pair of obverses are in the set of three marks, the four form a group, which may be proper or improper. If proper, the three may be origin and obverse with a mediate, or a pair of mediates with origin, obverse, or another mediate; 4 types. If improper, the three must be two origins and an obverse, or an origin and two obverses; 3 types.

Five marks evenly distant containing only one pair of obverses, must be a proper group with the obverse of one of its marks; see end of Art. 9. To these we may add the origin or obverse of the proper group with a mediate distant 1 or 3 from the extra mark, or else two mediates distant 11, 13 or 33 from that mark; 7 types.

15. A seven-fold statement with two pairs of obverses may have six marks evenly distant from one another and one oddly distant from them; in this case the six are an origin and five mediates in two different ways, or say two pairs and a two; the remaining mark may be distant 11, 13 or 33 from the two, which gives 3 types.

Otherwise the seven-fold statement must subdivide (as in the last case) into five and two or into four and three. If it subdivide into five and two, the two may be a pair or not. In the first case we have a proper group and the obverse of one of its marks, together with the origin and obverse of the group or a pair of mediates; 2 types. In the second case we have five mediates of an origin or its obverse, to which we may add two proximates distant 11, 13 or 33 from the odd mediate.

12. If a six-fold statement contains two pairs of obverses, these must be either evenly or oddly distant. If they are evenly distant we have an origin, obverse and two obverse mediates, to which two other marks are to be added. These may be both evenly distant; taking one of them as origin, it is associated with 5 mediates, so that there is 1 type only. Or both oddly distant; here there are two cases, according as the distances are 11, 33 or 13, 13. Or one oddly and one evenly distant; the latter is any one of the four remaining mediates, and then the former is distant 1 or 3 from it; 2 types. If the two pairs of obverses be oddly distant they form an aggregate which is related in the same way to all the remaining twelve marks; viz. any one of these being taken as origin, we have a pair of mediates and a proximate with its obverse ultimate. The thing to be considered, therefore, is the distance between the two marks to be added, which may be 1, 2 or 3, and each in two ways; 6 types.

A six-fold statement with three pairs of obverses is one of two types only; viz. these are all evenly distant when they are the mediates to one origin, or two evenly distant and one oddly distant from both of them.

13. A pure seven-fold statement must consist of a group and a triad; for it must contain a triad, by the same reasoning by which this was proved for a five-fold statement; and then either all the other four marks are oddly distant from this, and so form a group by themselves, or else one of them is evenly distant from the triad and so forms a group with it. If the group is proper, being the proximates to a certain origin, the triad must consist of two mediates and either the origin, the obverse or another mediate; and in the latter case the three mediates are distant 111 or 333 from some proximate; 4 types. If the group is improper, the triad is either all origins or all obverses, or two origins and an obverse, or an origin and two obverses; 4 types. In all, 8 types of pure seven-fold statement.

14. A seven-fold statement with one pair of obverses must consist either of four marks evenly distant from one another

group. This triad may be the origin or obverse together with two mediates distant 11, 13, 33 from the ultimate; 6 types; or else it may be three mediates distant 111, 113, 133, 333 from the ultimate; 4 types.

19. An eight-fold statement with two pairs of obverses must subdivide into four and four, or into five and three, or into six and two. In the first case the two pairs of obverses may be evenly distant, when the remaining marks form a group either proper, with its origin, obverse, and pair of mediates, or two pairs of mediates, or else improper; 3 types; or oddly distant, when the remainder form one of the six pure four-fold statements enumerated Art. 6. Two marks distant 2 from each other may be distant 11, 33 or 13, 13 from the pair of obverses which are oddly distant from them; thus each of the six four-fold statements gives 3 types of eight-fold statement, except the third, which gives 4; in all, 19. In the second case the three may be a triad, or may contain a pair of obverses. If it is a triad, the five are mediates to one origin and its obverse, and we may add three proximates distant 113 or 133 or two proximates distant 11, 13 or 33, with an ultimate distant respectively 11 or 33 from the odd mediate; 6 types. If the three contain a pair of obverses, the five make a proper group with obverse of one mark; to this we may add the origin and obverse of the group with mediate distant 1 or 3 from the ultimate, or a pair of obverse mediates with a mediate distant 1 or 3 as before; 4 types. In the third case the six must be an origin and five mediates, and we may add two proximates distant 11, 13, 33 from the odd mediate, or a proximate and an ultimate, or two ultimates, distant as before; 9 types.

20. In an eight-fold statement with three pairs of obverses these may be either all evenly distant, or two of them evenly distant and the other oddly distant from both. In the first case they are mediates to a certain origin and its obverse, and we may add the origin with a proximate or ultimate, two proximates, or a proximate and ultimate; 4 types. In the second case take the oddly distant pair for origin and obverse; then these are associated with two proximates and their ob-

or a proximate and an ultimate distant 11, 13 or 33 respectively from the odd mediate; 6 types.

If the seven-fold statement subdivide into four and three, the two pairs may be both in the four, or one in the four and one in the three. In the former case we have a triad, to which may be added the origin and obverse and a pair of mediates, or two pairs of mediates; 2 types. In the latter case the four consist of an origin and obverse and two mediates; we must add a pair consisting of a proximate and an ultimate, which may be distant 11, 33 or 13, 13 from the two mediates, and then another proximate or ultimate which may be distant 11, 13, or 33 from the two mediates; 6 types.

16. Three pairs of obverses in a seven-fold statement may be all evenly distant, or two evenly and the other pair oddly distant from each. If they are all evenly distant they are the mediates to a certain origin or its obverse, and the seventh mark may be the origin or a proximate; 2 types. In the other case we have an origin, obverse, and pair of mediates, together with a proximate and its obverse ultimate; we may add a proximate or a mediate; 2 types.

17. A pure eight-fold statement must consist of two groups, either both proper or both improper, or one of each. Two proper groups may have their origins distant 1 or 3; 2 types. To an improper group we may add a proper group made of one origin and three obverses, or of three origins and one obverse, or an improper group made of four origins or four obverses, or two origins and two obverses; 5 types. Altogether there are 7 types of pure eight-fold statement.

18. An eight-fold statement with one pair of obverses must subdivide into four and four, or into two and three. In the former case we have a pair of obverses, viz. an origin and its obverse, and two mediates; to which we must add a group formed out of the proximates and ultimates. This group may be proper, (1 type) or improper, the mediator being in regard to it two origins, two obverses, or an origin and an obverse; 3 types. In the latter case the five marks must be a proper group with the obverse of one mark, to which we must add a triad made out of the origin, obverse, and mediates of the

INVOLVING FOUR CLASSES.

13

| Art. | | TABLE (continued). | | | | | | |
|------|--|--------------------|-----|-----|-----|------------------|---------------|-----|
| 10 | 6-fold, pure, three and three four and two | ... | ... | ... | ... | 6 6 | | |
| | | | | | | 12 | 12 | |
| 11 | „ 1 pair obv., two and four three and three four and two five and one | ... | ... | ... | ... | 3 8 9 2 | | |
| | | | | | | 22 | | |
| 12 | „ 2 pair obv., odd dist., 6; even, 5 3 pair obv. | ... | ... | ... | ... | ... | 22 11 2 | |
| | | | | | | 47 | 47 | |
| 13 | 7-fold, pure; proper group, 4; improper, 4 | ... | ... | ... | ... | ... | 8 | |
| 14 | „ 1 pair obv., four and three three and four five and two | ... | ... | ... | ... | 10 7 7 | | |
| | | | | | | 24 | 24 | |
| 15 | „ 2 pair obv., six and one five and two four and three | ... | ... | ... | ... | 3 8 8 | | |
| | | | | | | 19 | 19 | |
| 16 | „ 3 pair obv. | ... | ... | ... | ... | ... | 4 | |
| | | | | | | 55 | 55 | |
| | Total of less-than-eight-fold statements | | | | | | ... | 159 |
| | Complementary more-than-eight-fold statements | | | | | | ... | 159 |
| 17 | 8-fold, pure | ... | ... | ... | ... | ... | 7 | |
| 18 | „ 1 pair obv., four and four five and three | ... | ... | ... | ... | 4 10 | | |
| | | | | | | 14 | 14 | |
| 19 | „ 2 pair obv., four and four five and three six and two | ... | ... | ... | ... | 22 10 9 | | |
| | | | | | | 41 | 41 | |
| 20 | „ 3 pair obv., all evenly dist. two evenly dist. | ... | ... | ... | ... | 4 8 | | |
| | | | | | | 12 | 12 | |
| | 4 pair obv. | ... | ... | ... | ... | ... | 4 | |
| | | | | | | 78 | 78 | |
| | Grand Total | ... | ... | ... | ... | ... | ... | 890 |

If the same substitutions are performed upon any two divisions their distance will remain unaltered. Hence *if two statements are of the same type the relative distances of the divisions denied by them will be the same*; but it does not follow that whenever these distances are the same the statements are of the same type.

The whole number of statements that can be made about n classes is $2^{2^n} - 2$, when we exclude the absence of any statement and its inverse the denial of the universe. The problem of enumeration consists in the distribution of these under a finite number of types; it requires an exhaustive method of describing the types, and may be checked by counting the number of statements belonging to each.

There is only one type of one-fold statement about n classes, namely, $A/B, \dots, N = 0$; and as the equation of any division to zero gives a statement of this type it follows that there are 2^n such statements.

There are n types of two-fold statement; for the two divisions equated to zero may have any distance from 1 to n . If this distance is r , the number of statements included in the type is $2^{n-1} \begin{Bmatrix} n \\ r \end{Bmatrix} \begin{Bmatrix} n \\ n-r \end{Bmatrix}$. The whole number of two-fold statements is of course $2^{n-1}(2^n - 1)$.

To determine the number of types of three-fold statements, let α, β, γ be the three divisions denied by such a statement, and suppose the distance $\beta\gamma$ to be not less than either of the distances $\alpha\beta, \alpha\gamma$. Let $\alpha\beta = r, \alpha\gamma = s$, and let t changes be common to the r and the s ; then $\beta\gamma = r + s - 2t$. Let r not be less than s ; we have $r + s - 2t \geq r$, therefore $s \geq 2t$, or t must not be greater than half the least of the numbers r, s . With the distances r, s then we shall have distinct types for the values of $t = 0, 1, 2, \dots, I \left(\frac{s}{2} \right)$ (I meaning *integral part of*); that is $I + I \left(\frac{s}{2} \right)$ types, provided that $r + s$ is not greater than

ENUMERATION OF THE TYPES OF COMPOUND STATEMENTS.

A SET of n classes A, B, C, \dots involve a number 2^n of cross-divisions A, b, C, d, \dots etc.; these shall be called simply *divisions*. A *statement* about these classes affirms the non-existence of one or more divisions; and it is called a one-fold, two-fold, etc., statement, according to the number of divisions whose existence it denies. A one-fold statement about a set of n classes is a 2^n -fold statement about a set of these and m other classes. Two statements are called *inverses* when each denies the divisions whose existence is allowed by the other; thus the inverse of a p -fold statement about n classes is a $(2^n - p)$ fold statement.

One division may be converted into another by the operation of changing certain classes into their complementary classes (A into a which is not- A). The number of these changes required is called the *distance* of the two divisions. Thus in four classes, the division $ABCD$ is distant two changes from the division $abcd$.

Two statements are said to be of the same type when one may be obtained from the other by a repetition of two processes: (1) the simultaneous substitution of a class and its complementary for another class and its complementary (A for B and a for b), (2) the interchange of a class and its complementary (A for a). It is clear that this amounts only to a use of new symbols for the classes involved.

III.

ON SOME PORISMATIC PROBLEMS*.

THE PROBLEM : To draw a polygon of a given number of sides, all whose vertices shall lie on one given conic, and all whose sides shall touch another given conic : is either not possible at all, or possible in an infinity of ways. This remark, originally made by Poncelet, has been shown by Professor Chayley to depend in a very beautiful manner upon the theory of elliptic functions; and in this way he has proved that an analogous theorem holds good wherever a $(2, 2)$ correspondence exists: that is to say, whenever two things are so related that to every position of either there correspond two positions of the other. Two points, x, y for instance, in a conic U , which are connected by the relation that the line xy touches a second conic V , have a correspondence of this kind: for if the point x be taken arbitrarily, two tangents can be drawn from it to V , determining two positions of y ; and conversely, the point y being fixed determines two positions of x . The theorem is then that in a $(2, 2)$ correspondence there is either no closed cycle of a given order, or an infinite number. In the present communication I propose first to prove this result by the method of correspondence alone, and then to extend the proof to higher orders of correspondence.

In a $(2, 2)$ correspondence there are $4(-2+2)$ united points, that is to say, four points, each of which coincides with one of its correspondents. In fact, if two numbers x and y are

* [From *Cambridge Philosophical Society's Proceedings*, n. 1670. Read Nov. 9, 1869, pp. 120-123.]

n ; in this case t cannot be less than $r + s - n$. First let r be even and equal to $2p$, where $4p$ is not greater than n . Then the number of types is

$$\begin{aligned} & 1 + 2 + 2 + 3 + 3 + \dots + p + p + \overline{p+1} \\ & = p + p (p+1) = p (p+2) \\ & = \frac{1}{2} r (r+1), \quad 2r \nless n. \end{aligned}$$

Next let $r = 2p + 1$, $2r \nless n$; then the

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[Prof. Cayley remarks that Prof. Clifford considers, in this fragment, the further question, how many are the statements of each type. In the case of 4 classes, the results up to the 3-fold statement, see Table, p. 12, are as follows:—

| | | | | | | No. of Statements. |
|-------------------|-----|-----|-----|-----|-----|--------------------|
| 1-fold | ... | ... | ... | ... | ... | 16 |
| 2-fold, dist. 1 | ... | ... | ... | ... | ... | 32 |
| " 2 | ... | ... | ... | ... | ... | 48 |
| " 3 | ... | ... | ... | ... | ... | 32 |
| " 4 | ... | ... | ... | ... | ... | 8 |
| 3-fold, dist. 112 | ... | ... | ... | ... | ... | 96 |
| " 222 | ... | ... | ... | ... | ... | 64 |
| " 123 | ... | ... | ... | ... | ... | 192 |
| " 233 | ... | ... | ... | ... | ... | 96 |
| " 134 | ... | ... | ... | ... | ... | 64 |
| " 224 | ... | ... | ... | ... | ... | 48 |
| | | | | | | 560 |

where the totals 16, 120, 560, &c. are, of course, the numbers for the combinations of 16 things 1, 2, 3, &c. together.

Prof. Jevons in *The Principles of Science* (Third Edition, p. 149) writes: "In the first edition (vol. i. p. 163), I asserted that some years of labour would be required to ascertain even the precise number of types of law governing the combination of four classes of things. Though I still believe that some years' labour would be required to work out the types themselves, it is clearly a mistake to suppose that the *numbers* of such types cannot be calculated with a reasonable amount of labour, Professor W. K. Clifford having actually accomplished the task." It was on reading Prof. Jevons's original statements that Prof. Clifford said laughingly, he couldn't let anyone say that and not do it straight off...it was 'his luck' to do quickly difficult things which were usually got at by long processes only.

Dr Hopkinson has drawn my attention to the fact that the problems treated of by Clifford and Jevons are different in this respect, that Clifford's result includes what Jevons calls 'inconsistent statements'.]

(which determines the improper solutions) will have more than four roots, and will therefore be identically satisfied by any number whatever; so that whatever point α we start with, the point w will come to coincide with it.

Precisely similar reasoning is applicable to the cycles of an even order. Thus, *e.g.* for a quadrilateral we get the four improper solutions $\gamma\beta\alpha\beta$, got by starting from the points γ . I pass to the consideration of correspondences of higher orders.

In an (r, r) correspondence there are

$2r$ united points a ;

their remaining correspondents form

$2r(r-1)$ points b ;

to these again correspond

$2r(r-1)^2$ points c , and so on.

Similarly, there are

$2r(r-1)$ points α ,

each of which is such that two of its correspondents coincide; viz. these are

$2r(r-1)$ points β ,

to which also correspond

$2r(r-1)^2$ points γ , and so on.

Now if we attempt to form a closed cycle of the n^{th} order, we shall be led to a correspondence

$$\{r(r-1)^{n-1}, r(r-1)^{n-1}\},$$

which has $2r(r-1)^{n-1}$ united points. From this number we shall have to subtract the number of improper solutions as given by our previous reasoning; thus we shall find

$(n=2m+1)$, $\{2r(r-1)^{2m} - 2r(r-1)^m\}$ proper solutions,

$(n=2m)$, $\{2r(r-1)^{2m-1} - 2r(r-1)^m\}$ proper solutions.

For example, the problem to inscribe in a conic a triangle whose sides shall touch a given curve of the third class admits of twelve proper and twelve improper solutions. If the number of proper solutions exceeds this number, the problem becomes porismatic: that is to say, there is an infinite number of solutions.

connected by an equation of the second degree in each of them, then when we make x and y coincide, there results an equation of the fourth degree (Chasles, *Comptes Rendus*, 1864). I call these united points the points a . Each point a has one of its correspondents coinciding with it; it has also another correspondent b . Each point b again has another correspondent c , and so on. There are also four points α , each of which is such that its two correspondents coincide in a point β . For let q be a correspondent of p , and r a correspondent of q ; then the relation between p and r is a $(2, 2)$ correspondence (since to each position of p there are two positions of r and *vice versa*), and therefore has four united points, viz. the points β . Each of these points β has another correspondent γ , and so on. We have thus two series of points, $abcd\dots a\beta\gamma\delta\dots$ each letter indicating a set of four generally distinct points.

Let us now endeavour to obtain a closed cycle of an odd order: for distinctness' sake we will try to draw a pentagon inscribed in one conic, U , and circumscribed to another, V^* . Start with a point x on the outer; pass to one of its correspondents, y ; y has another correspondent, z ; from z we go to u , from u to v , from v to w . If now w were the same point as x , we should have succeeded in our object. But the relation between w and x is a $(2, 2)$ correspondence, for we might have started from x in either of two directions. The united points of this correspondence should therefore apparently give solutions of our problem.

But these united points are no other than the four points c . For starting with one of these, we get the cycle $cbaabo$, which is a sufficient solution of the correspondence problem last enunciated. But it is *not* a solution of the original problem: for the series will go on $cbaabcde\dots$ and not repeat itself, so that the points $cbaab$ do not form a *proper* in-and-circumscribed pentagon. Thus the problem is in general impossible. If however there is any proper solution, the equation of the fourth degree

* [Supposing as before that the corresponding points x, y in the conic U are such that the line xy touches the conic V , then, as is easily seen, the points a are in fact the points of contact with U of the common tangents of U and V ; and the points α are the points of intersection of U and V .—C.]

IV.

PROOF THAT EVERY RATIONAL EQUATION HAS A ROOT*.

(Abstract.)

THE proof contained in the present communication depends on the determination of a quadratic factor of the rational integral expression

$$a^{2s} + a_1 a^{2s-1} + a_2 a^{2s-2} + \dots + a_{2s}.$$

On dividing this expression by $x^2 + p_1 x + p_2$, we obtain by the ordinary algebraic rules a remainder of the form $M_{2s-1} x + N_{2s}$, where M_{2s-1} and N_{2s} are functions of p_1 and p_2 , whose weights are $2s-1$ and $2s$ respectively, and which may accordingly be written in the forms

$$M_{2s-1} = b_{2s-1} + p_1 b_{2s-2} + \dots + p_s^{s-1} b_1, \\ N_{2s} = c_{2s} + p_1 c_{2s-1} + \dots + p_s^s c_0,$$

where the b, c are of an order in p_1 indicated by their suffixes. On writing down (by Professor Sylvester's Dialytic method) the result of eliminating p_2 between these equations, it is at once apparent that this resultant is of the order $s(2s-1)$. Thus the determination of a quadratic factor of an expression of degree $2s$ is reduced to the solution of an equation of order $s(2s-1)$. But this number is *one degree more odd* than the original number $2s$; that is to say, if the number $2s$ is 2^k multiplied by an odd number, then $s(2s-1)$ is 2^{k-1} multiplied by an odd number. Hence by a repetition of this process we shall ultimately arrive at an equation of odd order, which, as is well known, must have a real root. By then retracing our steps the existence of a quadratic factor of the original expression is demonstrated.

* [From *Cambridge Philosophical Society's Proceedings*, II. 1870. Read Feb. 21, 1870, pp. 156, 157.]

VI.

ON JACOBIANS AND POLAR OPPOSITES*.

1. THE word Jacobian is commonly understood to mean a determinant formed from the n^2 differential coefficients of n functions, each of n variables. For instance, given two homogeneous functions of the second degree in x and y , as U, V , each representing two points; the Jacobian is

$$\begin{vmatrix} \frac{dU}{dx} & \frac{dU}{dy} \\ \frac{dV}{dx} & \frac{dV}{dy} \end{vmatrix},$$

which is known to represent the foci of the involution determined by U, V . I propose to extend the meaning of this term so as to include *systems* of determinants, formed in precisely the same way from a number of functions *not* equal to the number of variables. For instance, if $U=0$ is the trilinear equation to a conic section, and $L=0$ that to a straight line, then the system of determinants

$$\begin{vmatrix} \frac{dU}{dx} & \frac{dU}{dy} & \frac{dU}{dz} \\ \frac{dL}{dx} & \frac{dL}{dy} & \frac{dL}{dz} \end{vmatrix} = 0,$$

will be found to represent the pole of the line L with respect to the conic U ; and I propose to call this system the Jacobian

* [From *The Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. II. pp. 229-239.]

flat; namely, that the ordinary laws of geometry are not valid in them.

(2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

(3) That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or etherial.

(4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity.

I am endeavouring in a general way to explain the laws of double refraction on this hypothesis, but have not yet arrived at any results sufficiently decisive to be communicated.

[In Paper XV. (infra) reference is made to a *proof that every rational equation has a root*, but I have not found any MS. of which IV. could be looked upon as the Abstract. Prof. Clifford once remarked to me, I think, that a paper on the same subject by Mr J. C. Malet (read before the Mathematical Society, June 14th, 1877, and printed in the *Transactions of the Royal Irish Academy*, Vol. XXVI. No. xiv.) treated the question from a somewhat similar point of view. The subject of V. was introduced at greater length to English mathematicians in a translation (IX. infra) of Riemann's *Habilitationsschrift*, 1854; see the *Gesammelte mathematische Werke*, pp. 254—269.]

which is the polar opposite of the corresponding vertex; such a triangle is merely the common self-conjugate triad of the two sections made by its plane. Again, the line BC' joining the two poles of any plane A , may be called the polar opposite line of the plane; and the cones drawn from the point A touching the two conicoids, will have BC' and A for polar line and plane; and the same will be true of the cone drawn touching any conicoid inscribed in the same developable as the two given ones. This again suggests a system of three planes, each the polar opposite of the line of intersection of the other two; but such a system is merely the self-conjugate triad of the two tangent cones drawn to the conicoids from the point of intersection of the planes.

IV. I pass to the case of *three* conicoids. Here a point will have three polar planes, meeting in its polar opposite. Let A and B be opposites, then it is easy to see, as before, that the line AB is harmonically divided by each of the conicoids, and therefore by any conicoid through their eight points of intersection; e.g. by any pair of planes which can be drawn to contain the eight points. Similarly, the three poles of a plane A are joined by its polar opposite plane B , and the line AB is subtended by the conicoids in an involution, of which A and B are the double or self-conjugate planes.

V. I proceed now to the interpretation of some Jacobians, and commence with those of binary quantities. I shall throughout use the letters L, M, X, \dots for equations of the first degree, and generally U, U', \dots for those of the second.

1. $J(L, M)$ is obviously the distance between the points L and M^* . In general, if L, M are linear functions of any number of variables, $J(L, M) = 0$ is the condition of their coincidence. The distance between two points may also be expressed thus: if Δ stands for $\left(\begin{smallmatrix} d \\ d_x d_y \end{smallmatrix} \right)$, then ΔL is the

* Strictly, the distance is $\frac{J(L, M)}{(d_x + v_1)(d_y + v_2)}$, between $L_x + v_1 y$ and $L_y + v_2 y$.

of U and L . In all cases I shall denote the Jacobian of a number of functions A, B, C, \dots whether equal or unequal to the number of variables, by the symbol $J(A, B, C, \dots)$.

II. As the theory of polar opposites is of constant occurrence in the interpretation of Jacobians, I here set down the heads of it. If the polars of a point A with respect to two conics meet in B , then it is clear that the polars of B will meet in A . On account of this reciprocal relation, I call the points A and B *polar opposites* with respect to the two conics. Consider first one conic; it is known that any line through a point A is harmonically cut by its polar at B and by the conic. The line AB , then, is harmonically divided by both conics, and therefore by any conic through their intersections; *e.g.* by any pair of common chords. It is thus evident that any straight line has a pair of polar opposites upon it, which are in fact the foci of the involution in which the line is cut by the conics.

In the same way it may be seen, that if a straight line A joins the poles of B with respect to two conics, B will join the poles of A , and the two lines may be called polar opposites of one another. The point AB is subtended by the two conics and the six intersections of common tangents in an involution of ten rays, of which A and B are the double or sibi-conjugate rays. And through any point we can draw a pair of polar opposite lines, which are in fact the double lines of the involution determined by the four tangents drawn from the given point to the conics.

III. Consider next the case of two conicoids: here the polar planes of any points will intersect on a line, which may be called the polar opposite line of the point. Let A be the point, and BC its opposite line, then it is easy to see that the plane ABC will cut either conicoid in a section, with respect to which A is the pole of BC , and that the same will be true of any conicoid passing through the curve of intersection of two given ones; *e.g.* of the four cones which can be so drawn. This naturally suggests the idea of a triangle, each side of

conjugates with respect to the quadric. If the quadric is the Hessian of the cubic, their Jacobian is no other than the cubi-covariant J . (Salmon's *Higher Algebra*, p. 99.) The Hessian represents two points, each of which is the polar quadric of the other; and if we take the harmonic conjugate of each point of the cubic with respect to its Hessian, we shall obtain the covariant J . Let A, B, C be the points of the cubic, and D, E its Hessian; then the three anharmonic ratios

$$[ADBE], [BDCE], [CDAE],$$

are all equal to one another, as appears readily from the canonical form of the cubic. The polar point of any of the points of J with regard to the cubic, being the same as its harmonic conjugate with respect to the Hessian, must be one of the points of the cubic itself; and in this case, the relation at the end of (4), shews that the four points form an harmonic range. We are thus led to Dr Salmon's construction for the covariant J , viz. it contains the harmonic conjugate of each point of the cubic relatively to the other two. In general the Jacobian of a cubic and a quadric does not represent the harmonic conjugates of the cubic with respect to the quadric. $J(S, T)$, S and T being cubics, of course represents the four points whose polar points are the same with regard to the two cubics. The Jacobian of a cubic and its covariant J is the discriminant multiplied by the square of the Hessian.

VI. I put together the most analogous forms of Jacobians in three and four variables, reserving part of the latter for separate consideration.

1. $J(L, M, N) = 0$ and $J(L, M, N, R) = 0$ are known to be respectively the conditions that the three lines and four planes may meet in a point*.

* This Jacobian enables us to express many metrical functions of lines and planes. Let $X=0$ denote the line or plane at infinity, $\phi(L)$ the condition that the line L shall pass through either circular point at infinity, and let

$$\phi(L + \kappa M) \equiv \phi(L) + 2\psi(L, M) \cdot \kappa + \phi(M) \cdot \kappa^2.$$

Consider now three straight lines L, M, N , and put J for the Jacobian,

distance between L and (ξ, η) , and generally, with any number of variables, ΔL is proportional to the distance between L and the point (ξ, η, ξ, \dots) .

2. $J(U, L)$, where U is a quadric, is the harmonic conjugate of L with respect to U . This may also evidently be expressed by ΔU . For instance, the identity

$$\Delta(\Delta U, \Delta U) = \Delta^2 U, \Delta U + \Delta U, \Delta^2 U$$

may be interpreted thus, if we remember that $\Delta^2 U$ is the product of the distances of (ξ, η) from the two points denoted by U : "Let P and Q be the harmonic conjugates of a point O with respect to A, B and C, D respectively, and let L be the harmonic conjugate of the same point with respect to P, Q ; then

$$\frac{OP}{OA + OB} + \frac{OQ}{OC + OD} = 0."$$

3. $J(U, V)$, U and V being quadrics, is well known to represent the foci of the involution determined by U and V .

4. $J(S, L)$, where S is a cubic, is the polar quadric of L with regard to S ; that is to say, if S represents the three points A, B, C , then $J(S, L)$ represents two points P , each of which possesses the property

$$\frac{LA}{PA} + \frac{LB}{PB} + \frac{LC}{PC} = 0.$$

They might also evidently have been represented by $\Delta S = 0$. The polar point, $\Delta^2 S$, is the harmonic conjugate of L with respect to the two points P ; and if we call it R , it possesses the property

$$\frac{RA}{LA} + \frac{RB}{LB} + \frac{RC}{LC} = 0.$$

5. The interpretation of all Jacobians not involving point is easy, when we remember that $J(A, B)$ is in fact the eliminant of $\Delta A, \Delta B$. Thus $J(S, U)$, S being a cubic and U a quadric, represents the three points, whose polar point with respect to the cubic, are the same as their harmonic

showing that it always circumscribes the common self-conjugate triangle. The discriminant of this is $ABClmn$, whose evanescence is the condition that L shall pass through one of the vertices of the triangle. Thus we see that *the discriminant with respect to x, y, z of the Jacobian of two conics and the line $x + ny + \xi z = 0$, is a contravariant representing the vertices of the common self-conjugate triangle.* So with four variables, (U, V, W, L) is both the locus of polar opposites of points in the plane L with regard to U, V, W , and the locus of the poles of L with regard to all the conicoids $lU + mV + nW = 0$. And with a similar notation to that employed above, in case the three conicoids have a common self-conjugate tetrahedron, the Jacobian may be written

$$\frac{lA}{x} + \frac{mB}{y} + \frac{nC}{z} + \frac{rD}{w} = 0,$$

showing that it contains the edges of the tetrahedron and has double points at the vertices. The discriminant is $ABCDlmnr$, which gives, as before, the theorem, that *when three conicoids have a common self-conjugate tetrahedron, the discriminant with respect to $xyzw$ of the Jacobian of the three conicoids and the line $\xi x + ny + \xi z + \omega w = 0$ is a contravariant representing the vertices of the tetrahedron.* When, in the plane case, the line L is at infinity, the Jacobian represents, as remarked by "Lancicensis" in the last *Messenger*, the nine-point conic of the quadrangle which is the intersection of U and V ; and we see that the polars of any fixed point on this conic, with regard to any conic $lU + mV = 0$, are parallel to a fixed line. In the solid case, it must be remembered that when three conicoids have a common self-conjugate tetrahedron, their eight points of intersection will form a figure which is the "projection" of a parallelepiped, and so $lU + mV + nW$ may in six ways be made to represent two planes. By sending the plane L to infinity, then, we learn that eight points so situated determine a surface of the third order, which bisects the line joining any two of the points, and contains the lines in which intersect the six pairs of planes containing the eight points. Moreover, the section of this surface made by any one of the planes consists of the

2. $J(U, L, M)$, U being a conic, and L, M lines, is the polar of the point LM with respect to U , or the locus of the pole of $UL + mM = 0$. Similarly, if U is a conicoid, and L, M, N planes, $J(U, L, M, N)$ may be interpreted either as the polar plane of the point LMN , or as the locus of the pole of $UL + mM + nN = 0$, l, m, n being arbitrary. $J(U, L)$ means, as before, the pole of L with respect to U .

3. $J(U, V, L)$ is, first, the locus of polar opposites of points on the line L ; and, secondly, the locus of the pole of L with respect to $UV + mV = 0$.

If we write U, V in the canonical forms

$$ax^2 + by^2 + cz^2 = 0,$$

$$ax'^2 + by'^2 + cz'^2 = 0,$$

then, putting A for the determinant $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$, &c., and

$$lx + my + nz = 0$$

for the equation to the line L , we find the Jacobian to be

$$\frac{lA}{x} + \frac{mB}{y} + \frac{nC}{z} = 0,$$

P for $J(X, M, N), J(L, X, N), J(L, M, X)$, and H for $\phi(L), \phi(M), \phi(N)$, then

$$\text{area of triangle } LMN = \frac{J^2}{P},$$

$$\text{product of sines} = 2\sqrt{3} \frac{J^2 \sqrt{(H)}}{P^2},$$

$$\frac{(\text{area})^2}{\text{product of sines}} = \frac{J}{2\sqrt{(2H)}}$$

(a function of some little importance),

$$\text{radius of circumscribed circle} = \frac{J}{P} \sqrt{(3H)},$$

$$\text{product of sines of angles} = \frac{P}{H}.$$

It is to be noticed that (as is indeed evident) $\phi(L), \phi(M), \phi(L, M)^2$ is identically equal to $J(L, M, X)^2$. See Mr. Grace's valuable "Notes," (*Quarterly Journal* for March, 1864).

Similarly, volume of tetrahedron $LMNR = \frac{J^3}{P}$, and so on. In all these results there is a factor present, which is easily determined by reference to the fundamental triangle or tetrahedron.

polar lines are identical with regard to the four conicoids, will lie on the Jacobian; as is implied by the property last stated.

VII. There are some Jacobians in four variables which have no analogues* in three variables. The simplest is:

1. $J(U, L, M)$, the polar line of LM .

2. $J(U, V, L)$ is the locus of points whose polar opposite lines lie in the plane L , and the locus of the poles of L with regard to all conicoids $UU + mV = 0$. It is a twisted cubic passing through the vertices of the tetrahedron self-conjugate with respect to U and V , and cutting the plane L in the vertices of the self-conjugate triad of the sections in which it meets L , V .

3. $J(U, V, L, M)$ contains the polar opposite lines of all points in the line LM , and the polar lines of LM with regard to all conicoids $UU + mV = 0$. It also contains the poles of all planes of the form $\lambda L + \mu M = 0$ with respect to $UU + mV = 0$. It is clearly a conicoid passing through the vertices of the tetrahedron self-conjugate with regard to U, V , and containing all the twisted cubics

$$J(U, V, U, \lambda + mM) = 0.$$

4. $J(U, V, W)$ is a curve of the sixth degree containing the vertices of all cones drawn through the eight intersections of U, V, W (Dr Salmon). Hence it is the locus of the vertices of tetrahedra self-conjugate with regard to two conicoids

$$UU + mV + nW, \quad \lambda U + \mu V + rW.$$

If the three conicoids have a common self-conjugate tetrahedron, this curve represents its edges.

I have given no account of the Jacobians of ternary or quaternary cubics, but their interpretation will be easy with the aid of the following theorem. Consider any number of groups, $A_1, A_2, A_3, \dots; B_1, B_2, \dots; C_1, C_2, \dots$, &c.; all the

* 1 and 2 seem closely analogous to the 2, 3 of p. 28. C.1

line where it meets the corresponding plane, and the nine-point conic of the quadrangle formed on it by the other four planes. And the polar planes of any fixed point on this surface, with respect to all conicoids $lU + mV + nW$, will be parallel to a fixed line.

4. $J(U, V)$ represents the vertices of the common self-conjugate triangle or tetrahedron, according as U and V are conics or conicoids.

5. $J(U, V, L, M)$ is the polar opposite of the point LM with regard to the conics U, V . It is clearly the point common to the Jacobians of U, V , and all lines through the point LM . Also the form of the Jacobian shews that it is common to the polars of LM with respect to all the conics $lU + mV = 0$. So, in Solid Geometry, $J(U, V, L, M, N)$ is the polar opposite line of the point LMN ; and so also $J(U, V, W, L, M, N)$ is the polar opposite point of LMN with respect to U, V, W .

6. $J(U, V, W)$, the Jacobian of three conics, will be found explained at the end of Dr Salmon's *Conic Sections* and in his *Higher Algebra*. I have only to add that it is the locus of the vertices of all triangles self-conjugate to two conics of the forms

$$lU + mV + nW = 0, \quad \lambda U + \mu V + \nu W = 0,$$

and that lines whose three poles lie in a straight line, meet the lines joining their poles on the Jacobian. So if U, V, W, X are four conicoids, $J(U, V, W, X)$ is the locus of points whose polar planes meet in a point, this latter being also a point on the Jacobian; and the locus of the vertices of all cones which can be represented by

$$lU + mV + nW + rX = 0;$$

and consequently the locus of the lines of intersection of all pairs of planes which can be represented by this equation. Moreover it contains the vertices of all tetrahedra self-conjugate with regard to two conicoids of the above form; and the edges of all tetrahedra self-conjugate with regard to three conicoids of the same form. And obviously, any line whose

† Theorem is a particular case of the theorem that three conics inscribed in the same quadrilateral subtend any vertex in revolution (*Educational Times* for December)*. It will be seen that Art. II. is only a development of this remark.

3. The last note on p. 34 of Dr Salmon's *Conics* will be a little plainer if stated as a property of the nine point conic of any quadrilateral inscribable in a circle. For

$$J(U, V, L) = L, J(U, V, L).$$

So, if U is a sphere, and L the plane at infinity, the twisted cubic $J(U, V, L)$ passes through the feet of the six normals that can be drawn from the centre of U to the conicoid V .

4. If A and B are polar opposites with respect to two circles, AB is bisected by the radical axis, and the circle on AB as diameter cuts both circles orthogonally. Hence, immediately, Dr Salmon's theorem, *Conics*, p. 314, Ex. II, that the Jacobian of three circles is the circle cutting them orthogonally, together with the line at infinity. It is thus seen that the polars of a point A on the Jacobian meet in the opposite extremity of the diameter through A .

* (*Mathematical Questions with their Solutions*, From the *Edinburgh and London*, Vol. I. (p. 30).)

A 's being quantities of the same degree, and all the B 's, &c., but the A 's not necessarily of the same degree as the B 's; and form any number of quantities in involution with these, as

$$l_1 A_1 + m_1 A_2 + n_1 A_3 + \dots (a_1),$$

$$l_2 A_1 + m_2 A_2 + n_2 A_3 + \dots (a_2),$$

$$\dots\dots\dots \&c.$$

Now suppose that we have a geometrical interpretation for $J(A, B, C)$, for $J(A_1, A_2, B, C)$, and so on; then

$$J(A_1, A_2, \dots; B_1, B_2, \dots; C_1, C_2, \dots)$$

will include all the loci formed in the same way as

$$J(a, b, c)^*; \quad J(a_1, a_2, \dots; b_1, b_2, \dots; c_1, c_2, \dots), \&c.$$

VIII. I add an example or two to shew that the theory of Polar Opposites is not wholly barren of results.

1. A pair of conjugate diameters cuts the line at infinity in two conjugate points, *i.e.* two points each of which lies in the polar of the other. Hence any two conics have two pairs of conjugate diameters parallel, which cut the line at infinity in a pair of polar opposites†. A particular case is when the axes are parallel, when we see at once, by (II.), that any pair of common chords will be equally inclined to them, and so a circle will pass through the quadrangle of intersection. Similarly, if three conicoids have their principal planes parallel, the eight points of intersection will lie on a sphere.

2. If from the centre O of any circle a normal ON be drawn to a conic, and if two common tangents of the circle and conic intersect in A , and the other two in B , then AN and BN are equally inclined to ON . For it is easily seen that ON and the tangent at N are polar opposite lines. When the point O is on the conic, Professor Cayley has shewn that this

* [$J(a_1, b_1, c_1)$.]

† [*i.e.* parallel to each other, and each cutting the line at infinity in one and the same pair of polar opposites. O .]

axis whatever, is equal to the moment about a parallel axis through the centre of gravity, together with the moment which the whole mass, if collected at the centre of gravity, would have about the original axis.

These things being so, the following is a construction for the moment of inertia about any axis PQ through a point P . Draw a plane PT through P perpendicular to the axis; then determine λ so that the surface

$$\frac{x^2}{A + \lambda M} + \frac{y^2}{B + \lambda M} + \frac{z^2}{C + \lambda M} = \frac{1}{M},$$

may touch this plane PT ; this gives a simple equation* for λ , which has consequently only one value. Then the required moment of inertia is $M(OP^2 - \lambda)$, O being the centre of gravity, which is the origin.

For, draw OT perpendicular to the plane PT ; and let Ot be the perpendicular on the parallel plane which touches the ellipsoid of gyration. Then OT being a parallel axis through the centre of gravity, and PT the perpendicular distance of O from PQ ; the moment of inertia about PQ is

$$M(Ot^2 + PT^2).$$

$$\begin{aligned} \text{But}^\dagger \quad OT^2 &= l^2 \left(\frac{A}{M} + \lambda \right) + m^2 \left(\frac{B}{M} + \lambda \right) + n^2 \left(\frac{C}{M} + \lambda \right) \\ &= \frac{l^2 A + m^2 B + n^2 C}{M} + \lambda \\ &= Ot^2 + \lambda. \end{aligned}$$

$$\begin{aligned} \text{Hence} \quad Ot^2 + PT^2 &= OT^2 + PT^2 - (OT^2 - Ot^2) \\ &= OP^2 - \lambda, \end{aligned}$$

the moment of inertia is therefore $M(OP^2 - \lambda)$.

Now, consider the confocal ellipsoid which passes through P , and its tangent plane (A) at P . Let λ_1 be the value of λ for this ellipsoid. Then if the plane A be turned ever so little round P , it will begin to cut the ellipsoid in a small ellipse;

* Salmon's *Geometry of three dimensions*, p. 121.

† Salmon, *l. c.* p. 58.

VII.

ON THE PRINCIPAL AXES OF A RIGID BODY*.

THE object of this note is to simplify the manner in which the theory of principal axes is made to depend on the theory of confocal surfaces of the second order.

It is well known†, that if A, B, C are the principal moments of inertia of a rigid body at the centre of gravity, the moment of inertia about an axis through the centre of gravity, whose direction-cosines referred to the principal axes are l, m, n is

$$l^2A + m^2B + n^2C \dots\dots\dots (1).$$

But if M be the mass of the body, and we draw the ellipsoid

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{1}{M} \dots\dots\dots (2),$$

(ellipsoid of gyration), this amounts to saying that the moment of inertia is Mp^2 , where p is the perpendicular from the centre on the tangent plane

$$lx + my + nz = p \dots\dots\dots (3),$$

of the ellipsoid (2); that is, if we draw a plane perpendicular to the given axis to touch the ellipsoid (2), then p is the central perpendicular on this plane.

It is also known‡ that the moment of inertia about any

* [From *The Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. iv. pp. 78-81.]

† Routh's *Rigid Dynamics*, p. 7.

‡ Routh, *l. c.* p. 9.

any point on the focal hyperbola, the momental ellipsoid is a prolate spheroid.

Again*, two confocal surfaces can be drawn to touch any given straight line, and the two corresponding tangent planes are at right angles. Hence, on every straight line there are two points at which one principal axis is perpendicular to the line, and these two axes are at right angles. If the two points of contact coincide, then at that point two principal axes are perpendicular to the line, which is therefore itself a principal axis. The condition, therefore, that an axis may be a principal axis at some point of its length, is that the two points of contact of confocal surfaces touching it must coincide; which is obvious, for in that case the line is normal to the third surface passing through the common point of contact.

* Salmon, *l. c.* p. 127.

and so the confocal ellipsoid which touches the plane λ in its new position, will lie wholly within the other. Its axes will therefore be less than those of the other, and therefore λ will be less than λ_1 . That is to say, the value of λ is a *maximum* for the plane which touches the confocal ellipsoid through P . Therefore the moment of inertia about the axis perpendicular to this plane, namely $M(OI^2 + \lambda_1)$, is a *minimum*. Now the axis of least moment at any point is a principal axis. It follows therefore, that *the normal to the confocal ellipsoid through P is the principal axis of least moment at P .*

In a manner precisely similar, it may be shown that if we draw through P an hyperboloid of two sheets confocal to the ellipsoid of gyration, the value of λ for its tangent plane at P is a *minimum*; and therefore that the normal to this surface at P is the principal axis of greatest moment at P .

This being so, we know that the remaining principal axis is perpendicular to these two, and is therefore normal to the confocal hyperboloid of one sheet which passes through P .

We have proved then that the principal axes at any point P , are the normals to the three surfaces confocal to the ellipsoid of gyration which pass through P ; and if λ , μ , ν are the values of λ for these three surfaces, or, as we may say, if λ , μ , ν are the curvilinear co-ordinates of P in respect of the ellipsoid of gyration, then the three moments of inertia are

$$M(OI^2 + \lambda), \quad M(OI^2 + \mu), \quad M(OI^2 + \nu).$$

This connection having been established, all the usual propositions about principal axes follow at once from the known theory of confocal surfaces. Thus, the locus of points where two of the principal moments are equal, is the focal conics of the ellipsoid of gyration. For*, of the three confocal surfaces which pass through any point on a focal conic, two coincide with the focal conic itself. For any point on the focal ellipso, the momental ellipsoid is an oblate spheroid; and for

* Salmon, *l. c.* p. 114.

VIII.

SYNTHETIC PROOF OF MIQUEL'S THEOREM*.

IN the note to page 235 of Dr Salmon's *Conics*, mention is made of a theorem originally given by M. Auguste Miquel, in Liouville's *Journal*, Vol. x., p. 349. The theorem may be stated as follows. It is known that we can draw exactly one parabola to touch four given lines. If now we have five lines given, we can draw five parabolas, each of which touches four of the given lines. The theorem is that the foci of these five parabolas lie all on one circle. M. Miquel's proof, reproduced by Catalan, depends on the fact that the circle circumscribing the triangle formed by three tangents to a parabola passes through the focus. Since, as above remarked, a parabola can be drawn to touch any four straight lines, it follows from this that the four circles, circumscribing the four triangles which we get by leaving out each of the lines in turn, all meet in a point, the focus of the parabola. The theorem can thus be stated as a property of straight lines and circles, without any mention of parabolas; and accordingly M. Miquel proves it by using the ordinary (Euclid) geometry appropriate to such theorems.

I find, however, that the theorem connects itself in a somewhat interesting way with that general conception of geometrical facts which regards all the properties of a curve or other continuous figure as depending upon its order alone, and which it has become usual to call Synthetic Geometry. I am not

* [From *The Oxford, Cambridge and Dublin Messenger of Mathematics*, Vol. v. pp. 124—141.]

The assemblage of points lying upon two straight lines is of the second order, since two of them can be found upon another arbitrary straight line. And generally the assemblage of all the points lying upon n straight lines is of the n^{th} order.

Similarly, an assemblage of lines is said to be of a certain *class*, when a certain number of the lines can be drawn through an arbitrary point. Thus the assemblage of lines passing through a point is of the *first class*, because *one* of the lines can be drawn through another arbitrary point. The assemblage of lines passing through two points is of the second class, because two of the lines can be drawn through another arbitrary point. And generally the assemblage of all the lines passing through n points is of the n^{th} class.

We have now a number appertaining to the aggregate of points, viz. its *order*, and a number appertaining to the aggregate of lines, viz. its *class*. Neither of these numbers belong in strictness to the curve itself; but there is a number—the *Geschlecht-zahl* or *deficiency*—which does belong to the curve, and not immediately to the points or tangents*. It is not however my business to speak of that now. I am going to investigate certain cases in which these two things—the assemblage of points, and the assemblage of tangents—change continuously together; and in which it is very important to observe the modifications which *both* of them undergo.

A curve of the third class is, in general, of the shape represented in fig. 1; that is to say, it consists of a tricuspid surrounded by an oval. It is very easy to see that from any point within the tricuspid we can draw three tangents to it and none to the oval; from any point in the intermediate space we can draw one tangent to the tricuspid and none to the oval; and from any point outside the oval we can draw two tangents to it and one to the tricuspid. Such a curve is of the sixth order; and it is singled out among curves of the

* [This incidental remark seems noteworthy: the number in question belongs as much, and in the same way, to the tangents as to the points, that is, not peculiarly to either: but the author's point of view seems to be a different one. C.]

C. The assemblage of all the positions of the moving line.

It is very easy but at the same time very important to observe that these are three distinct things. We are accustomed to say that A is the *locus* of B (the points) and the *envelope* of C (the tangents). Every curve of course has an assemblage of points upon it, and an assemblage of lines touching it; but it is not *the same thing* as either of these, any more than the assemblage of points is the same thing as the assemblage of lines. I shall illustrate this further by taking the two simplest and most fundamental cases.

Namely, suppose first that the point, instead of moving, remains always at rest while the line turns round it. Then the figure A is merely the point itself and no longer a curve. B has entirely disappeared; there is no assemblage of positions of the moving point. For by an *assemblage* of positions [of a point] we mean at least a line; now a point is absolutely *no* line, as a line is *no* surface, and a surface *no* space. C is now the assemblage of lines through the point. So then of our three things, A and C remain, while B has entirely disappeared; but it is exceedingly obvious that a point and the assemblage of straight lines through it are two different things. Next suppose that the line remains still while the point moves along it. Then A is the straight line, which we are not accustomed to call a curve. B is the assemblage of all the points on this line. C has entirely disappeared, for there is no assemblage of lines. But a straight line is not the same thing as all the points on it, though you may think so at first. To be convinced, contemplate the other case just considered; you have just as much right to say that a point is the same thing as all the lines through it. And then read S. Thomas Aquinas on this question, if you can find the reference, which I have forgotten.

An assemblage of points is said to be of a certain *order*, when a certain number of the points can be found upon an arbitrary straight line. Thus the assemblage of points lying upon a straight line is of the *first* order, because *one* of the points can be found upon another arbitrary straight line.

For a double parabola, being a curve of the third class, may break up into a conic and a point; namely, into an ordinary parabola and a point at infinity. Among the double parabolas which touch the five lines are to be reckoned five such degenerate cases, consisting of the point at infinity on one of the lines and the conic parabola which touches the other four. The focus of this degenerate form is (as is obvious from the definition of a focus) simply the focus of the conic parabola; whence the corollary follows, and Miquel's theorem is proved.

COR. 2. If we take six lines to start with, we may in this way determine six circles, omitting the lines one by one. These six circles all meet in a point, the focus of the double parabola which touches the six lines.

The transformation (as to its tangents) of a cardioid curve into a conic and a point is illustrated by fig. 11, which represents such a curve very nearly consisting (as to its visible points) of a conic and a doubled finite portion of one of its tangents.

IV.

DEVELOPMENTS.

So far we have considered the following series of propositions:

(1). Given three lines, a circle may be drawn through their intersections. (Euc. IV. 5.)

(1'). Given four lines, the four circles so determined meet in a point. (Well known.)

(2). Given five lines, the five points so found lie on a circle. (Miquel.)

(2'). Given six lines, the six circles so determined meet in a point. (Sect. III, Cor. 2.)

I shall now shew that the series is interminable; that is, that $2n$ lines determine $2n$ circles all meeting in a point, and

parallel to the asymptotes of the conic, the point f for focus, and the line C for focal tangent.

According as the conic is ellipse or hyperbola, the double parabola will be elliptic or hyperbolic. If the conic is a parabola, it assumes an intermediate form, the semi-cubical parabola $ay^3 = x^3$. If the conic is a circle, the envelope reduces to a point, as it ought to, being a hypercycloid whose focus is at an infinite distance compared with the dimensions of the figure.

III.

THEOREM. *The locus of the foci of all the double parabolas which touch five fixed lines is a circle.*

A double parabola is uniquely determined by six tangents, for a curve of the third class is determined in general by nine tangents; and to be given that a particular line is a double tangent is equivalent to three linear tangential conditions. The double parabolas touching five fixed lines are therefore a singly infinite series like the conics inscribed in a quadrilateral, and there is only one of them that touches another arbitrary line. Let an arbitrary line be drawn through the circular point i ; there is then one double parabola of the series that touches this line. From the point j one tangent only can be drawn to the curve, for it is of the third class, and the line infinity already counts for two tangents. When therefore the tangent from i to the curve is given, the tangent from j is uniquely determined; and so likewise when the tangent from j is given, the tangent from i is uniquely determined. These two tangents are therefore corresponding rays of two homographic pencils, and the locus of their intersection (*i.e.* of the focus f) is consequently a conic section passing through the points i, j , that is to say, a circle.

COR. 1. The foci of the five parabolas which touch every four of the five lines are on this circle.

not a point of contact, there may be drawn one other tangent to the curve; the line infinity counting for n , and the class being $n+1$. Such a curve therefore has always one and only one focus. Now a curve of class $n+1$ is in general determined by $\frac{1}{2}(n+2)(n+3)-1$ or $\frac{1}{2}(n^3+5n+4)$ tangents; but an n -fold tangent of given position is equivalent to $\frac{1}{2}n(n+1)$ single tangents in its effect upon the determination of the curve. The number of tangents finite in position which determine an n -fold parabola is the difference of these numbers, $2(n+1)$. All the n -fold parabolas, therefore, which touch $2n+1$ fixed lines form a singly infinite series; and it is easy to see that the locus of their foci is a circle. For if we draw an arbitrary line through the circular point i , one curve of the series can be drawn to touch it, and this determines uniquely the tangent from the other point j . These two tangents then, as before, are corresponding rays of two homographic pencils, and their intersection must trace out a conic through the points i, j , that is to say, a circle.

Now among these n -fold parabolas are included $2n+1$ degenerate cases, each consisting of an $(n-1)$ fold parabola and a point, viz. the point at infinity on one of the lines, and the $(n-1)$ fold parabola determined by the other $2n$. The foci of these are therefore points on the circle in question, and we may enunciate the following propositions:

(n). Given $2n+1$ lines, the foci of the $2n+1$ $(n-1)$ fold parabolas each of which touches $2n$ of the lines are on the same circle.

This circle is the locus of the foci of n -fold parabolas touching the lines, and therefore

(n'). Given $2n+2$ lines, the $2n+2$ circles so determined meet in a point, the focus of the n -fold parabola touching the lines.

By successive applications of this theorem with suitable values of n , we are able to see that the series of statements at the beginning of this section may be continued indefinitely.

Let us now inquire what is the *pedal* of an n -fold parabola with regard to the focus, that is to say, what is the locus of the

that for $2n+1$ lines the $2n+1$ points so found lie on the same circle.

Connected with Prop. 1, however, there is another theorem which is susceptible of generalization. If from any point in the circumscribing circle we draw perpendiculars to the sides of a triangle, the feet of these perpendiculars are in one straight line. I call this (1p); the corresponding pendant to Miquel's theorem is,

(2p). If from any point p on the circle in Prop. (2) we draw perpendiculars on the five lines, their feet lie on a conic passing through p .

So again we have

(3). Given seven lines, the seven points obtained as in (2'), are all in one circle.

(3p). If from any point p on this circle we draw perpendiculars on the seven lines, their feet will lie on a cubic having a node at p .

And generally

(np). If from any point p on the circle determined by $2n+1$ lines we draw perpendiculars to them, the feet of the perpendiculars will lie on a curve of order n passing $n-1$ times through p .

To prove these results, it is necessary to consider a curve of class $n+1$, touching the line infinity n times. I shall call such a curve an n -fold parabola. It is in fact of order $2n$, and has n pairs of parabolic branches*. From any point at infinity

* The tangential equation to an n -fold parabola is always

$$f(a_1 a_2 \dots a_n + X, i j c_1 c_2 \dots c_{n-1}),$$

where a_1, a_2, \dots are points of contact with the line infinity, and c_1, c_2, \dots points at infinity on tangents from the focus to the curve. From this we may deduce at once a construction analogous to (3), p. 49. In the case of the triple parabola this construction takes the simplified form

$$\text{intercept on } X \text{ made by } A_1, A_2 = \sin(X, C_1) \sin(X, C_2) \sin(X, C_3),$$

where A_1, A_2 are the focal tangents, and C_1, C_2, C_3 the asymptotic directions, μ may be determined by making X coincide with A_1, A_2 successively.



Fig. 1.



Fig. 2.

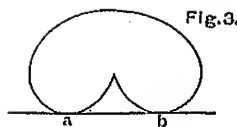


Fig. 3.

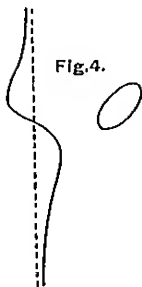


Fig. 4.

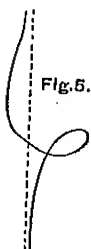


Fig. 5.

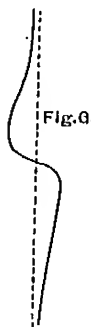


Fig. 6.

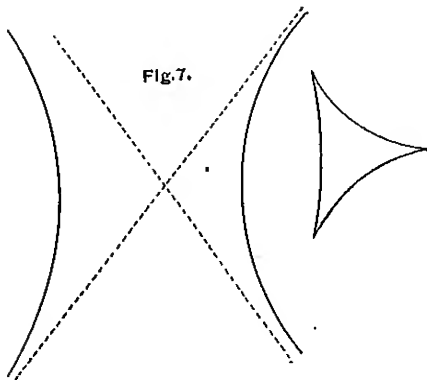


Fig. 7.

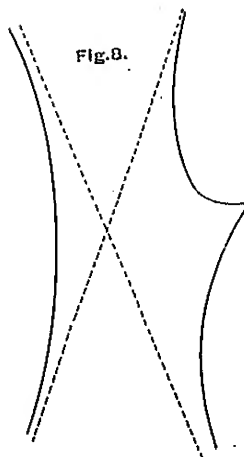


Fig. 8.

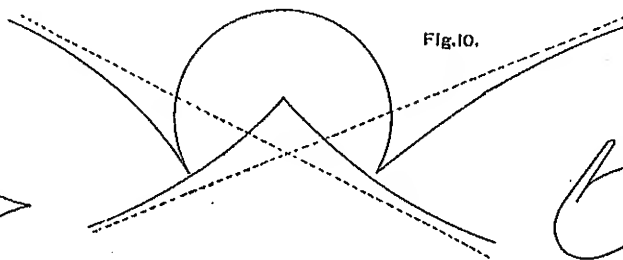


Fig. 10.

Fig. 9.



Fig. 11.



foot of the perpendicular from the focus on the tangent. It is easy to prove—as was pointed out by Dr Hirst—that this pedal is the inverse of the polar reciprocal of the curve, both taken with regard to a circle whose centre is the focus. Now the reciprocal of an n -fold parabola in regard to the focus is a curve of the $(n+1)^{\text{th}}$ order passing n times through the focus (because the original curve touches n times the line infinity) and once through each of the circular points (because the original curve touches the lines $f\bar{f}$, $f\bar{f}$). Its inverse is then a curve of order $2(n+1) - n - 2 = n$, passing $n+1 = 2 - n + 1$ times through f , and not at all through the circular points²⁹. Hence we have the proposition

(np). From any point which is the focus of an n -fold parabola touching $2n+1$ lines, if perpendiculars be drawn to the lines, their feet will lie on a curve of order n passing $n+1$ times through the point in question.

* (Circular inversion (or transformation by reciprocal radii vectors) is a particular case of triangular or quadri- inversion; the pole and the two circular points forming the triangle employed. Now in general the triangular inverse of a curve of order n , passing α , β , γ times respectively through the vertices of the triangle, is a curve of order $2n - \alpha - \beta - \gamma$ passing $n - \beta - \gamma$, $n - \gamma - \alpha$, $n - \alpha - \beta$ times respectively through the vertices of the triangle. (Dr Hirst, *Proceedings of the Royal Society*, March, 1865; and *Educational Times*, Reprint, Vol. i, p. 41; Vol. iii, p. 91.)

n ; in this case t cannot be less than $r + s - n$. First let r be even and equal to $2p$, where $4p$ is not greater than n . Then the number of types is

$$\begin{aligned} & 1 + 2 + 2 + 3 + 3 + \dots + p + p + \overline{p+1} \\ & = p + p(p+1) = p(p+2) \\ & = \frac{1}{2}r(r+1), \quad 2r \nless n. \end{aligned}$$

Next let $r = 2p + 1$, $2r \nless n$; then the

* * * * *

[Prof. Cayley remarks that Prof. Clifford considers, in this fragment, the further question, how many are the statements of each type. In the case of 4 classes, the results up to the 3-fold statement, see Table, p. 12, are as follows:—

| | | | | | | No. of Statements. | |
|-------------------|-----|-----|-----|-----|-----|--------------------|-----|
| 1-fold | ... | ... | ... | ... | ... | 16 | 16 |
| 2-fold, dist. 1 | ... | ... | ... | ... | ... | 32 | |
| " 2 | ... | ... | ... | ... | ... | 48 | |
| " 3 | ... | ... | ... | ... | ... | 32 | |
| " 4 | ... | ... | ... | ... | ... | 8 | |
| 3-fold, dist. 112 | ... | ... | ... | ... | ... | 96 | 120 |
| " 222 | ... | ... | ... | ... | ... | 64 | |
| " 123 | ... | ... | ... | ... | ... | 192 | |
| " 233 | ... | ... | ... | ... | ... | 96 | |
| " 134 | ... | ... | ... | ... | ... | 64 | |
| " 224 | ... | ... | ... | ... | ... | 48 | 560 |
| | | | | | | | |

where the totals 16, 120, 560, &c. are, of course, the numbers for the combinations of 16 things 1, 2, 3, &c. together.

Prof. Jevons in *The Principles of Science* (Third Edition, p. 149) writes: "In the first edition (vol. i. p. 163), I asserted that some years of labour would be required to ascertain even the precise number of types of law governing the combination of four classes of things. Though I still believe that some years' labour would be required to work out the types themselves, it is clearly a mistake to suppose that the *numbers* of such types cannot be calculated with a reasonable amount of labour, Professor W. K. Clifford having actually accomplished the task." It was on reading Prof. Jevons's original statements that Prof. Clifford said laughingly, he couldn't let anyone say that and not do it straight off...it was 'his luck' to do quickly difficult things which were usually got at by long processes only.

Dr Hopkinson has drawn my attention to the fact that the problems treated of by Clifford and Jevons are different in this respect, that Clifford's result includes what Jevons calls 'inconsistent statements'.]

To make this equation yield us a relation of lengths and angles serving for geometrical construction, let us take lines A, B, C at a finite distance passing through the points a, b, c respectively, and denote by the letter O the line at infinity. The points a, b, c may now be denoted by AO, BO, CO ; viz. a is the intersection of the lines A, O and so on. Let X be a variable tangent to the curve; then substituting its coordinates in the equation (1) thus modified, we have

$$XAO, XBO, Xf = X, Xf, Xf, XCO, \dots \dots \dots (2),$$

where XAO means the determinant formed with the coordinates of the lines X, A, O , which vanishes when they meet in a point; and Xf means the result of substituting the coordinates of the line X in the equation of the point f , or (which is the same thing) the result of substituting the coordinates of the point f in the equation of the line X . Now if we observe that*

$$\sin(X, A) = \frac{XAO}{\sqrt{(Xf, Xf, Af, Af)}},$$

$$\text{perpendicular from } f \text{ on } X = \frac{Xf}{\sqrt{(Xf, Xf, Of, Of)}},$$

* These formulæ are justified as follows. The coordinates of the circular points are taken to be

$$\text{of the point } i, x : y : z = 1 : -1 : 0,$$

$$\text{of the point } j, x : y : z = 1 : 1 : 0,$$

and the equation of the line O at infinity is taken to be

$$0, x + 0, y + 1 = 0.$$

Then if the equation of X be $lx + my + n = 0$, we have

$$Xf, Xf = (l + m)(l - m)z^2 + n^2,$$

and if the equation of A be $lx + my + n' = 0$, then

$$\begin{aligned} XAO &= l^2 + m^2 + n^2 - l^2 - l^2 \\ &= l^2 + m^2 + n^2 \\ &= 0, 0, 1 \end{aligned}$$

while $Of = 1$, whatever are the coordinates of f . Making these substitutions, the formulæ become

$$\sin(X, A) = \frac{l^2 + m^2 + n^2}{\sqrt{(l^2 + m^2 + n^2)(l^2 + m^2)}},$$

$$\text{perpendicular from } f \text{ on } X = \frac{l^2 + m^2 + n^2}{\sqrt{(l^2 + m^2)(l^2 + m^2)}},$$

which are the ordinary ones.

IX.

ON THE HYPOTHESES WHICH LIE AT THE BASES OF GEOMETRY*.

[Translation of a paper by Riemann, see V. *supra*.]

Plan of the Investigation.

It is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor, *a priori*, whether it is possible.

From Euclid to Legendre (to name the most famous of modern reforming geometers) this darkness was cleared up neither by mathematicians nor by such philosophers as concerned themselves with it. The reason of this is doubtless that the general notion of multiply extended magnitudes (in which space-magnitudes are included) remained entirely unworked. I have in the first place, therefore, set myself the task of constructing the notion of a multiply extended magni-

* [From *Nature*, Vol. VIII. Nos. 183, 184, pp. 14—17, 36, 37. For a Bibliography of Hyper-space and Non-Euclidean Geometry, see Articles by George Bruce Halsted in the *American Journal of Mathematics*, *Pure and Applied*, Vol. I. pp. 261—276, 381, 385; Vol. II. pp. 65—70.]

tude out of general notions of magnitude. It will follow from this that a multiply extended magnitude is capable of different measure-relations, and consequently that space is only a particular case of a triply extended magnitude. But hence flows as a necessary consequence that the propositions of geometry cannot be derived from general notions of magnitude, but that the properties which distinguish space from other conceivable triply extended magnitudes are only to be deduced from experience. Thus arises the problem, to discover the simplest matters of fact from which the measure-relations of space may be determined; a problem which from the nature of the case is not completely determinate, since there may be several systems of matters of fact which suffice to determine the measure-relations of space—the most important system for our present purpose being that which Euclid has laid down as a foundation. These matters of fact are—like all matters of fact—not necessary, but only of empirical certainty; they are hypotheses. We may therefore investigate their probability, which within the limits of observation is of course very great, and inquire about the justice of their extension beyond the limits of observation, on the side both of the infinitely great and of the infinitely small.

I. *Notion of an n-ply extended magnitude.*

In proceeding to attempt the solution of the first of these problems, the development of the notion of a multiply extended magnitude, I think I may the more claim indulgent criticism in that I am not practised in such undertakings of a philosophical nature where the difficulty lies more in the notions themselves than in the construction; and that besides some very short hints on the matter given by Privy Councillor Gauss in his second memoir on Biquadratic Residues, in the *Göttingen Gelehrte Anzeige*, and in his Jubilee-book, and some philosophical researches of Herbart, I could make use of no previous labours.

§ 1. Magnitude-notions are only possible where there is an antecedent general notion which admits of different specialisa-

tions. According as there exists among these specialisations a continuous path from one to another or not, they form a *continuous* or *discrete* manifoldness: the individual specialisations are called in the first case points, in the second case elements, of the manifoldness. Notions whose specialisations form a *discrete* manifoldness are so common that at least in the cultivated languages any things being given it is always possible to find a notion in which they are included. (Hence mathematicians might unhesitatingly found the theory of discrete magnitudes upon the postulate that certain given things are to be regarded as equivalent.) On the other hand, so few and far between are the occasions for forming notions whose specialisations make up a *continuous* manifoldness, that the only simple notions whose specialisations form a multiply extended manifoldness are the positions of perceived objects and colours. More frequent occasions for the creation and development of these notions occur first in the higher mathematic.

Definite portions of a manifoldness, distinguished by a mark or by a boundary, are called Quanta. Their comparison with regard to quantity is accomplished in the case of discrete magnitudes by counting, in the case of continuous magnitudes by measuring. Measure consists in the superposition of the magnitudes to be compared; it therefore requires a means of using one magnitude as the standard for another. In the absence of this, two magnitudes can only be compared when one is a part of the other; in which case also we can only determine the more or less and not the how much. The researches which can in this case be instituted about them form a general division of the science of magnitude in which magnitudes are regarded not as existing independently of position and not as expressible in terms of a unit, but as regions in a manifoldness. Such researches have become a necessity for many parts of mathematics, e.g., for the treatment of many-valued analytical functions; and the want of them is no doubt a chief cause why the celebrated theorem of Abel and the achievements of Lagrange Pfaff, Jacobi for the general theory of differential equations have so long remained unfruitful. Out of this general part of the science of extended magnitude in which nothing is assumed

On these suppositions, the linear element may be any homogeneous function of the first degree of the quantities dx , which is unchanged when we change the signs of all the dx , and in which the arbitrary constants are continuous functions of the quantities x . To find the simplest cases, I shall seek first an expression for manifoldnesses of $n-1$ dimensions which are everywhere equidistant from the origin of the linear element; that is, I shall seek a continuous function of position whose values distinguish them from one another. In going outwards from the origin, this must either increase in all directions or decrease in all directions; I assume that it increases in all directions, and therefore has a minimum at that point. If, then, the first and second differential coefficients of this function are finite, its first differential must vanish, and the second differential cannot become negative; I assume that it is always positive. This differential expression, then, of the second order remains constant when ds remains constant, and increases in the duplicate ratio when the dx , and therefore also ds , increase in the same ratio; it must therefore be ds^2 multiplied by a constant, and consequently ds is the square root of an always positive integral homogeneous function of the second order of the quantities dx , in which the coefficients are continuous functions of the quantities x . For Space, when the position of points is expressed by rectilinear co-ordinates, $ds = \sqrt{\sum (dx)^2}$; Space is therefore included in this simplest case. The next case in simplicity includes those manifoldnesses in which the line-element may be expressed as the fourth root of a quartic differential expression. The investigation of this more general kind would require no really different principles, but would take considerable time and throw little new light on the theory of space, especially as the results cannot be geometrically expressed; I restrict myself, therefore, to those manifoldnesses in which the line-element is expressed as the square root of a quadric differential expression. Such an expression we can transform into another similar one if we substitute for the n independent variables functions of n new independent variables. In this way, however, we cannot transform any expression into any other; since the expression contains $\frac{1}{2}n(n+1)$ coefficients which are

this by the square of the infinitesimal triangle, whose vertices are $(0, 0, 0, \dots)$, (x_1, x_2, x_3, \dots) , $(dx_1, dx_2, dx_3, \dots)$. This quantity retains the same value so long as the x and the dx are included in the same binary linear form, or so long as the two geodesics from 0 to x and from 0 to dx remain in the same surface-element; it depends therefore only on place and direction. It is obviously zero when the manifold represented is flat, *i.e.*, when the squared line-element is reducible to $\sum dx^2$, and may therefore be regarded as the measure of the deviation of the manifoldness from flatness at the given point in the given surface-direction. Multiplied by $-\frac{1}{3}$ it becomes equal to the quantity which Privy Councillor Gauss has called the total curvature of a surface. For the determination of the measure-relations of a manifoldness capable of representation in the assumed form we found that $\frac{1}{2}n(n-1)$ place-functions were necessary; if, therefore, the curvature at each point in $\frac{1}{2}n(n-1)$ surface-directions is given, the measure-relations of the continuum may be determined from them—provided there be no identical relations among these values, which in fact, to speak generally, is not the case. In this way the measure-relations of a manifoldness in which the line-element is the square root of a quadric differential may be expressed in a manner wholly independent of the choice of independent variables. A method entirely similar may for this purpose be applied also to the manifoldness in which the line-element has a less simple expression, *e.g.*, the fourth root of a quartic differential. In this case the line-element, generally speaking, is no longer reducible to the form of the square root of a sum of squares, and therefore the deviation from flatness in the squared line-element is an infinitesimal of the second order, while in those manifoldnesses it was of the fourth order. This property of the last-named continua may thus be called flatness of the smallest parts. The most important property of these continua for our present purpose, for whose sake alone they are here investigated, is that the relations of the twofold ones may be geometrically represented by surfaces, and of the morefold ones may be reduced to those of the surfaces included in them; which now requires a short further discussion.

§ 3. In the idea of surface, together with the intrinsic measure-relations in which only the length of lines on the surface is considered, there is always mixed up the position of points lying out of the surface. We may, however, abstract from external relations if we consider such deformations as leave unaltered the length of lines, i.e., if we regard the surface as bent in any way without stretching, and treat all surfaces so related to each other as equivalent. Thus, for example, any cylindrical or conical surface counts as equivalent to a plane, since it may be made out of one by means of bending, in which the intrinsic measure-relations remain, and all elements are about a plane; therefore the whole of planimetry retains their validity. On the other hand they count as essentially different from the sphere, which cannot be changed into a plane without stretching. According to our previous investigations the intrinsic measure-relations of a curved surface in which the line element may be expressed as the square root of a quadratic differential, which is the case with surfaces, are determined by the total curvature. Now this quantity, or the value of surfaces in the idea of a visible interpretation, can be also perceived as the total curvature of the surface, or multiplied by the area of a small geodesic triangle, it is equal to the spherical excess of this triangle. The first definition assumes the comparison of all these products of the two radii of curvature as determined by Gauss from integrals; the second, that in the same plane the area of a small hexagon is proportional to its spherical excess. The given are intelligible in meaning to the curvature of an *extended* manifold at a given point and in a given surface line from the fact that our result stands beside the fact that a geodesic geodesic triangle is given is entirely determined when its initial element is so given. According to this we obtain a determination of surfaces of our geometry all the geodesics proceeding from the given point and lying essentially in the given surface direction. This surface has at this given point a definite curvature, which is also the curvature of the whole continuum at the given point in the given neighbourhood.

§ 4. Before we make the applications to spheres, spheres with considerations about that manifold which is the general one necessary.

i.e., about those in which the square of the line-element is expressible as a sum of squares of complete differentials.

In a flat n -fold extent the total curvature is zero at all points in every direction; it is sufficient, however (according to the preceding investigation), for the determination of measure-relations, to know that at each point the curvature is zero in $\frac{1}{2}n(n-1)$ independent surface-directions. Manifoldnesses whose curvature is constantly zero may be treated as a special case of those whose curvature is constant. The common character of these continua whose curvature is constant may be also expressed thus, that figures may be moved in them without stretching. For clearly figures could not be arbitrarily shifted and turned round in them if the curvature at each point were not the same in all directions. On the other hand, however, the measure-relations of the manifoldness are entirely determined by the curvature; they are therefore exactly the same in all directions at one point as at another, and consequently the same constructions can be made from it: whence it follows that in aggregates with constant curvature figures may have any arbitrary position given them. The measure-relations of these manifoldnesses depend only on the value of the curvature, and in relation to the analytic expression it may be remarked that if this value is denoted by α , the expression for the line-element may be written

$$\frac{1}{1 + \frac{1}{4}\alpha\sum x^2} \sqrt{\sum dx^2}.$$

§ 5. The theory of *surfaces* of constant curvature will serve for a geometric illustration. It is easy to see that surfaces whose curvature is positive may always be rolled on a sphere whose radius is unity divided by the square root of the curvature; but to review the entire manifoldness of these surfaces, let one of them have the form of a sphere and the rest the form of surfaces of revolution touching it at the equator. The surfaces with greater curvature than this sphere will then touch the sphere internally, and take a form like the outer portion (from the axis) of the surface of a ring; they may be rolled upon zones of spheres having less radii, but will go round

more than once. The surfaces with less positive curvature are obtained from spheres of larger radii, by cutting out the bit bounded by two great half circles and bringing the section line together. The surface with curvature zero will be a cylinder standing on the equator; the surfaces with negative curvature will touch the cylinder externally and be formed like the inner portion (towards the axis) of the surfaces of a ring. If we regard these surfaces as *locus in quo* for surface regions moving, then, as Space is *locus in quo* for bodies, the surface regions can be moved in all these surfaces without stretching. The surfaces with positive curvature can always also be formed if surface regions may also be moved arbitrarily about upon the without *bending*, namely they may be formed into sphere surfaces; but not those with negative curvature. Besides the independence of surface regions from position there is, in all cases of zero curvature also an independence of direction in position, which in the former surfaces does not exist.

III. Application to Space.

§ 1. By means of these inquiries into the determinations of the measure-relations of an n -fold extent the conditions may be declared which are necessary and sufficient for determination of metric properties of space, if we assume the independence of line-length from position and expressibility of this line-length as the square root of a quadratic differential, that is to say, if none in the smallest parts.

First, they may be expected that the curvature at each point is zero in three surface directions; and therefore metric properties of space are determined if the sum of angles of a triangle is always equal to two right angles.

Secondly, if we assume with Euclid not merely independence of lines independent of position, but of Isotropy also, that the curvature is everywhere constant; and therefore the sum of the angles is determined in all triangles whatever it is, namely one.

Thirdly, one might, instead of taking the length of lines to be independent of position and direction, assume also an independence of their length and direction from position. According to this conception changes or differences of position are complex magnitudes expressible in three independent units.

§ 2. In the course of our previous inquiries, we first distinguished between the relations of extension or partition and the relations of measure, and found that with the same extensive properties, different measure-relations were conceivable; we then investigated the system of simple size-fixings by which the measure-relations of space are completely determined, and of which all propositions about them are a necessary consequence; it remains to discuss the question how, in what degree, and to what extent these assumptions are borne out by experience. In this respect there is a real distinction between mere extensive relations, and measure-relations; in so far as in the former, where the possible cases form a discrete manifoldness, the declarations of experience are indeed not quite certain, but still not inaccurate; while in the latter, where the possible cases form a continuous manifoldness, every determination from experience remains always inaccurate; be the probability ever so great that it is nearly exact. This consideration becomes important in the extension of these empirical determinations beyond the limits of observation to the infinitely great and infinitely small; since the latter may clearly become more inaccurate beyond the limits of observation, but not the former.

In the extension of space-construction to the infinitely great, we must distinguish between *unboundedness* and *infinite extent*, the former belongs to the extent relations, the latter to the measure-relations. That space is an unbounded three-fold manifoldness, is an assumption which is developed by every conception of the outer world; according to which every instant the region of real perception is completed and the possible positions of a sought object are constructed, and which by these applications is for ever confirming itself. The unboundedness of space possesses in this way a greater empirical certainty than any external experience. But its infinite extent by no means

follows from this; on the other hand if we assume independence of bodies from position, and therefore ascribe to space constant curvature, it must necessarily be finite provided this curvature has ever so small a positive value. If we prolong all the geodesics starting in a given surface-element, we should obtain an unbounded surface of constant curvature, i.e., a surface which in a *flat* manifoldness of three dimensions would take the form of a sphere, and consequently be finite.

§ 3. The questions about the infinitely great are for the interpretation of nature useless questions. But this is not the case with the questions about the infinitely small. It is upon the exactness with which we follow phenomena into the infinitely small that our knowledge of their causal relations essentially depends. The progress of recent centuries in the knowledge of mechanics depends almost entirely on the exactness of the construction which has become possible through the invention of the infinitesimal calculus, and through the simple principles discovered by Archimedes, Galileo, and Newton, and used by modern physics. But in the natural sciences which are still in want of simple principles for such constructions, we seek to discover the causal relations by following the phenomena into great minuteness, so far as the means of observation permit. Questions about the causal relations of space as the infinitely small are not therefore superfluous questions.

If we suppose that bodies exist independently of position, the curvature in everywhere constant, and at the same time from astronomical measurements that it cannot be different from zero; or at any rate its reciprocal must be only very very near to zero with which the range of our telescopes may be neglected. But if this independence of bodies from position does not exist, we cannot draw conclusions from nature relations of the great, to those of the infinitely small; in that case the curvature at each point may have an arbitrary value in three dimensions, provided that the total curvature of every symmetrical part of space does not differ sensibly from zero. Still more complicated relations may exist if we no longer suppose that bodies are element expressible as the square root of a quadratic differential

Now it seems that the empirical notions on which the metrical determinations of space are founded, the notion of a solid body and of a ray of light, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena.

The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space. In this last question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in a discrete manifoldness, the ground of its metric relations is given in the notion of it, while in a continuous manifoldness, this ground must come from outside. Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek the ground of its metric relations outside it, in binding forces which act upon it.

The answer to these questions can only be got by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by tracking in this conception the successive changes required by facts which it cannot explain. Researches starting from general notions, like the investigation we have just made, can only be useful in preventing this work from being hampered by too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices.

This leads us into the domain of another science, of physics, into which the object of this work does not allow us to go today.

Synopsis.

PLAN of the Inquiry:

I. Notion of an *n*-ply extended magnitude.

§ 1. Continuous and discrete manifolds. — Defined parts of a manifoldness are called *Quantia*. — Division of the theory of continuous magnitude into the theories,

- (1) Of mere region-relations, in which an independence of magnitudes from position is not assumed;
- (2) Of size-relations, in which such an independence must be assumed.

§ 2. Construction of the notion of a one-fold, two-fold, *n*-fold extended magnitude.

§ 3. Reduction of place being in a given measured line to quantity-fixings. — Time character of an *n*-fold extended magnitude.

II. Measure-relations of which a manifoldness of *n* dimensions is capable on the assumption that time has a length independent of position, and consequently that every time may be measured by every other.

§ 1. Expression for the line-element. — Manifolds in which the line-element is expressible as the square root of a sum of squares of *n* single differentials.

§ 2. Investigation of the manifolds of *n* dimensions in which the line-element may be represented as the square root of a quadratic differential. — Measure of the deviation from flatness (curvature) at a given point and in given directions. — For the determination of the measure-relations it is allowable and sufficient that the curvatures be arbitrarily given at every point in $\frac{1}{n!}$ of the $n!$ possible directions.

§ 3. Geometric illustration.

§ 4. Flat manifoldnesses (in which the curvature is everywhere 0) may be treated as a special case of manifoldnesses with constant curvature. These can also be defined as admitting an independence of n -fold extents in them from position (possibility of motion without stretching).

§ 5. Surfaces with constant curvature.

III. Application to Space.

§ 1. System of facts which suffice to determine the measurement-relations of space assumed in geometry.

§ 2. How far is the validity of these empirical determinations probable beyond the limits of observation towards the infinitely great?

§ 3. How far towards the infinitely small? Connection of this question with the interpretation of nature.

strate one or two other properties of the figure. The present communication is confined for the most part to the case in which $n=4$; in a second we hope to notice some peculiarities of the higher plane cases, and to state the *true* analogues of Pascal's Theorem in Geometry of Higher Dimensions

2. *Of the mn -intersections of two curves of the m th and n th degrees, if pn lie on a curve of the p th degree, the remaining $(m-p)n$ will lie on a curve of the $(m-p)$ th degree (p being less than m , and n not greater than m).*

Let U_m, U_n, U_p be three curves of degrees m, n, p , respectively. Since U_m passes through all the intersections of U_n and U_p , U_m must be of the form

$$\phi U_n + \psi U_p \dots \dots \dots (2);$$

but U_m is of the m th degree and U_p of the p th, therefore ψ must be of the degree $m-p$; but $\psi=0$ passes through all the remaining intersections of U_m and U_n . Therefore, &c. Q.E.D.

This includes (when $m=n$) the well known theorem of Salmon's *Higher Plane Curves*, Chap. II. Sect. 1, Art. 24. And the latter at once gives the property that the n exeunts of (1) lie on a right line. For we have two systems of n right lines, and a curve of the $(n-1)$ th degree passing through $n(n-1)$ of their intersections; therefore the other n intersections lie on a right line. In its general form the theorem shows that we may erase any of the rows of the symbol (1) (which we call the "*index*" of the figure) without destroying the property of the exeunts.

3. We shall have occasion to use Prof. Cayley's theorem, that "*every curve of the m th order, (m not being less than n or p , nor greater than $n+p-3$), which passes through all but $\frac{1}{2}(n+p-m-1)(n+p-m-2)$ of the intersections of two curves of the n th and p th orders, passes also through the remaining intersections.*" The form (2) contains $\frac{1}{2}(m-n)(m-n+3)$ constants in ϕ , $\frac{1}{2}(m-p)(m-p+3)$ in ψ , and one constant of mul-

we form the equation $A + B + C = 0$ to express that the points A, B, C lie on a right line, and so for the rest. Now let us take a as an abbreviation for the quantity $A + B + C$, and so on. Thus $a + b + c + d$ includes all the twelve points, and so does $\alpha + \beta + \gamma + \delta$. Hence

$$a + b + c + d \equiv \alpha + \beta + \gamma + \delta,$$

and therefore if seven of these quantities vanish, the eighth must vanish also. But we know that $a, b, c, d, \beta, \gamma, \delta$ all $= 0$, therefore $\alpha = 0$ also, or FHL is a straight line.

6. *An octolateral has six diagonals which intersect by pairs in three points of the curve lying on a right line.*

With the notation of the last article, each of the quantities $a, b, c, d; \alpha, \beta, \gamma, \delta$ is identically zero. So therefore is any quantity formed from them by addition or subtraction. Now consider the identities

$$\left. \begin{aligned} (a + b + \alpha + \beta) - (c + d + \gamma + \delta) &\equiv (A + F) - (N + G) \\ (a + c + \alpha + \gamma) - (b + d + \beta + \delta) &\equiv (B + H) - (E + M) \\ (a + d + \alpha + \delta) - (b + c + \beta + \gamma) &\equiv (C + L) - (D + K) \end{aligned} \right\} \dots\dots (4).$$

If AF meet the curve in P , so that $A + F + P = 0$, we must have also $N + G + P = 0$, since $A + F = N + G$ identically. That is, AF and NG meet the curve in the same point. Similarly for BH and EM , CL and DK . Call these points P, Q, R . Then, if we form the six equations like $A + F + P = 0$, and add them all together, we have

$$2(P + Q + R) + A + B + C + D + \dots = 0;$$

therefore $P + Q + R = 0$, or PQR is a straight line.

7. By aid of this property we may consider the octolateral from an entirely different point of view.

The cubic passes through all the intersections of AF , γ, δ with NG , a, b . Therefore its equation may be written

$$AF. \gamma \delta + NG. ab = 0 \dots\dots\dots (5),$$

where, for shortness, AF represents the perpendicular from a current point on AF , so that $AF=0$ is the equation to AF . A constant multiplier is of course supposed.

Now if we consider this equation generally, it contains *thirteen* independent constants, viz. two for each of the six lines, and one constant of multiplication. But the general equation of the third degree contains *nine* constants; we may therefore, as the cubic is given, assume *four* points in (5) and the rest will be determined. Assume then P , the intersection of AI' and NG , and the points A, N . These determine the straight lines PAI' , PNG , and we have still one point at our disposal. Choose any point B for the intersection of a and γ ; join AB , which is a , and then C , the third point where it cuts the curve, will be a point on δ . To find γ, δ we have a choice of two constructions; we may join either BN, CG , or BG, CN ; we adopt the former. If BN meet the curve in D , and CG in E , then the form of equation (5) shews that FDE is a straight line, namely, b . We have constructed then the following portion of the index (3),

$$\left\{ \begin{array}{cccc} *, & A, & B, & C \\ F, & *, & D, & E \\ & & *, & G \\ & & N, & * \end{array} \right\} \dots\dots\dots (6).$$

The point P is of course not represented.

Now choose another point M , and use it as B was used before to obtain three more points H, K, L corresponding to ECD . We have then a second equation to the cubic

$$AF.cd + NG.a\beta = 0 \dots\dots\dots (7),$$

and we may complete the symbol (6) in the form

$$\left\{ \begin{array}{cccc} 1, & A, & B, & C \\ F, & 2, & D, & E \\ H, & K, & 3, & G \\ L, & M, & N, & 4 \end{array} \right\},$$

the exponents 1 2 3 4 being now determined.

axes of these three meet in a point. For we write down the equations

$$AP, \gamma\delta + NG, ab = 0 \dots\dots\dots AX,$$

$$AP, cd + NG, \alpha\beta = 0 \dots\dots\dots AY,$$

$$AP, xy + NG, zm = 0 \dots\dots\dots AZ,$$

$$\frac{NG}{AP} = \frac{xy}{zm} = \frac{cd}{\alpha\beta} = \frac{\gamma\delta}{ab}.$$

The last statement expressing the property in question.

10. The proposition proved in (6) may be extended to *six diagonals which include all the inpoints meet the cubic in six new points which lie on a conic*. This is proved precisely as in (6), the addition of six equations giving the result

$$P + Q + R + S + T + U = 0,$$

which is the condition. The number of these conics we have not had the courage to count. There are nine quadrilaterals of inpoints which yield three pairs of diagonals each, and by combinations of these we obtain 96 conics, including the straight line PQR , Art. 6. But there are a great many other quadrilaterals which cannot be so divided into quadrilaterals. The diagonals may be classified; there are 6 of the class $af\delta, b\gamma$, 12 of the class $af\delta, c\delta$, and 24 of the class $af\delta, b\gamma$. The property of the six points of intersection of a cubic and a conic is this; *if they be divided into three pairs, the lines joining these will meet the cubic in three points on a right line*. For since

$$P + Q + R + S + T + U = 0,$$

if

$$P + Q + X = 0,$$

$$R + S + Y = 0,$$

$$T + U + Z = 0,$$

we must have $X + Y + Z = 0$. To every such conic corresponds *at least* 16 of these lines; for if we take any pair PQ , the remaining four may be divided into pairs in three ways, $RS, TT', UU', SU, RU, ST'$; and so for each of the pairs PR, PT', PT, PT' . Suppose now that P, Q are the points where the diagonals

RI , KI meet the same property, and let RI , KI meet at P_1 , Q_1 , and RI , LI at P_2 , Q_2 . Then $K \cup L = P_1 \cup P_2$ is a line, while the sum of the two relations R , L is

$$P = Q = P_1 \cup Q_1 = P_2 \cup Q_2$$

on the lines PQ , P_1Q_1 , P_2Q_2 , and on the plane. If there are more than one line, we count card^2 as the number of lines and multiply it by 15.

(1) If at any point A we have a line KL , then $2A \neq K = L$. First, clearly, with the point A is at least one L . It is easily seen that the relation of second card^2 is an another relational, and that any points x and y with the second card^2 relation of points satisfy $x \leq y$ and $y \leq x$. Many other properties may easily be proved.

Yon, 20, 1963

¹ This is not meant as the definition of a line, but as a way to show that the definition is not too restrictive.

ANALYTICAL METRICS*.

I. Introduction.

1. ANY one must have observed that there are two kinds of theorems in Geometry; one kind having reference to *position* only, the other kind having reference to *magnitude*. The first theorem is an example of the first, or *graphic* geometry; Euclid I. 47 is an example of the second, or *metric* geometry. It may be possible to state the same theorem in two ways, so as to make it either metric or graphic. In such a case the graphic statement may be distinguished by the fact that it is *unaltered by projection*. In fact, the word *graphic* is co-extensive with *projective*. And so, bearing in mind the properties of projection, we may define Metric Geometry as that science which has to do with the magnitudes of angles, distances, areas, and volumes. It seems, at first sight, as if the method of coordinates, especially in its more complete homogeneous form, were almost wholly applicable to Graphic Geometry, and altogether unfit for the study of Metrics. And this idea is strengthened by the fact that nearly the whole of Graphic Geometry is due to the method of coordinates, while the science of Metrics has hitherto been fitted by it very little indeed. But there are two considerations which go against this view. The first is derived from Poncelet's discovery of the true nature of metrics. Poncelet showed that all circles pass through the same two points at infinity, and that

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all angles, lengths, &c. may be expressed as *graphic* functions of these two points. For example, the angle between two straight lines is a certain function of the anharmonic ratio in which they cut the line joining the circular points. The general principle may be thus stated:—Whenever we speak of the metric properties of loci, we consider the loci, not by themselves, but in connection with another *fixed* locus, called the Absolute. Thus, in the case of Plane Geometry, the Absolute is the two circular points at infinity.

The second consideration is derived from the Higher Algebra. M. Magnus has proved that linear transformation is virtually equivalent to projection, and that graphic properties are in fact those which are unaltered by linear transformation. Now it is one of the propositions of the Higher Algebra, that when we consider any set of loci, the invariants, or functions of the coefficients, and covariants, or connected loci, which are unaltered by linear transformation, are *limited in number*. All graphic properties, therefore, may be stated in terms of a *finite number* of expressions. Now combine these considerations; first, that Metric Geometry may be analytically reduced to Graphic; and, secondly, that Graphic Geometry is necessarily of finite extent and exhaustible; and it will, I think, be abundantly evident that Analytical Metrics ought to be studied.

As to the mode of proof, I have freely made use of known results both in Pure Geometry and in Cartesian Coordinates, though the formulæ are to be regarded as referring primarily to systems of coordinates in which the equations are homogeneous. It is, of course, possible to *start* Coordinate Geometry from definitions or axioms, without referring at all to any other science, such as Pure Geometry; and in this way one may arrive at the results even of Metrics. But the course here adopted seemed to be, in existing circumstances, shorter. Those who wish to see the subject handled scientifically, must refer to Prof. Cayley's Sixth Memoir upon Quantics.

2. I give here two sample modes of proof, which together will suffice to prove most of the propositions which follow. Proofs which are similar to these will hereafter be omitted.

(A) To find the area of the triangle Δ at which the three points whose equations are given in the form

$$lx + my + nz = 0,$$

(the area of the fundamental triangle being 1)

The area vanishes only when the points are on a straight line, and becomes infinite only when one of the points is at infinity. Now the condition that they shall be in a straight line is

$$D = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

and the product of the conditions that they shall be at infinity is

$$P = (l_1 + m_1 + n_1)(l_2 + m_2 + n_2)(l_3 + m_3 + n_3)$$

Now the expression for the area must be of the third degree in the coefficients of either point, and when the coordinates of the area are given, the locus of the vertex is a straight line. Hence

required expression is then found by $\frac{D}{P}$, which is a constant.

By reference to the fundamental triangle we get $\frac{1}{P}$ as the area, therefore the area is $\frac{D}{P}$.

(B) To find the cosine of the angle between two straight lines $A = 0$, $B = 0$,

Let $\phi(A)$ and $\psi(B)$ be the equations of the tangents to the circles of the circular points; we know from Chapter IV that these are of the second order in the coefficients of A and B . From the nature of the second order $\phi(\lambda A + \mu B)$ must be of the form

$$\lambda^2\phi_1 A + 2\lambda\mu\phi_2 AB + \mu^2\phi_3 B \quad (1)$$

Now when we put for A and B their coordinates respectively, the coefficients of powers of λ , μ in (1) must be equal to zero. From the evanescence of (1) gives the two values of λ/μ , $\lambda/\mu = \frac{\phi_2 \pm \sqrt{\phi_2^2 - 2\phi_1\phi_3}}{\phi_1}$. $\lambda A + \mu B = 0$ represents a line the angle between whose sides is θ , and these lines are the same in both cases. If (x, y, z) are the coordinates, the expression

$$\frac{\phi_2 \pm \sqrt{\phi_2^2 - 2\phi_1\phi_3}}{\phi_1} \frac{\psi_2 \pm \sqrt{\psi_2^2 - 2\psi_1\psi_3}}{\psi_1}$$

is a product of the cosine of

the angle between A and B ; this is therefore its value in all other systems of coordinates.

The function ϕ is called the Absolute, and the determination of its form is an important part of our subject.

NOTE. (α) The function used in proof (A), and called J , will occur very often in the sequel. Suppose we have a number of linear equations equal to the number of variables in each, and that we form the determinant which is the result of eliminating the variables from these equations; then this determinant is called their Jacobian, and will be denoted by $J(LMN\dots)$, where $L=0$, $M=0$, &c., ... are the linear equations.

(β) The abbreviation $\infty=0$ will be used to represent the equation to the line, &c. at infinity; so that $J(LM\infty)$ means the result of eliminating the variables between $L=0$, $M=0$, and the equation to the line at infinity.

(γ) In the demonstration (B) it is important to remember, that when we put for A and B their Cartesian equivalents, these expressions are transformed by the *same* substitution; so that if A is changed to A' , and B to B' , $LA+mB$ will be changed to $LA'+mB'$. The proof depends on this. It would not be sufficient to know that $A=0$, in a homogeneous system, and $A'=0$, in Cartesians, represent the same straight line, because then a constant factor might be introduced, of which we should know nothing.

(δ) The method of comparison of dimensions, made use of almost exclusively in the following sections, requires a little explanation. In the first place, all the functions employed are integral functions; so that it is quite legitimate to say "Because A always vanishes when B does, therefore A contains B as a factor." Secondly, every expression must contain some factor depending on the coefficients of the "Absolute," but these factors will be systematically disregarded, as they are only numerical, and can easily be determined by reference to any simple particular case. These things being so, the method of comparison of dimensions amounts simply to this; when we have proved the existence of factors enough to account for all the dimensions

of an expression, we may be sure that it has sense. And, an analytical equivalence being thus proved, it only to divide both sides by such quantities as will make it sensible geometrically.

II. Geometry of One Dimension. *Geophysics*

3. Geometry of one dimension consists of points located on a straight line, which are dots marked by their distances from two fixed points $ax + by = 0$, given in the form $lx + my = 0$. The equation for the point at infinity of the line is clearly $ax + by = 0$; and as the coordinates of the point $lx + my$ may be at infinity is $lx + my = 0$. The coordinates of points $A = l_1x + m_1y$ and $B = l_2x + m_2y$ may be expressed by

$$J(A, B) = \frac{l_1^2 - l_2^2}{m_1^2 - m_2^2}.$$

So the condition that A may be at infinity may be $J(A, \infty) = 0$.

The distance between A and B is $\frac{J(A, B)}{J(A, \infty) + J(B, \infty)}$.

Geophysics

4. Let A, B, C, D be four points on a straight line, appears, by Art. 3, that

$$\begin{aligned} J(R, C) &= J(A, B) \cdot J(A, D), \\ J(C, D) &= J(A, B) \cdot J(A, C). \end{aligned}$$

Now it is known that in linear geometry, $J(A, B)$ is only affected with a constant factor, and this factor disappears in the expression $\frac{J(A, B)}{J(A, C)}$. This expression, therefore, is unaltered by linear transformations. It is called an harmonic ratio of the points A, B, C, D .

Now this function belongs to the Mathematics of Geometry, being expressed by means of lengths, as $\frac{AB \cdot CD}{AC \cdot BD}$. But as it belongs to Graphic Geometry, to which it is equivalent, $\frac{J(A, B)}{J(A, C)}$

or linear transformation. On these accounts I propose to call it a *Graphometric* function. The reason why a new name is needed, is that there are other Graphometric functions, which are naturally presented by analytical metrics. For instance, it will be proved that if $ABCDEF$ are six points in a plane, the ratio $\frac{(ABC)(DEF)}{(ABD)(CEF)}$, between the products of the triangles they contain, is graphometric, or independent of projection though expressed in terms of areas. And it is natural to suppose that these and similar functions will be important in Solid Geometry, because the analogous function, namely, anharmonic ratio, is so important in Plane Geometry. For this reason I shall, in what follows, pay particular attention to Graphometrics.

III. *Points, Lines, and Circles.*

5. In respect of one line $L=0$, the expression considered is $\phi(L)$, whose evanescence is the condition that the line shall pass through one of the circular points. If the line be, in Cartesians, $lx + my + n = 0$, then

$$\phi L \equiv l^2 + m^2,$$

that is, it is of the second order in the coefficients of L , and its discriminant vanishes. We have, also,

$$\phi(\lambda L + \mu M) \equiv \lambda^2(l_1^2 + m_1^2) + 2\lambda\mu(l_1l_2 + m_1m_2) + \mu^2(l_2^2 + m_2^2);$$

$$\text{or} \quad \lambda^2 \phi L + 2\lambda\mu\psi(L, M) + \mu^2 \phi M \dots\dots\dots (3),$$

so that in any system of coordinates the function $\psi(LM)$, thus formed, is the condition that L and M may be at right angles. In the same way, or from the nature of the case, it is seen that the condition that (3) may be a perfect square is the square of the condition that L and M may be parallel, or may meet on the line at infinity; that is

$$\phi L \cdot \phi M - (\psi LM)^2 \equiv J(LM \infty)^2 \dots\dots\dots (4)*.$$

In respect of three lines the function $J(LMN)$ will be considered.

* The equation to the line at infinity will be found from ϕ .

The principal function of a point in $L, A, \text{ or } N$, the condition that it may be at infinity. Given two points $(L_1, m_1, n_1), (L_2, m_2, n_2)$, the line joining them is

$$\left| \begin{array}{cc} m_1 & n_1 \\ m_2 & n_2 \end{array} \right| x + \left| \begin{array}{cc} n_1 & l_1 \\ n_2 & l_2 \end{array} \right| y + \left| \begin{array}{cc} l_1 & m_1 \\ l_2 & m_2 \end{array} \right| z = 0,$$

from this it will be easy to understand $\phi(LH)$, the condition that LH may pass through a circular point, and $\phi(LH, CH)$, the condition that LH and CH may be at right angles.

Angles

6. The cosine of the angle between L and M has been proved to be $\frac{\phi(LM)}{\sqrt{\phi(LL)\phi(MM)}}$; by (4) the sine is $\frac{J(LM)}{\sqrt{\phi(LL)\phi(MM)}}$.

If then we consider four lines, L, M, N, R , we have

$$\begin{aligned} \sin(LM), \sin(NR) &= J(LM) \times J(NR) \\ \sin(LN), \sin(RM) &= J(LN) \times J(RM) \end{aligned}$$

From the identity*

$$J(LM) \times J(NR) + J(LN) \times J(RM) + J(LR) \times J(MN) = 0,$$

we may therefore deduce

$$\begin{aligned} \sin(LM), \sin(NR) + \sin(LN), \sin(RM) \\ + \sin(LR), \sin(MN) = 0. \end{aligned}$$

If we make N perpendicular to R , this becomes the formula

$$\sin(LM) + \sin(LR) \times \sin(MN) = 0.$$

Next, consider the determinant

$$\begin{vmatrix} \phi(LN) & \phi(LM) \\ \phi(MN) & \phi(MM) \end{vmatrix}$$

This vanishes when L is parallel to M or when the two lines become identical, and also when N is parallel to R , or when the

* A method of proving these theorems with the greatest ease is that they may also be treated in the same way as the next.

columns become identical. Hence, by a comparison of dimensions, we can see that

$$\begin{vmatrix} \psi LN, & \psi LR \\ \psi MN, & \psi MR \end{vmatrix} = J(LM\infty), J(NR\infty),$$

or, which is the same thing,

$$\begin{aligned} \cos(LN) \cdot \cos(MR) - \cos(LR) \cdot \cos(MN) \\ = \sin(LM) \cdot \sin(NR) \dots\dots\dots(6). \end{aligned}$$

If we make M parallel to R , this becomes the formula

$$\cos(A \perp B) = \cos A \cos B \mp \sin A \sin B.$$

Since $\psi(LM)$ is of the *first order* in the coefficients of each of the lines, we must have

$$\psi(D, lA + mB + nC) = l\psi AD + m\psi BD + n\psi CD,$$

so that, if l, m, n are regarded as variables, the tangential equation to the point at infinity in a direction perpendicular to D is

$$l \cdot \psi AD + m \cdot \psi BD + n \cdot \psi CD = 0.$$

We have similar equations for points so connected with B and C . But these three points, being all at infinity, are in the same straight line; we must therefore have identically

$$\begin{vmatrix} \psi AD, & \psi BD, & \psi CD \\ \psi AE, & \psi BE, & \psi CE \\ \psi AF, & \psi BF, & \psi CF \end{vmatrix} = 0;$$

that is to say

$$\begin{vmatrix} \cos(AD), & \cos(BD), & \cos(CD) \\ \cos(AE), & \cos(BE), & \cos(CE) \\ \cos(AF), & \cos(BF), & \cos(CF) \end{vmatrix} = 0 \dots\dots\dots(7),$$

where A, B, C, D, E, F are any six lines. By making A, B, C coincide with D, E, F respectively, we obtain the relation which exists among the cosines of the angles of a triangle; but the signs are undetermined.

The theorem (7) may also be proved indirectly as follows. Form the reciprocal determinant, expressing the minors by the help of (6); it will be found to vanish identically.

Distances.

7. The distance between two points A and B is $\sqrt{\phi(A, B)}$.

The length of the perpendicular from a point A on a line L is $\frac{AL}{AN}$, where AL denotes the result of substituting the coefficients of A for the variables in L , or conversely.

So the length of a line N , cut off by two lines L and M is $\frac{\sqrt{\phi(N, L)N}}{\sqrt{\phi(LN\infty) \cdot \phi(MN\infty)}}$.

And the length of the perpendicular from A on AB is

$$\frac{J(A, AB)}{C \cdot \sqrt{\phi(A, B)}}.$$

The point where the line AB cuts any line U is of course represented by $\frac{A}{UA} = \frac{B}{UB}$. So that if L and M are homogeneous, the evanescence of the determinant

$$\begin{vmatrix} L, A, AB \\ M, A, AB \end{vmatrix}$$

is the condition that L, M, AB may meet in a point, in fact, by comparing dimensions, we may see that it is either $J(L, M, AB)$ or $J(LM, A, AB)$. The interpretation of this result will be given afterwards. Now let U, U' be two different points, the point U to the circular points at infinity. Then the product of the

$$\begin{vmatrix} U, A, UB \\ U', A, UB \end{vmatrix} = \begin{vmatrix} U, A, UB \\ U', A, UB \end{vmatrix}$$

must vanish when $\phi(UB) = 0$, and also when $\phi(UA) = 0$. In fact, by comparison of dimensions, it is seen that $\phi(UB) = 0$, $\phi(UA) = 0$, $\phi(UA) = 0$, $\phi(UB) = 0$, and

$$U, A, UB + U', A, UB = 2\sqrt{\phi(UA) \cdot \phi(UB)}.$$

* Suppose A and B to be (x_1, y_1, z_1) and (x_2, y_2, z_2) , and suppose U and U' to be (x_3, y_3, z_3) and (x_4, y_4, z_4) . Then $2\sqrt{\phi(U, UB) \cdot \phi(U', UB)} = 2\sqrt{\phi(U, UB) \cdot \phi(U', UB)} = 2\sqrt{\phi(U, UB) \cdot \phi(U', UB)}$.

thus we have

$$O\infty^2 \cdot \phi AB \equiv A\infty^2 \cdot \phi BO + 2A\infty \cdot B\infty \cdot \psi(AC, BO) \\ + B\infty^2 \cdot \phi AC \dots (8),$$

or $AB^2 = BO^2 + OA^2 + 2BO \cdot OA \cos(BO, OA).$

By means of the identity (8) I propose to find the value of the determinant

$$\begin{vmatrix} \phi AE, \phi BE, \phi CE, \phi DE, E\infty^2 \\ \phi AF, \phi BF, \phi CF, \phi DF, F\infty^2 \\ \phi AG, \phi BG, \phi CG, \phi DG, G\infty^2 \\ \phi AH, \phi BH, \phi CH, \phi DH, H\infty^2 \\ A\infty^2, B\infty^2, C\infty^2, D\infty^2, 0 \end{vmatrix}.$$

Take any point O , and substitute for the several constituents by the formula (8). Then by subtracting the last row and column, with proper multipliers, from each of the others, we may reduce the determinant to the form

$$\begin{vmatrix} \psi(OA, OE), \psi(OB, OE), \psi(OC, OE), \psi(OD, OE), E\infty \\ \psi(OA, OF), \psi(OB, OF), \psi(OC, OF), \psi(OD, OF), F\infty \\ \psi(OA, OG), \psi(OB, OG), \psi(OC, OG), \psi(OD, OG), G\infty \\ \psi(OA, OH), \psi(OB, OH), \psi(OC, OH), \psi(OD, OH), H\infty \\ A\infty, B\infty, C\infty, D\infty, 0 \end{vmatrix}$$

multiplied by $\frac{A\infty \cdot B\infty \cdot C\infty \cdot D\infty \cdot E\infty \cdot G\infty \cdot H\infty}{O\infty^4}$. But by

(7) every term of this vanishes identically. Hence we obtain the relation which connects the distances of four points, 1234, from four other points, 5678 :

$$\begin{vmatrix} 15^2, 25^2, 35^2, 45^2, 1 \\ 16^2, 26^2, 36^2, 46^2, 1 \\ 17^2, 27^2, 37^2, 47^2, 1 \\ 18^2, 28^2, 38^2, 48^2, 1 \\ 1, 1, 1, 1, 0 \end{vmatrix} \equiv 0.$$

When 1234 are identical with 5678, this gives the relation between the distances of any four points in a plane.

Triangles.

8. The area of the triangle formed by three points L, M, N is the same as that formed by the points ML, NL, LM ; that is

$$\frac{J(MN, NL, LM)}{MN \phi, NL \phi, LM \phi} = \frac{J(LM, MN, NL)}{J(MN, NL, LM)} = J(MN, NL, LM).$$

This I call $\frac{I}{P}$. I write also Π for $\frac{I}{P}$ if M, N, L are the vertices. In this notation it is easy to see that

$$a = \frac{I}{P} \phi(L, J(MN, NL, LM)), \text{ etc.} \quad \text{where } \frac{I}{P} = \frac{I}{J(M, N, L)}.$$

$$\sin A \sin B \sin C = \frac{I}{\Pi} = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

and so on.

In the case of six points, 123456, it is clear that if A is the area of the triangle 123,

$$\frac{A(123)A(456)}{A(123)A(456)} = \frac{J(123)J(456)}{J(123)J(456)} = \frac{I}{P} = \frac{I}{J(123, 456)}.$$

and therefore that this function is proportional to $\frac{I}{P}$. But the proposition is not true in respect of $\frac{I}{P}$ and the general metric function of three lines, which function is not proportional multiplied by some power of Π , since $\frac{I}{P}$ is not a metric function. Now Π is of two dimensions because of the fact that $\frac{I}{P}$ of one; the function must therefore be $\frac{I}{P} \Pi^2$. We express this in terms of the sides and angles, we have that $\frac{I}{P} \Pi^2 = \frac{I}{P} \sin A \sin B \sin C$

$$\frac{(\text{area})^2}{abc} = \frac{1}{4} \sin A \sin B \sin C \frac{a^2 b^2 c^2}{\sin A \sin B \sin C} = \frac{a^2 b^2 c^2}{4 \sin A \sin B \sin C}.$$

$$\frac{a^2}{2 \sin A} \sqrt{(\Sigma(N - \sin A))^2 - \Sigma \sin A \sin B \sin C} = \frac{a^2 b^2 c^2}{4 \sin A \sin B \sin C}.$$

if

$$2 \Sigma \sin A + \sin A + \sin B + \sin C = 4 \sin A \sin B \sin C.$$

This function I call for convenience the *Projector* of the triangle, and denote by $P(ABC)$. We have, of course,

$$\frac{P(ABC) \cdot P(DEF)}{P(ABD) \cdot P(CEF)} = \frac{J(ABC) \cdot J(DEF)}{J(ABD) \cdot J(CEF)},$$

and this ratio is therefore unaltered by projection.

By means of the transformation (8), and its reciprocal

$$\begin{aligned} \phi O \cdot J(AB\infty)^2 &= J(BO\infty)^2 \cdot \phi A \\ &\quad + 2J(BO\infty) \cdot J(AO\infty) \cdot \psi AB + J(AO\infty)^2 \cdot \phi B, \end{aligned}$$

the following theorems may be easily proved:

$$\begin{aligned} (\alpha) \quad & \left| \begin{array}{cccc} \phi AD, & \phi AE, & \phi AF, & A\infty^2 \\ \phi BD, & \phi BE, & \phi BF, & B\infty^2 \\ \phi CD, & \phi CE, & \phi CF, & C\infty^2 \\ D\infty^2, & E\infty^2, & F\infty^2, & 0 \end{array} \right| \\ & \equiv J(ABC) \cdot J(DEF) \cdot A\infty \cdot B\infty \cdot C\infty \cdot D\infty \cdot E\infty \cdot F\infty \\ & \quad \text{(in points).} \end{aligned}$$

$$\begin{aligned} (\beta) \quad & \left| \begin{array}{cccc} J(AD\infty)^2, & J(AE\infty)^2, & J(AF\infty)^2, & \phi A \\ J(BD\infty)^2, & J(BE\infty)^2, & J(BF\infty)^2, & \phi B \\ J(CD\infty)^2, & J(CE\infty)^2, & J(CF\infty)^2, & \phi C \\ \phi D, & \phi E, & \phi F, & 0 \end{array} \right| \\ & \equiv J(BC\infty) \cdot J(CA\infty) \cdot J(AB\infty) \cdot J(EF\infty) \cdot J(FD\infty) \cdot J(DE\infty) \\ & \quad \text{(in lines).} \end{aligned}$$

And, in the process of proving them, we obtain

$$\begin{aligned} (\gamma) \quad & \left| \begin{array}{cccc} \psi(OA, OD), & \psi(OA, OE), & \psi(OA, OF), & A\infty \\ \psi(OB, OD), & \psi(OB, OE), & \psi(OB, OF), & B\infty \\ \psi(OC, OD), & \psi(OC, OE), & \psi(OC, OF), & C\infty \\ D\infty, & E\infty, & F\infty, & 0 \end{array} \right| \\ & \equiv J(ABC) \cdot J(DEF) \cdot O\infty^4 \text{ (in points).} \end{aligned}$$

$$(\delta) \begin{vmatrix} \psi AD, & \psi AE, & \psi AF, & \frac{\phi A}{J(OA\infty)} \\ \psi BD, & \psi BE, & \psi BF, & \frac{\phi B}{J(OB\infty)} \\ \psi CD, & \psi CE, & \psi CF, & \frac{\phi C}{J(OC\infty)} \\ \frac{\phi D}{J(OD\infty)}, & \frac{\phi E}{J(OE\infty)}, & \frac{\phi F}{J(OF\infty)}, & 0 \end{vmatrix} \\
 \equiv \phi^2 \frac{J(BC\infty).J(CA\infty).J(AB\infty).J(EF\infty).J(FD\infty).J(DE\infty)}{J(OA\infty).J(OB\infty).J(OC\infty).J(OD\infty).J(OE\infty).J(OF\infty)} \\
 \text{(in lines),}$$

To these may be added

$$(\epsilon) \begin{vmatrix} AI, & BL, & CL \\ AM, & BM, & CM \\ AN, & BN, & CN \end{vmatrix} \equiv J(ABC).J(LMN) \\
 \text{(in points and lines).}$$

The interpretations of these are :

$$(\alpha) \begin{vmatrix} 14^2, & 15^2, & 16^2, & 1 \\ 24^2, & 25^2, & 26^2, & 1 \\ 34^2, & 35^2, & 36^2, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \equiv \text{area } (123) \times \text{area } (456)$$

(where 14 denotes the distance between the points 1 and 4).

$$(\beta) \begin{vmatrix} \sin^2 14, & \sin^2 15, & \sin^2 16, & 1 \\ \sin^2 24, & \sin^2 25, & \sin^2 26, & 1 \\ \sin^2 34, & \sin^2 35, & \sin^2 36, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}$$

$$\equiv \sin 23 . \sin 31 . \sin 12 . \sin 56 . \sin 64 . \sin 45$$

(where 14 denotes the angle between the lines 1 and 4).

$$(\gamma) \begin{vmatrix} \cos AOD, & \cos AOE, & \cos AOF, & \frac{1}{OA} \\ \cos BOD, & \cos BOE, & \cos BOF, & \frac{1}{OB} \\ \cos COD, & \cos COE, & \cos COF, & \frac{1}{OC} \\ \frac{1}{OD}, & \frac{1}{OE}, & \frac{1}{OF}, & 0 \end{vmatrix} \\
 \equiv \frac{\text{area } ABC}{OA . OB . OC} \cdot \frac{\text{area } DEF}{OD . OE . OF} \text{ (in points).}$$

$$(d) \begin{vmatrix} \cos AD, & \cos AE, & \cos AF, & \frac{1}{\sin OA} \\ \cos BD, & \cos BE, & \cos BF, & \frac{1}{\sin OB} \\ \cos CD, & \cos CE, & \cos CF, & \frac{1}{\sin OC} \\ \frac{1}{\sin OD}, & \frac{1}{\sin OE}, & \frac{1}{\sin OF}, & 0 \end{vmatrix} \\ \equiv \frac{\sin BC \cdot \sin CA \cdot \sin AB \cdot \sin EF \cdot \sin FD \cdot \sin DE}{\sin OA \cdot \sin OB \cdot \sin OC \cdot \sin OD \cdot \sin OE \cdot \sin OF}.$$

$$(e) \begin{vmatrix} AL, & BL, & CL \\ AM, & BM, & CM \\ AN, & BN, & CN \end{vmatrix} \equiv \text{area } ABC \times \text{projector of } LMN,$$

where ABC are points, and LMN lines, and AL is the perpendicular from the point A on the line L .

The interpretation of the identity

$$\begin{vmatrix} AL, & BL \\ AM, & BM \end{vmatrix} \equiv J(L, M, AB) \equiv J(LM, A, B)$$

is $\begin{vmatrix} AL, & BL \\ AM, & BM \end{vmatrix} \equiv \text{projector of } L, M, AB \times \text{distance } AB,$
 $\equiv \text{area of } LM, A, B \times \sin LM,$

where AL is the perpendicular as before.

Circles.

9. To find the equation to the circle whose diameter is the line joining the intersection of A and B to the intersection of C and D .

The circle is the locus of the intersection of two lines at right angles to one another drawn through the two points. Now if $A = kB$ and $C = k'D$ are at right angles to one another, we must have

$$\psi(A - kB, C - k'D),$$

or $\psi AC - k \cdot \psi BC - k' \psi AD + kk' \psi BD = 0,$

and then, eliminating k , k' by the equations $A = k/B$, $C = k'/B$, we have for the equation to the circle

$$BD, \psi AC = AD, \psi BC = BC^2, \psi AD = AC^2, \psi BD = 0,$$

that is to say

$$\begin{vmatrix} \psi AC & \psi AD & A \\ \psi BC & \psi BD & B \\ C & D & 0 \end{vmatrix} = 0$$

10. *The circles whose diameters are the diagonals of the quadrilateral have a common radical axis.*

For, denote the equation just found by $\psi(A/B, C/D) = 0$, then it is easy to verify the identity

$$(\overline{AB}, CD) + (\overline{AD}, BC) + (\overline{AC}, BD) = 0,$$

which expresses the property in question.

11. If a, b, c, d are the perpendiculars to a fixed line containing p, q, r on the lines A, B, C, D , the equations for circles can be expressed equally well by

$$\begin{vmatrix} \cos(p/q) & \cos(p/r) & p \\ \cos(p'/q) & \cos(p'/r) & p' \\ q & r & 0 \end{vmatrix} = 0$$

The translation of these circles into the ordinary system of rectangular coordinates requires the determination of the A 's, B 's, C 's, D 's. I now proceed.

IV. Determination of the A 's, B 's, C 's, D 's.

12. As the title of this section might have given, there is a mixed feeling. I will say at once that "the determination of the Absolute" means "to find the focus of \mathfrak{S} ." It is not precisely that the method here used is supposed to find the \mathfrak{S} generally given; but only that it is more *convenient* and *practical* application.

Arcs.

The Arcal coordinates of a point are quantities proportional to the triangles it determines with the sides of a fixed triangle. Let α, β, γ be the sides of this triangle then we have

$$x, y, z = \frac{\alpha Y}{\sqrt{\beta}X + \sqrt{\gamma}Z} : \frac{\beta Y}{\sqrt{\alpha}X + \sqrt{\gamma}Z} : \frac{\gamma Z}{\sqrt{\alpha}X + \sqrt{\beta}Z},$$

and so

$$\phi (lx + my + nz)$$

$$= \alpha^2 l^2 + \beta^2 m^2 + \gamma^2 n^2 + 2\beta\gamma mn \cos A + 2\gamma\alpha nl \cos B + 2\alpha\beta lm \cos C,$$

the sign being determined as before.

Interpretation of Constants.

13. Let P_1, P_2, P_3 be the three perpendiculars of angle of reference, and $\omega_1, \omega_2, \omega_3$ the perpendiculars the angular points on the line (line). Then by the in Art. 7, we have, in Trilinears,

$$\omega_1 : \omega_2 : \omega_3 = P_1 : P_2 : P_3,$$

since, in this system, the coordinates of the angular are as

$$P_1, 0, 0; 0, P_2, 0; 0, 0, P_3.$$

Consequently,

$$l : m : n = \frac{\omega_1^2}{P_1^2} : \frac{\omega_2^2}{P_2^2} : \frac{\omega_3^2}{P_3^2}.$$

In the Arcal system, the coordinates of the angular are as 1, 0, 0; 0, 1, 0; 0, 0, 1; so that in this case we

$$l : m : n = \omega_1 : \omega_2 : \omega_3.$$

It is of the greatest importance to notice that there are *different* systems of Tangential Coordinates, corresponding Trilinears and Arcs respectively. In the former, the coordinates of a line are proportional to $\frac{\omega_1^2}{P_1^2}, \frac{\omega_2^2}{P_2^2}, \frac{\omega_3^2}{P_3^2}$; in the latter they are proportional to $\omega_1, \omega_2, \omega_3$. It may be

distinction in mind, we may say generally that the *coefficients* in the *equation* of a point are proportional to the *coordinates* of the point; and that the *coefficients* in the *equation* of a line are proportional to the *coordinates* of the line. And then we may get rid of coordinates altogether, and consider nothing but equations.

Line at Infinity.

14. Form the reciprocal of ϕ by the ordinary method; it is, in Trilinears,

$$(x \sin A + y \sin B + z \sin C)^2,$$

and in Areals,

$$4E^2 \sin^2 A (x + y + z)^2.$$

It must be remembered that we are not here finding the *equation* of the line at infinity, but the value of the function Ax^2 of the point (x, y, z) . In the case of Trilinears, if we suppose x, y, z to be actually *equal* to the perpendiculars from a point on the circle, then the quantity

$$x \sin A + y \sin B + z \sin C$$

is four times the projector of the triangle of reference.

Quadrilaterals.

15. Consider four straight lines $A, B, C, D = 0$. Since the equation of *any* line can be expressed in terms of the equations of three others, there must be an identical relation

$$lA + mB + nC + sD = 0 \dots\dots\dots (1),$$

and because this is independent of the values of x, y, z , we must have

$$lA_x + mB_x + nC_x + sD_x = 0,$$

$$lA_y + mB_y + nC_y + sD_y = 0,$$

$$lA_z + mB_z + nC_z + sD_z = 0,$$

where A_x denotes the coefficient of x in A , &c. Eliminating therefore l, m, n, s from these four equations, we have

$$\begin{vmatrix} A_x & B_x & C_x & D_x \\ A_y & B_y & C_y & D_y \\ A_z & B_z & C_z & D_z \\ A & B & C & D \end{vmatrix} = 0.$$

But, by definition of a Jacobian, this is equal to zero:

$$A, J(R'D) = B, J(C'D) + C, J(DAB) - D, J(CAB) = 0,$$

and this therefore is the identical relation between the given lines.

Now it is often convenient to use a set of four coordinates, w, y, z, w , connected by the identical relation

$$w + y + z + w = 1,$$

we have then only to put

$$\begin{aligned} w, y, z, w &= A, J(R'D), B, J(C'D), \\ &C, J(DAB), D, J(CAB), \end{aligned}$$

and we shall have such a system. We then find that

$$\phi(lc + my + nz + sw) = \phi(lA + mB + nC + sD) = \phi(lA, J(R'D)^2 + \dots + 4D, J(CAB)^2 + \dots + 4D, J(CAB)^2).$$

Now let $\alpha, \beta, \gamma, \delta$ be the projections of the lines $AB, AC, AD, CAD, ABD, CBA$; and divide the results previously obtained by $\phi A, \phi B, \phi C, \phi D$; then we may write

$$\phi(lc + my + nz + sw) = \alpha A^3 + \beta B^3 + \gamma C^3 + \delta D^3 + 3\alpha\beta B^2C + \dots$$

We have here in fact made a projection of the line passing through from a point on A , multiplied by the projection of AB^2 . And this might be taken as the definition of the projection of coordinates.

If for D we write ∞ , the identity becomes

$$\infty, J(AB^2) = A, J(B^2C) + B, J(C^2A) - C, J(A^2B) = 0.$$

Hence we find, in Table III, for instance,

$$\frac{J(AB^2)}{\alpha\phi A, \phi B, \phi C} = \frac{J(AB^2)}{\alpha\phi A, \phi B, \phi C} = \frac{J(AB^2)}{\alpha\phi A, \phi B, \phi C} = \frac{J(AB^2)}{\alpha\phi A, \phi B, \phi C},$$

which agrees with the result before obtained, namely

$$\frac{J(AB^2)}{\alpha\phi A, \phi B, \phi C}$$

is really the ratio of the projection of AB^2 to the projection of the triangle of reference.

Equations.

16. The condition that a point A shall lie on a line L is $AL = 0$. Now if we consider the coefficients of the point as coordinates, then $AL = 0$ is the equation of the line L ; if we consider the coefficients of the line as coordinates, $AL = 0$ is the equation of the point A . It is not difficult to see how this notion may be generalized. Consider the form

$$L = L_1y_1 + L_2y_2$$

which may represent either a point or a line, according as (L_1, L_2) or (y_1, y_2) are regarded as variables. I shall denote this form by an asterisk $(*)$, and use it exclusively to represent equations. For instance, the equation $J(AB^*) = 0$ is the equation to the point or line AB , according as A and B are lines or points. I give one or two examples of equations found by this method.

(a) *Lines of a pencil subtended by four given points in a given standard conic curve.*

We want a point P such that $\frac{\sin APB}{\sin AP'P} = \frac{\sin C'PB}{\sin C'P'B} = k$. The locus is then immediately written in double six form:

$$\sin APB = \frac{P \cdot (ABPP)}{\sin \phi_1 \cdot AP_1 \cdot \phi_2 \cdot BP_2}.$$

It is

$$J(AB^*) \cdot J(A'B^*) = k \cdot J(P^*) \cdot J(PB^*),$$

which is clearly a conic through the points $ABCD$.

(b) *Envelope of a line cut by four given lines in a given standard conic curve.*

If the conic is AX^2 , the envelope is (in Tangential coordinates)

$$J(LM^*) \cdot J(XN^*) = k \cdot J(LX^*) \cdot J(MN^*),$$

a conic containing the four given lines.

(γ) *Envelop of a line, the product of whose distances from two fixed points is constant.*

$$\text{Ans.} \quad (A^*) (B^*) = k^2 A \infty . B \infty . \phi (*),$$

a conic inscribed in the quadrilateral formed by joining A and B to the two points of ϕ .

(δ) *Envelop of a line of constant length resting on two given lines.*

$$\text{Ans.} \quad \phi (*) . J(LM^*)^2 = k^2 . J(L\infty^*)^2 . J(M\infty^*).$$

Let α be the point LM , β the point $L\infty$, γ the point $M\infty$; then if we remember that any point on the line $\alpha\beta$, i.e. at infinity, has its equation of the form $lx + m\beta = 0$, this equation may be written

$$a^2 (a\beta^2 + 2b\beta\gamma + c\gamma^2) = k^2 \beta^2 \gamma^2,$$

$$\text{or} \quad \frac{a}{\gamma^2} + \frac{2b}{\beta\gamma} + \frac{c}{\beta^2} = \frac{k^2}{a^2};$$

the envelop is therefore the tangential inverse, in respect of the triangle $\alpha\beta\gamma$, of a conic in respect of which α is the pole of $\beta\gamma$. It is obvious from the figure that each of the lines L , M , touches at two cusps, so that the curve is of the sixth order. The equation shews that the line at infinity ($\beta\gamma$) is also a double tangent, and when L , M are at right angles, $b = 0$, and the line at infinity touches at two cusps.

(ϵ) *Envelop of a line cut by three given lines in given ratios.*

$$\text{Ans.} \quad \lambda J(MN^*) . J(L\infty^*) + \mu J(NL^*) . J(M\infty^*) \\ + \nu J(LM^*) J(N\infty^*) = 0,$$

a parabola touching the three given lines.

The utility of the method in questions of this sort is still more evident in the case of curves and surfaces of the second order.

V. *Planes and Points in Space.**Expressions Considered.*

17. In Geometry of Three Dimensions the absolute is an imaginary circle in which the plane at infinity cuts any sphere. The condition that a plane $A = 0$ may touch this circle, is of the second order in the coefficients of A , and may be written $\phi(A) = 0$. This being so, we have, as before,

$$\phi(\lambda A + \mu B) \equiv \lambda^2 \phi(A) + 2\lambda\mu \cdot \psi(A, B) + \mu^2 \cdot \phi(B),$$

where the function $\psi(A, B)$ is of the first order in the coefficients of each of the two planes, and vanishes only when they are at right angles.

The condition that a point $a = 0$ may be at infinity is of the first order in the coordinates or coefficients of the point, and may be written $a \infty = 0$.

The condition that four planes $A, B, C, D = 0$ may meet in a point, is that the determinant formed with their coordinates or coefficients, that is to say, their Jacobian, shall vanish. This I express by the equation

$$J(ABCD) = 0.$$

The condition that three planes $A, B, C = 0$ may meet at infinity, or, which is the same thing, may be parallel to the same line, is accordingly

$$J(ABC \infty) = 0.$$

18. The form of the absolute ϕ is found exactly as in plane geometry; I will therefore anticipate a formula, and give it here. We have generally,

$$\phi(lx + my + nz + sw) \equiv l^2 \phi(x) + \dots + 2mn \psi(y, z) + \dots$$

Now by means of the formula

$$\cos(A, B) = \frac{\psi(A, B)}{\sqrt{\phi(A) \cdot \phi(B)}},$$

this becomes, in quadriplanar coordinates,

$$\phi (lw + my + nz + sw) \equiv l^2 + m^2 + n^2 + s^2 - 2mn \cos (y, z) - \dots,$$

and in tetrahedral coordinates

$$\phi (lw + my + nz + sw) \equiv l^2 \alpha^2 + \dots - 2mn \cdot \beta \gamma \cdot \cos (y, z) - \dots,$$

in both of which $\cos (y, z)$ means the cosine of the internal angle between the planes $y=0$, $z=0$ of the fundamental tetrahedron, and α , β , γ , δ are the areas of the faces. The equation to the plane at infinity is then found to be, in the quadriplanar system,

$$Px + Qy + Rz + Sw = 0,$$

$$\text{where } P^2 = \begin{vmatrix} 1, & -\cos (y, z), & -\cos (y, w) \\ -\cos (y, z), & 1, & -\cos (z, w) \\ -\cos (y, w), & -\cos (z, w), & 1 \end{vmatrix}.$$

It is convenient to call P the *sine* of the solid angle yzw (M. Paul Serret). If we write A , B , C , D for the solid angles of the tetrahedron, then

$$\frac{\sin A}{\alpha} = \frac{\sin B}{\beta} = \frac{\sin C}{\gamma} = \frac{\sin D}{\delta},$$

and the equation to the plane at infinity, in tetrahedral coordinates, is

$$\beta \gamma \delta \sin A (x + y + z + w) = 0.$$

Fundamental Propositions.

19. The following propositions may be proved by Cartesian coordinates.

A. If a plane touch the imaginary circle at infinity,

(a) Every area measured on the plane is zero.

(b) Every perpendicular distance from it is infinite.

(c) Of every angle on the plane, the sine is zero and the cosine unity.

(d) Of every angle made with the plane, the sine and cosine are infinite.

- B.* If a straight line meet the imaginary circle at infinity,
 (a) Every length measured on the line is zero.
 (b) Every perpendicular distance from it is infinite.
 (c) Of the angle between any two planes through it, the sine is zero and the cosine unity.
 (d) Of every angle made with it, the sine and cosine are infinite.

- C.* If a point be at infinity,
 (a) Its perpendicular distance from any plane or straight line not passing through it is infinite.
 (b) The volume of the tetrahedron which it forms with any other three points not in the same plane with it is infinite.
 (c) The area of the triangle which it forms with any other two points, not such that the plane of the triangle touches the absolute (see prop. *A*, *a*), is infinite.
 (d) Its distance from any other point, not at infinity, is infinite.

Formulae of Adaptation.

20. These are to be deduced from the propositions just stated precisely in the same way as the corresponding formulæ in Plane Geometry were proved. I postpone for the present the consideration of formulæ relating to straight lines.

The volume contained by four points *a*, *b*, *c*, *d* is

$$\frac{J(abcd)}{a \infty . b \infty . c \infty . d \infty}.$$

If the points are given as the intersections of four planes, *A*, *B*, *C*, *D*, the expression becomes

$$\frac{\{J(ABCD)\}^3}{J(BCD \infty) . J(ODA \infty) . J(DAB \infty) . J(ABC \infty)}.$$

The area contained by three points *a*, *b*, *c* is

$$\frac{\sqrt{\phi(abc)}}{a \infty . b \infty . c \infty}.$$

be perpendicular at once to each of the planes D, E unless these planes are parallel to the same line, the which is perpendicular to all three of them is at infinity. If therefore the condition is satisfied, all meet at infinity, or D, E, F meet at infinity. Thus

$$\Psi \left\{ \frac{ABC}{DEF} \right\} = J(ABC, \infty) \cdot J(DEF, \infty).$$

The condition $\Psi \left\{ \frac{AB}{CD} \right\} = 0$ will be satisfied if we take plane through the line (A, B) perpendicular to the line (C, D) . It is easy to see that this can only be the case line (A, B) is perpendicular to the line (C, D) . This may be expressed in the form

$$\Psi \left\{ \frac{AB}{CD} \right\} = \Psi(AB, CD) = 0.$$

To interpret the theorems (1) and (2), I observe that, multiply any single row or column of a determinant by a quantity, is to multiply the whole determinant by that. Thus we have

$$\cos \left\{ \frac{ABCD}{EFGH} \right\} = \Psi \left\{ \frac{ABCD}{EFGH} \right\} = \sqrt{\phi A} \cdot \phi B \cdot \phi C \cdot \phi D \cdot \phi E \cdot \phi F \cdot \phi G \cdot \phi H$$

$$\text{and } \cos \left\{ \frac{ABC}{DEF} \right\} = \Psi \left\{ \frac{ABC}{DEF} \right\} = \frac{J(ABC, \infty)}{\sqrt{\phi A} \cdot \phi B \cdot \phi C} = \frac{J(A, B, C)}{\sqrt{\phi A} \cdot \phi B \cdot \phi C}.$$

If now we make A, B, C identical with D, E, F respectively we get

$$\{J(ABC, \infty)\}^2 = \cos \left\{ \frac{ABC}{ABC} \right\} = \sin^2(A, B, C) \text{ by definition}$$

$$\text{so } \cos \left\{ \frac{ABC}{DEF} \right\} = \sin(A, B, C) \cdot \sin(D, E, F).$$

The theorems (1) and (2) will be found to embody a number of results.

22. I now prove certain known theorems of determinants, which are useful in this subject.

Consider five planes A, B, C, D, E , whose equations are

$$\begin{aligned} L_1x + m_1y + n_1z + s_1w &= 0, \\ &\dots\dots\dots \\ L_5x + m_5y + n_5z + s_5w &= 0. \end{aligned}$$

We know that, since each one can be expressed in terms of the other four, there must exist some identical relation

$$PA + QB + RC + SD + TE = 0,$$

And because this is true for all values of x, y, z, w , we must have

$$\begin{aligned} PL_1 + Qm_1 + Rn_1 + Ss_1 &= 0, \\ &\text{&c.} \qquad \qquad \qquad \text{&c.} \end{aligned}$$

From these five equations we can eliminate $PQRST$, and we find

$$\begin{aligned} A \cdot J(BCDE) + B \cdot J(CDEA) + C \cdot J(DEAB) + D \cdot J(EABC) \\ + E \cdot J(ABCD) \equiv 0 \dots\dots\dots (4), \end{aligned}$$

the identical relation in question.

Now if we substitute in A the coordinates of the intersection of B, C, D, E , the result is clearly $J(ABCDH)$. But the equation (4) is true for all values of the variables; we may therefore substitute in it the coordinates of the point (P, Q, R) . In this way we obtain

$$\begin{aligned} J(ABCDH) \cdot J(BCDE) + J(BFCH) \cdot J(CDEA) \\ + J(CFCH) \cdot J(DEAB) + J(DFCH) \cdot J(EABC) \\ + J(EFCH) \cdot J(ABCD) \equiv 0 \dots\dots\dots (5). \end{aligned}$$

Making H identical with E , and transposing, we have

$$\begin{aligned} J(ABCE) \cdot J(DECE) \equiv J(AECE) \cdot J(BCDE) \\ + J(BFCE) \cdot J(CADE) + J(CFCE) \cdot J(ABDE), \dots\dots (6). \end{aligned}$$

Write ∞ for E , and use theorem (2); thus

$$\cos \begin{vmatrix} ABC \\ DEF \end{vmatrix} \equiv \cos \begin{vmatrix} AFG \\ BCD \end{vmatrix} + \cos \begin{vmatrix} BEG \\ CAD \end{vmatrix} + \cos \begin{vmatrix} CFG \\ ABD \end{vmatrix} \dots\dots (7).$$

Again, let E be the plane $w=0$ of the fundamental tetrahedron; and write (1, 2, 3, 4, 5, 6) for the traces on this plane of the planes (A, B, C, D, E, F) respectively; then we get the theorem in plane geometry

$$J(123) \cdot J(456) \equiv J(156) \cdot J(234) + J(164) \cdot J(235) \\ + J(145) \cdot J(236).$$

If we make the lines 1, 2 identical, and write ∞ for each, this becomes the theorem referred to in Art. 6.

Spheres.

23. To find the equation to the sphere whose diameter is the line joining the point (A, B, C), to the point (D, E, F).

The sphere may be defined as the locus of the foot of a perpendicular from the point (A, B, C) on a variable plane passing through the point (D, E, F). Now the equations to a line through (A, B, C) perpendicular to a plane L are

$$\frac{A}{\psi(A, L)} = \frac{B}{\psi(B, L)} = \frac{C}{\psi(C, L)} = \frac{1}{k}, \text{ suppose.}$$

Let now $L \equiv \lambda D + \mu E + \nu F$; then at the foot of the perpendicular,

$$\lambda \psi(A, D) + \mu \psi(A, E) + \nu \psi(A, F) - kA = 0,$$

$$\lambda \psi(B, D) + \mu \psi(B, E) + \nu \psi(B, F) - kB = 0,$$

$$\lambda \psi(C, D) + \mu \psi(C, E) + \nu \psi(C, F) - kC = 0,$$

$$\lambda D + \mu E + \nu F = 0,$$

we eliminate λ, μ, ν, k , and we get for

$$\begin{vmatrix} \psi AD, & \psi AE, & \psi AF, & A \\ \psi BD, & \psi BE, & \psi BF, & B \\ \psi CD, & \psi CE, & \psi CF, & C \\ D & E & F & 0 \end{vmatrix} = 0.$$

Call this $(\psi) \begin{vmatrix} DEFO \\ ABCO \end{vmatrix} = 0$; then since theorem (6) is a general theorem of determinants, we have,

$$(\psi) \begin{vmatrix} ABCO \\ DEFO \end{vmatrix} + (\psi) \begin{vmatrix} AEF O \\ BCDO \end{vmatrix} + (\psi) \begin{vmatrix} AFD O \\ BCEO \end{vmatrix} + (\psi) \begin{vmatrix} ADEO \\ BCFO \end{vmatrix} = \dots (8).$$

these four spheres have therefore a common radical axis. Whence this geometrical theorem:

If the faces of a tetrahedron A, B, C, D are met by a straight line in the points a, b, c, d ; then the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis.

It will be observed that the *faces* of the tetrahedron are $A, D, E, F = 0$, and that the *straight line* is the intersection of the planes $B = 0, C = 0$. The relation (8) is very easily verified.

*(To be continued.)**

* [No more on this subject was published in the *Quarterly Journal*.]

ON THE GENERAL THEORY OF ANHARMONY

1. The theory of Anharmonicity on the straight line is stated in the following symmetrical form:

(i) There is an identical relation connecting the four points, 1, 2, 3, 4, on a right line, viz.,

$$12 \cdot 34 + 13 \cdot 42 + 14 \cdot 23 = 0.$$

(ii) The ratios of the three terms in this identity altered by projection or integral linear transformation is a corresponding theory of four straight lines meeting in a point.

2. In applying this theory to geometry of two dimensions we meet with this third proposition:

(iii) If a straight line meet four fixed straight lines, then the distances of the points which they determine satisfy a relation of the form

$$\lambda \cdot 12 \cdot 34 + \mu \cdot 13 \cdot 42 + \nu \cdot 14 \cdot 23 = 0,$$

then the envelope of the line is of the second class, from four given lines. There is, of course, a correlative proposition, the locus of a point subtended by four points in a manner. The propositions (i), (ii), (iii), each including *verse* and *correlative* propositions, may be regarded as stating the entire theory of anharmonicity in geometry of two dimensions, and its application to geometry of two dimensions.

* [From the *Proceedings of the London Mathematical Society*, pp. 35-46.]

3. I proceed to state the theory of anharmonics in geometry of two dimensions.

(i) There are six identical relations connecting the areas of triangles formed by six points, 1, 2, 3, 4, 5, 6, in a plane, viz.,

$$123 \cdot 456 + 124 \cdot 563 + 125 \cdot 634 + 126 \cdot 345 = 0,$$

with five others obtained from this by permutation.

(ii) The ratios of the terms in these identities are not altered by projection or integral linear transformation. Under each of these propositions are included *three* correlatives, to explain which I must introduce three new definitions:

DEF. 1st. The *projector* of a plane triangle is the square of the area divided by the continued product of the sides.

DEF. 2nd. In a solid angle considered as determined by three concurrent straight lines, the sine of the angle between the first line and the plane of the other two, multiplied by the sine of the angle between the other two lines, is a symmetrical quantity in respect of the three lines, and is called the *sine* of the three lines.

DEF. 3rd. In a solid angle considered as determined by three planes, the sine of the angle between the first plane and the intersection of the other two, multiplied by the sine of the angle between the other two planes, is a symmetrical quantity in respect of the three planes, and is called the *sine* of the three planes.

4. It is convenient for several purposes to use the word "distance" as including all these notions: thus I shall speak of the *distance* of two lines in a plane, meaning the sine of the angle between them;

the distance of three points is the area of their triangle;

" " of three straight lines in a plane, the projector of their triangle;

" " of three planes, the sine of the planes;

" " of three concurrent lines in space, the sine of the lines.

And I shall have occasion afterwards to define the four points and of four planes in space. By these definitions, the propositions (i), (ii) may be interpreted different ways, corresponding to the four aspects of anisotonal extension: the symbol 123 being always intended to mean the *distance* of the three things considered.

5. The propositions (i), (ii), including their correlative propositions, constitute the entire theory of anisotony in geometry of two dimensions. To apply to geometry of three dimensions, I state the following proposition, which has only *one* correlative.

(iii) If a plane meet six fixed planes so that the six lines they determine upon it satisfy a relation of the form

$$\lambda \cdot 123 \cdot 456 + \mu \cdot 124 \cdot 365 + \nu \cdot 156 \cdot 234 + \dots = 0,$$

then the envelope of the plane is of the second class and the six given planes.

6. The theory of anharmonicity in three dimensions is entirely analogous to the two former theories, that it further discussion. The distance of four points in \mathcal{E} of their tetrahedron, and the distance of four planes in \mathcal{E} of the volume divided by the product of the areas of \mathcal{E}

7. Two finite lines $11', 22'$, measured on the same line, are said to be *harmonically situate* when

$$12 \cdot 1'2' + 12' \cdot 1'2 = 0;$$

and when this is the case, if the two pairs of points are ruled by the equations $U=0$, $V=0$, there is an involution connecting the quadrics U , V which may be denoted

$$[1](U, V)=0;$$

the notation indicating that if we expand the determinant $\lambda U + \mu V$, or

$$[\lambda](\lambda U + \mu V),$$

then the coefficient of $\lambda\mu$ in the expansion vanishes.

There is a similar relation between two angles having a common vertex; when this relation holds, I say that the two covertical angles are harmonically situate. I also speak of an harmonic pair of angles, or of an harmonic pair of finite lines, or lengths; meaning in this case covertical angles, or collinear lengths.

8. This being so, it is known that two lengths (I use the word *length* to denote a pair of points), anyhow placed in a plane, determine a conic passing through their ends, which conic is the locus of points which the lengths subtend harmonically, that is, in a pair of harmonic angles. I call this the harmonic conic of the two lengths. This (with the correlative propositions) completes the theory of harmonics in one dimension, and its application to two dimensions. I now come to consider harmonics in two dimensions.

9. If the harmonic conic of two lengths 11', 22' divides harmonically a third length 33', then the relation between the three lengths is symmetrical, and I say that the three lengths are *harmonically situate* in the plane. The following relation subsists among their distances, viz.,

$$123, 1'2'3' + 1'23, 123' + 12'3, 1'2'3' + 123', 1'2'3 = 0.$$

And if each length, or point pair, be considered as a degenerate conic, so that the equations to the three point-pairs are $U, V, W = 0$, then, when the three lengths are harmonically situate, there is an invariant relation connecting the quadrics U, V, W , which may be denoted by

$$\Delta(U, V, W) = 0;$$

$\Delta(U, V, W)$ denoting the coefficient of $\lambda\mu\nu$ in the expansion of Discriminant of $(\lambda U + \mu V + \nu W)$.

It is obvious that a similar relation may subsist among three angles in a plane, three pairs of lines through a point in space, or three pairs of planes through a point in space.

10. Three lengths anyhow placed in space determine a quadric surface passing through their ends, which surface is the locus of points which the lengths subtend harmonically, that is,

in a triad of harmonic angles. I call this the harmonic quadric of the three lengths.

11. If the harmonic quadric of three lengths 11', 22', 33' divides harmonically a fourth length 44', then the relation between the four lengths is symmetrical, and I say that the four lengths are *harmonically situate* in space. The following relation subsists among their distances, viz.,

$$\Sigma . 1234 . 1'2'3'4' = 0 \text{ (eight terms).}$$

And if each length, or point-pair, be considered as a degenerate quadric, so that the equations to the four point-pairs are $U=0$, $V=0$, $W=0$, $T=0$, then, when the four lengths are harmonically situate, there is an invariant relation connecting the quadrics U , V , W , T , which may be denoted by

$$\square (U, V, W, T) = 0;$$

$\square (U, V, W, T)$ denoting the coefficient of $\lambda\mu\nu\rho$ in the expansion of

$$\text{Discriminant of } (\lambda U + \mu V + \nu W + \rho T).$$

It is obvious that a similar relation may subsist among four pairs of planes.

12. It only remains to explain the meaning of the conditions

$$\square (U, V, W) = 0,$$

$$\square (U, V, W, T) = 0,$$

when the quadrics do *not* break up into factors. Two conics U , V determine an harmonic conic H , locus of points which they subtend in a pair of harmonic angles. If a triangle self-conjugate of H can be inscribed in W , then the relation between UVW is symmetrical, and $\square (U, V, W) = 0$. The conics may then be spoken of as three mutually harmonic conics; a similar relation may hold between three covertical cones. Thus, in fact, three quadric *surfaces* U , V , W determine an harmonic quadric H , locus of points which they subtend in three harmonic cones. If a tetrahedron self-conjugate of H can be inscribed in T , then the relation between U , V , W , T is symmetrical, and $\square (U, V, W, T) = 0$.

XIII.

ON A GENERALIZATION OF THE THEORY OF POLARS*.

(THE present Note establishes the idea of the polar curve of a curve of given class in respect of a curve of given order, the class being less than the order; and of the polar curve of a curve of given order in respect of a curve of given class, the order being less than the class. It also deals with a certain invariant of two curves, such that the order of one is equal to the class of the other; and with certain other invariants and contravariants arising out of the theory of polars. I desire to present these ideas by themselves to the Society, because they seem likely to be useful for other purposes than that to which I propose to apply them subsequently, viz., the extension of Grassmann's Geometric Analysis.)

1. Let B_n be a curve of the n th order, and c_m a curve of the m th class. Let the equations of the curves be

$$B_n \equiv (A, B, C, D, \dots, \mathfrak{X}xy z)^n$$

$$c_m \equiv (a, b, c, d, \dots, \mathfrak{X}\xi \eta \zeta)^m$$

in point and line coordinates respectively.

In c_m write $\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z}$ in place of ξ, η, ζ respectively, and operate on B_n with the symbol thus formed. I denote the result by merely writing c_m as an operator before B_n ; thus

$$c_m B_n \equiv \left(a, b, c, \dots, \mathfrak{X} \frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z} \right)^m \cdot (A, B, C, \dots, \mathfrak{X}xy z)^n;$$

* [From the *Proceedings of the London Mathematical Society*, Vol. II. No. 16, pp. 116—118.]

then we find

(i) If m is less than n , $c_n R_m$ is a covariant, which I call the *polar curve* of c_n in respect of R_m . It is given in the point coordinates x, y, z , and is of order $n-m$.

(ii) If m is equal to n , $c_n R_m$ is an invariant of the two curves. When this invariant vanishes, I say that the curves are *harmonic* of each other.

(iii) If m is greater than n , $c_n R_m = 0$ always.

2. It is important to shew that these new meanings of the words *polar* and *harmonic* include the old meanings. Now the m th polar of a point (x, y, z) whose tangential equation is $x^2 + y^2 + z^2 = 0$, say the point P , is

$$\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}\right)^m R_2 = 0,$$

which will be denoted in our present notation by $p^m R_2 = 0$. But this is, according to the new definition, the polar of m times the point p ; the point p taken m times being at once a particular case of a curve of the m th class. Again, the two curves

$$(a, b, c, f, g, h) \text{ and } (A, B, C, F, G, H) \text{ in } (x, y, z)^2$$

have been called *harmonic* when

$$Aa + Bb + Cc + 2Ef + 2Gg + 2Hh = 0,$$

the invariant being obviously the result of turning the first into an operator and applying it to the second.

3. Returning to the curves c_n, R_m , we may convert R_m into an operator by writing in it $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in place of x, y, z respectively. The result of the operation on c_n may be denoted by $R_m c_n$. As before, if $n = m$, the result is zero; if $n = m$, it is the harmonic invariant; if $n < m$, it is a curve of class $m-n$, which may be called the *polar curve* of R_m in respect of c_n .

4. Another definition of polar curves may be given. Suppose that the curves g_m, h_n together make up a curve which is harmonic of R_{m+n} . It is convenient to say that the curves g_m, h_n

are complementary. Then it is clear that *all the curves complementary to a given curve are harmonic of its polar*. This may be regarded as a generalization of the theorem: All the points conjugate to a fixed point in regard to a conic, lie on the polar of the fixed point.

5. Consider a curve of even order H_{2n} . There is an invariant of the order $\frac{1}{2}(n+1)(n+2)$ in the coefficients, which vanishes when the n th differentials are in involution; this invariant is a symmetrical determinant. Its reciprocal is a contravariant of the order $2n$ in the variables, and $\frac{1}{2}(n^2+3n)$ in the coefficients, which may be called h_{2n} . The curves H_{2n} , h_{2n} are so related that if X_n be the polar of y_n in respect of H_{2n} , then the y_n is the polar of X_n in respect of h_{2n} . That is to say, if $X_n = y_n H_{2n}$, then $X_n h_{2n} = I y_n$ where I is the invariant just defined. For example, if we consider a series of conics c_2 and their polars C_2 in regard to a given quartic curve Q_4 , then there exists a curve q_4 of the fourth class, such that in respect to it the conics c_2 are the polars of C_2 .

6. Two curves whose order and class are different may be made susceptible of the harmonic relation by taking each a proper number of times. Thus, curves H_n , c_n have an invariant $(H_n)^3 (c_n)^2$ of the order 4 in the coefficients of H_n and of the order 3 in the coefficients of c_n^* . It is to be observed that the equation $H_{2nn}(c_n)^2 = 0$, is the most general relation of the n th order that can subsist among the coefficients of c_n .

7. The following remarks relate to theorems in Dr HURWITZ's paper "On certain formulae concerning the Theory of Discriminants":

* Thus we may have an invariant which vanishes when a curve is harmonic to itself. Let P_n be the curve, n its reciprocal; then $P_n^2 c_n^2$ is the invariant in question, or a power of it. For a circle, it is the discriminant; for a cubic, the invariant T of the sixth order. See Salmon's *Higher Algebra*, 1st ed., note to p. 67.

If the curve P_n has no node, the class = order = 3, and the invariant is $(P_n)^2 (c_n)^2$. If however P_n has a node d , d^2 is part of the reciprocal, and the invariant is $d^2 (c_n)^2$. Cusps may be similarly dealt with.

† [Proceedings of the London Mathematical Society, Vol. II, Nov. 15, 16.]

If the polar of C_n in respect of R_n has a node, C_n is harmonic of a curve of order $\frac{1}{2}n(n+1)$, which is De Henrici's curve $S^{(2)}$. In fact when C_n is a point taken on line g , the point is on the curve $S^{(2)}$; this is De Henrici's theorem.

In general, $x'y^nR_{m+n}$ denotes a straight line. If it vanishes identically, x is a node on the n th polar of g , and y is a node on the m th polar of x . In the case $x''y''R_{m+n}$ and $x'''y''R_{m+n}$ are conics having nodes at y and x respectively*. In the relation

$$x'y^nR_{m+n} = 0$$

write $x = \delta x$ for x and $y = \xi y$ for y . This is equivalent to supposing δx and ξy to be points on the tangents at x and y to the curves $S^{(2)}$ $S^{(2)}$ which are the loci of those points respectively. Then we have

$$(my\delta x + nx\xi y)x''y^nR_{m+n} = 0,$$

operate on this with y ; we know that

$$nyy'x''y^nR_{m+n} = 0,$$

it follows that

$$m\xi x'x''y^{n+1}R_{m+n} = 0,$$

that is to say, the tangent at x to $S^{(2)}$ is the line polar of y in respect of the $(m+1)$ th polar of x . This is another of De Henrici's theorems. I have added this part as an example of the readiness with which the operative notation lends itself to such investigations.

* Viz., these are the poles of tangents at the base points. It is observable that the tangents at x to the n th polar of g , the tangents to the curve C_n , and the line g , form a harmonic pencil.

Here, for example, in the first table opposite the word Cubic and under the power 5 we find the number 15; the theorem corresponding to this is—

If 15 points are such that every quintic through 14 of them passes through the remaining one, all these points must lie on a cubic curve.

Now if we take 15 points arbitrarily on a cubic curve, it is not in general true that the fifth powers of their distances from an arbitrary line satisfy a linear homogeneous relation. That this may be the case, the 15 points must be intersections of the cubic with a quintic; and these are not arbitrary points, but 14 of them being given, the 15th is determined, by a theorem of Jacobi and Pflücker. The theorem immediately derived from the table, then, must be completed by this statement; the points are not only all on a cubic, but they are intersections of a cubic and a quintic.

It is to be understood also that if we take a number N of points lying between any two adjacent numbers in the same vertical column of the table, then the same theorem is true about N that is true about the greater of these numbers. Thus we are informed by the first table that a syzygy among the 4th powers of the distances of 12 points makes them lie on a cubic, and that a similar syzygy for 15 points makes them lie on a quintic; this latter theorem is true for the intermediate numbers 13 and 14. It is not however *all* that is true in either of these cases; the 14 points are points of intersection of two quartics, and the 13 points are (I believe) points on a cubic such that no twelve of them are intersections of the cubic with a quartic. I wish particularly to draw attention to these intermediate cases, where it appears that more is true than can be proved by the method to be presently explained.

Method of Demonstration. Let the tangential equation of a point be

$$0 = \alpha\xi + \beta\eta + \gamma\xi (\pm p, \text{ say})$$

and let the equation of a curve of the n^{th} order be

$$0 = (\star \phi(x, y, z))^n (\pm B_n, \text{ say})$$

then I say that

$$(*\chi \frac{\delta}{\delta \xi}, \frac{\delta}{\delta \eta}, \frac{\delta}{\delta \zeta})^n \cdot (\alpha \xi + \beta \eta + \gamma \zeta)^n = (*\chi \alpha, \beta, \gamma)^n | n ;$$

that is to say, if we operate with B_n on the n^{th} power of p , we shall obtain the result of substituting the coordinates of p for x, y, z in B . If, then, this result vanishes, the point p is on the curve B_n .

I will now prove that if the 12th powers of the *nil-facta* in the tangential equations of 43 points are connected by a linear syzygy, the 43 points are on a quartic curve. We can draw a quartic B_4 through 14 of the points; operate with B_4 on the syzygy, then these 14 points are cleared away, and there remains a syzygy between the 8th powers of the remaining 29 points. We have therefore now to prove that these 29 points are on a quartic. Draw a curve C_4 through 14 of them, and operate on the new syzygy with C_4 . This clears away 14 more points, and we are left with a syzygy among the 4th powers of 15 points. But then by Serret's theorem these lie on a quartic. Hence, *any* 15 of the original 43 points are on the same quartic; therefore all the 43 are on the same quartic.

To prove that if the cubes of 13 points in space are connected by a syzygy they lie on a quadric surface, operate with the plane through three of them; we are then left with a syzygy among the squares of 10 points, and Serret's theorem again applies.

The application of this method to the remaining cases will now be easy.

* XV.

ON SYZYGISTIC RELATIONS CONNECTING THE
POWERS OF LINEAR QUANTICS.

I THINK the first treatment of this subject is to be found in some very interesting articles of M. Paul Serret's; *Nouvelles Annales*, t. IV. (1863), pp. 145, 193, and 433. M. Serret's attention was confined to the *squares* of linear quantics; and in regard to these he established such propositions as the following:—If the squares of the characteristics of the equations of six lines satisfy a syzygetic relation, the six lines touch a conic section. That is to say, if $P_1^2=0$, $P_2^2=0$, ..., $P_6^2=0$ are the equations of the lines, and if we have an identical relation

$$\lambda_1 P_1^2 + \lambda_2 P_2^2 + \lambda_3 P_3^2 + \lambda_4 P_4^2 + \lambda_5 P_5^2 + \lambda_6 P_6^2 = 0;$$

or, as he finds it convenient to write

$$\sum_1^6 \lambda P^2 = 0$$

where the λ 's are numerical coefficients, then the lines P_1 , P_2 , ..., P_6 touch the same conic. Another of his propositions is that if eight planes P_1 , P_2 , ..., P_8 satisfy an identical relation

$$\sum_1^8 \lambda P^2 = 0,$$

then the eight planes are such that any quadric surface touching seven of them touches also the eighth. These propositions are arrived at by a somewhat circuitous path, though the steps severally are elegant. From the latter M. Serret obtains a very beautiful and immediate proof of Hesse's theorem that two tetrahedra self-conjugate to the same quadric are such that every quadric touching seven of their faces touches also the eighth. Namely, the equation of the first quadric may be written in either of the forms

$$\lambda_1 P_1^2 + \lambda_2 P_2^2 + \lambda_3 P_3^2 + \lambda_4 P_4^2 = 0, \quad \lambda_5 P_5^2 + \lambda_6 P_6^2 + \lambda_7 P_7^2 + \lambda_8 P_8^2 = 0,$$

and these two being identical to a factor *prods*, we have a syzy-

getic relation among the eight squares, from which by the second of the above propositions the theorem in question at once follows.

Having, by an application of Prof. Sylvester's most powerful method of contravariant differentiation, succeeded in extending these propositions to higher powers of linear quantities, and to curves and surfaces of any order, I found as a particular result that two quadrilaterals of the same system totally inscribed in a cubic are such that every curve of the third class containing seven of their sides touches also the eighth. Doubtful of this proposition, I communicated it to the Mathematical Society, and was subsequently informed by Mr. Cotterill that the eight lines in question touch the same cone. This is equivalent to the analytic theorem, "if the cubes of eight linear quantities are syzygetic, the squares of any six of them are syzygetic." The proof of this by contravariant differentiation and the statement of a series of analogous propositions occupy the following notes.

I.

If in the tangential equation of a curve

$$c_p = (\xi, \eta, \zeta)^p = 0$$

we write

$$\frac{d}{dx} = \frac{d}{dy} = \frac{d}{dz} \text{ for } \xi, \eta, \zeta$$

and operate upon

$$(lx + my + nz)^p,$$

we shall get p multiplied by the result of substituting ξ, η, ζ in c_p ; that is to say

$$\left(\frac{d}{dx} = \frac{d}{dy} = \frac{d}{dz} \right)^p \cdot (lx + my + nz)^p = p \cdot (\xi, \eta, \zeta)^p.$$

For shortness, denote $(lx + my + nz)$ by Q , and let c_p mean also the differential operator

$$\left(\frac{d}{dx} = \frac{d}{dy} = \frac{d}{dz} \right)^p.$$

Then if the operator c_p reduces Q^p to zero, the line Q touches the curve c_p , and conversely.

Suppose now that there are several lines Q_{00}, Q_{01}, \dots and that there is an identical relation,

$$\sum \lambda Q^p = 0,$$

connecting the p^{th} powers of the quantities Q . Let also c_p be a curve touching all but one of the lines, so that the operator c_p reduces to zero all but one of the quantities Q^p . The expression $\sum \lambda Q^p$ being identically zero, the result of operating upon it with c_p must be zero, or we have

$$\sum \lambda c_p Q^p = 0.$$

But of the terms $\lambda c_{00} Q_{00}^p, \lambda c_{01} Q_{01}^p, \dots$ we know that all vanish but one; it follows that this last one also vanishes, or the curve c_p touches the remaining line. We may therefore enunciate the proposition: *If there are n lines $Q_1, Q_2, \dots, Q_n = 0$, and if there is an identical relation*

$$\sum_1^n \lambda Q^p = 0,$$

then every curve c_p of class p which touches $n-1$ of the lines will also touch the n^{th} .

It will be sufficient merely to enunciate the obviously corresponding proposition in three dimensions:—

If there are n planes $Q_1, Q_2, \dots, Q_n = 0$, and if there is an identical relation

$$\sum_1^n \lambda Q^p = 0,$$

then every surface c_p of class p which touches $n-1$ of the planes will also touch the n^{th} .

In solid geometry, however, as usual, the analogy branches off into two distinct directions, and we are led to consider a somewhat different theory.

Let the number of straight lines which can be drawn through a fixed point and in a fixed plane to touch a given surface be called the *rank* of the surface (viz. this is both the class of a general plane section and the order of a general tangent cone), then that relation between the six coordinates of a line which expresses that the line touches the surface will be of a degree equal to the rank of the surface. I shall denote the expression equated to zero in this equation by a Greek letter whose suffix

indicates the rank; thus, for example, $\beta_2 = 0$ is the rank-equation* of a quadric surface.

The six coordinates being a, b, c, f, g, h , where $af + bg + ch = 0$, it is very easy to prove that if in β_n an expression of the n^{th} degree in these coordinates, we substitute for a, b, c, f, g, h respectively

$$\frac{d}{df}, \frac{d}{dg}, \frac{d}{dh}, \frac{d}{da}, \frac{d}{db}, \frac{d}{dc},$$

we shall get an invariant symbol of operation. I shall use β_n to mean not only the function of the coordinates, but also this operator obtained by the substitution just defined. This being so, if we call the condition that the line $(abcfgh)$ shall meet a given line σ or $(lmnpqr)$, the equation of the line σ (namely the equation is

$$\sigma \equiv pa + qb + rc + lf + mg + nh = 0),$$

then the condition that σ shall touch the surface β_n is

$$\beta_n \sigma^n = 0.$$

From this it follows that

If there are n straight lines $\sigma_{(1)}, \sigma_{(2)}, \dots, \sigma_{(n)}$, and if there is an identical relation

$$\sum_1^n l \sigma^n = 0,$$

then every surface β_p of rank p which touches $n - 1$ of the lines will also touch the n^{th} .

II.

At this point I digress somewhat to consider the interpretation of what I have elsewhere called the harmonic invariant of two curves or surfaces, the order of one being equal to the class of the other. First, in the case of two conics, the point-equation of the first being $B_2 = 0$, and the line-equation of the second $c_2 = 0$, the harmonic invariant is $c_2 B_2$, which is commonly called the invariant Θ . Suppose that this vanishes; then if B_2 can be written in the form

$$X^2 + Y^2 + Z^2$$

* [The expression "line-equation" would have been the more natural one, but a confusion might arise between this line-equation of a surface, and the line-equation of a plane curve.—G.]

(so that the lines XYZ form a self-conjugate triangle), since the operator e_3 reduces this to zero we see that if the conic e_3 touch two of the lines it must touch also the third. Similarly, if B_2 can be written in the form

$$X^2 + Y^2 + Z^2 + W^2$$

(so that $XYZW$ form a self-conjugate quadrilateral), if e_4 touches three of the lines it must touch also the fourth. Hence $e_4 B_4 = 0$ is the condition both (1) that e_4 shall be inscribed in an infinite number of *triangles* self-conjugate to B_4 , and (2) that e_4 shall be inscribed in an infinite number of *quadrilaterals* self-conjugate to B_4 . These are known interpretations; the latter, given first I think by Dr Salmon under a slightly different form (*Conics*, §375), was reduced to this more simple and natural shape by Professor Cremona (*Educational Times*, Reprint)*. Next, in the case of two quadric surfaces $e_6 = 0$ and $B_6 = 0$, if $e_6 B_6 = 0$, and B_6 can be expressed in either of the forms

$$\Sigma_1^4, X^2, \Sigma_1^6, X^2, \Sigma_1^8, X^2,$$

we see at once that if e_6 touch all but one of the planes X it must touch also that other. Hence $e_6 B_6$ is the condition that e_6 shall be inscribed (1) in an infinity of *tetrahedra* self-conjugate to B_6 , (2) in an infinity of *pentahedra* self-conjugate to B_6 , (3) in an infinity of *hexahedra* self-conjugate to B_6 . The terms *conjugate hexahedron*, *conjugate pentahedron*, are introduced by M. Serret, and seem likely to be of considerable use.

Passing now to the rank-equations of the two quadrics, which I shall write $\beta_6 = 0, \gamma_6 = 0$, I observe that if β_6 can be thrown into the form $\Sigma_1^6 a_1^2$, then the 6 lines a are such that each is conjugate to all the rest; or the lines are the edges of a self-conjugate tetrahedron. If then the harmonic invariant $\gamma_6 \beta_6$ (Dr Salmon's invariant T) vanishes, and γ_6 touches five of these lines, it will touch the sixth; or γ_6 can touch the edges of an infinite number of tetrahedra self-conjugate to β_6 . We have not yet studied the properties relative to β_6 of a system of lines such that β_6 may be expressed in the form $\Sigma_1^p a^2$ when p is greater than 6; yet it is

* [cf. Vol. ix, p. 199; Vol. ix, pp. 69, 71.]

obvious that such systems will give new interpretations of the invariant T .

The general extension of this method of interpretation is now perfectly easy. If B_n is a curve of n^{th} order harmonic of c_n a curve of n^{th} class, and if B_n can be written in the form $\Sigma_1^n X^n$, then if c_n touch $p+1$ of the lines X it will touch also the p^n . Similarly for surfaces in regard to lines and planes. The converse proposition is

The curve or surface $\Sigma_1^n X^n = 0$ is harmonic of every curve or surface c_n of the n^{th} class which touches all the lines or planes X .

I forbear to state the correlative propositions in which lines and planes are replaced by points.

III.

Let us return to the original question.

If there is an identical relation

$$\Sigma_1^3 \Lambda P^3 = 0$$

between the cubes of the 8 linear quantities P , there shall also be an identical relation between the squares of any 6 of them.

For we know that we can find a linear differential operator which shall reduce any two of the quantities to zero; namely, let h be the point of intersection of the lines P_{12} , P_{13} , having for coordinates a, b, c , then the operator

$$a \frac{d}{dx} + b \frac{d}{dy} + c \frac{d}{dz}$$

being also denoted by h , we have $hP_{12} = 0$, $hP_{13} = 0$; and so if we operate with h on the given syzygy we obtain therefore

$$3\Sigma_1^2 \Lambda, hP, P^2 = 0,$$

a syzygy connecting the squares of the quantities $P_{12}, P_{13}, P_{14},$

If there is an identical relation

$$\Sigma_1^4 \Lambda P^4 = 0$$

between the fourth powers of the 11 linear quantities P , there shall also be an identical relation between the squares of any 6 of them.

Let c_p be the conic section touching the five lines $P_0 \dots P_{10}$; then $c_p P_{11}^5 = 0, \dots, c_p P_{10}^5 = 0$. If then we operate with c_p on the given syzygy we shall obtain

$$12\Sigma_1^6 \lambda_i c^2 P^{10} \cdot P^{12} = 0,$$

a syzygy connecting the squares of the quantities $P_0 \dots P_{10}$; provided all the $c_p P^{12}$ do not vanish. If however all these vanish, all the lines touch the conic c_p , and there is again a syzygy connecting the squares of any six of them. Using these two demonstrations as examples, we are enabled to construct the following table.

| Powers P^r | 0-2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10-11 |
|-----------------|-----|----|----|----|----|----|----|----|-------|
| 3 | 0 | | | | | | | | |
| 3 | 0 | 10 | | | | | | | |
| 4 | 11 | 12 | 15 | | | | | | |
| 5 | 13 | 15 | 17 | 21 | | | | | |
| 6 | 15 | 19 | 20 | 24 | 25 | | | | |
| 7 | 15 | 21 | 24 | 25 | 30 | 30 | | | |
| 8 | 21 | 24 | 29 | 30 | 36 | 36 | 45 | | |
| 9 | 24 | 29 | 34 | 35 | 42 | 42 | 47 | 55 | |
| 10 | 25 | 30 | 34 | 41 | 42 | 45 | 50 | 57 | 63 |
| 11 | 25 | 33 | 36 | 42 | 46 | 50 | 54 | 60 | 70 |

* [This is in effect the Table A of paper XIV, where it is explained that the number in the Table is

$$N = \frac{n}{2} n(n+3) + \frac{1}{2} q(q+1) q(t+2),$$

where n is the quotient and q the remainder of the division of p by n , so that $p = nq + q$, and $q < n$.

It may be remarked that no bar as p is not greater than n , that is down to the bar in each column of the Table, the number

$$= \frac{1}{2} n(n+3) + \frac{1}{2} q(q+1) q(t+2),$$

and that the several columns are then continued as follows:

| End, $n = 2$ | $n = 3$ | Δn |
|--------------|-------------|------------|
| 0 | | |
| 3 | 10 | |
| 11 = 6 + 5 | 12 | |
| 13 = 9 + 5 | 15 | |
| 15 = 11 + 5 | 19 = 10 + 9 | |
| 16 = 13 + 5 | 24 = 12 + 9 | |
| 21 = 16 + 5 | 24 = 15 + 9 | |
| Δn | Δn | |

It is easy to see that this is in fact an equivalent construction of the table. C.]

[ON THE THEORY OF DISTANCES.]

[PRELIMINARY*.]

I EXPLAIN in the first place the notation employed, which is an extension of the Geometric Analysis of CHASSMANN, explained by him in the "Ausdehnungslehre" and in Crelle's Journal, and founded in part on a remark of Leibnitz.

CHASSMANN employs single large letters, A, B, C , to represent straight lines in a plane, and single small letters, a, b, c , to represent points. When two large letters come together, as AB , the notation is taken to mean the point of intersection of the lines A, B . So when two small letters come together, as ab , the notation is taken to mean the line joining the points a, b . The equation $AB^2C = 0$ means that the lines A, B, C meet in a point; the equation $abc = 0$, that the points a, b, c lie in a line; and the equation aB or $Ba = 0$, that the point a lies on the line B . No signification is given to the separated symbols AB^2 , abc , aB , except as equated to zero. The main principle of the application of this method is that the order of an equation in any letter contained is measured by the number of times that letter occurs; a remark which will be further explained in the sequel.

I now explain the extensions of this notation which I have found it convenient to make.

* (The *Preliminary* matter forms the substance of notes given to Prof. Clifford by the Author at the British Association Meeting in the year 1861. The paper itself (pp. 151 - 152), without a title, appears to have been written independently. The title I have given to the two connected sections has been taken from that of a paper Prof. Clifford read at the above meeting, an abstract of which is given below (p. 164). I have employed ϵ to represent $\omega^{\frac{1}{2}}$ [1].

A curve may be given by its points, or by its tangents; that is to say, we may know its equation in point-coordinates (x, y, z) , the degree of the equation being the *order* of the curve; or we may know its equation in the contravariant, tangential, or line-coordinates (ξ, η, ζ) , the degree of the equation being then the *class* of the curve. This being so, I denote by a large letter a quantic in (x, y, z) , and I write the order of the quantic in the form of a suffix. Thus C_a denotes a cubic in (x, y, z) , and $C_a = 0$ is the equation to a curve of the third order. Next, I use a small letter with a suffix to denote a quantic in (ξ, η, ζ) , the order of the quantic being denoted by the suffix. Thus b_a denotes a cubic in (ξ, η, ζ) , and $b_a = 0$ is the equation to a curve of the third class. When there is no suffix, the suffix 1 is to be understood; in this respect the notation coincides with that of GRASSMANN.

When two large letters come together, each is raised to the power denoted by the suffix of the other, as $A_m^n B_a^m$. The symbol then denotes a quantic which, equated to zero, gives the equation in tangential coordinates of the mn intersections of the curves A_m, B_a . Similarly $a_m^n b_a^m = 0$ is the equation of the mn common tangents of the curves a_m, b_a . The reason of the indices is now apparent; each equation being of the degree n in the coefficients of the first curve, and of the degree m in those of the second.

When three large letters or three small letters come together, each is raised to a power denoted by the product of the suffixes of the other two; the symbol then denotes the resultant of the three quantities.

When a small letter comes before a large one, as $b_a C_a$, the notation is taken to mean the result of changing (ξ, η, ζ) in b_a into $(0, 1, 1)$ and performing the operation thus indicated on C_a . So, finally, when a large letter comes before a small one, as $B_a c_a$, the notation is taken to mean the result of changing (x, y, z) in B_a into $(0, 1, 1)$ and performing the operation thus indicated on c_a .

In the use of these symbols to investigate the relations of geometrical magnitudes, it is to be observed that the absolute

values of (xyz) or $(\xi\eta\zeta)$ or the coefficients of a quantic are not given, but only their ratios; and consequently that the symbols defined above can have no special value but zero. If however we form a fraction such that every letter mentioned occurs an equal number of times in the numerator and denominator, this *will* have a definite numerical value, being a function of the known ratios aforesaid; and may accordingly represent a geometrical magnitude. This theory of *characteristics* is due to Prof. SYLVESTER.

It is further to be observed that the metric properties of figures in plane geometry depend upon the circular points at infinity, which I denote by i, j ; and those of figures in spherical geometry upon the imaginary circle at infinity, which I denote by O_s or o_s according as it is given by points or tangents. The points i, j in the one case, and the circle O_s in the other, have received the name of "the Absolute" from Prof. CAYLEY, to whom this theory is due*.

FORMULÆ FOR A PLANE CONIC.

Expressions are obtained below for the *distance* of a point from a conic given tangentially, and for the *distance* of a line from a conic given by its points. Two different geometrical definitions are obtained for each of these; their ratio is a quantity which I have called the *distance of the curve from the absolute*.

The asymptotes of the conic are denoted by P, Q ; a pair of foci, viz. either the two real or the two imaginary foci, are denoted by p, q ; the conic is called C_2 or c_2 .

DISTANCE OF THE POINT a FROM c_s .

Let any straight line B be drawn through the point a , meeting the conic in b, m ; let also the tangents from a to the conic be L, M .

* [\overline{ab} =distance between points a and b , ab =line joining a and b .]

First Dist. $a, c_2 = \sin^2 LM, ap^2, aq^2$

$$= \frac{a^2 C_2, (aj)^2}{a^2 c_2, aj^2 c_2} \cdot \frac{(aj)^2, (ij^2 c_2)^2}{(aj)^2, (ij^2 c_2)^2} \cdot \frac{a^2 C_2}{(ij^2 c_2)^2}.$$

Second Dist. $a, c_2 = al, am, \sin BP, \sin BQ$

$$= \frac{a^2 C_2, (Bj)^2}{(aj)^2, (Bj)^2 C_2} \cdot \frac{(Bj)^2 C_2}{(Bj)^2, \sqrt{(i^2 C_2, j^2 C_2)}} \cdot \frac{a^2 C_2}{(aj)^2, \sqrt{(i^2 C_2, j^2 C_2)}}.$$

The ratio of these two is

$$\frac{\sin^2 LM, ap^2, aq^2}{al, am, \sin BP, \sin BQ} = \frac{\sqrt{(i^2 C_2, j^2 C_2)}}{(ij^2 c_2)^2} = p q^2.$$

DISTANCE OF THE LINE A FROM C_2 .

Let any point b be taken on the line A , the tangents from b to the conic being LM ; also let A meet the conic in the points l, m .

First Dist. $A, C_2 = lm^2, \sin^2 AP, \sin^2 AQ$

$$= \frac{A^2 c_2, Al, Aj}{(Aj)^2 C_2} \cdot \frac{(Aj)^2 C_2}{(Al, Aj)^2, i^2 C_2, j^2 C_2} \cdot \frac{A^2 c_2}{Al, Aj, i^2 C_2, j^2 C_2}.$$

Second Dist. $A, C_2 = \sin AL, \sin AM, bp, bq$

$$= \frac{A^2 c_2, (lj)^2}{Al, Aj, \sqrt{(b^2 c_2, bj^2 c_2)}} \cdot \frac{\sqrt{(b^2 c_2, bj^2 c_2)}}{(lj)^2, ij^2 c_2} \cdot \frac{A^2 c_2}{Al, Aj, ij^2 c_2}.$$

The ratio of these two is

$$\frac{lm^2, \sin^2 AP, \sin^2 AQ}{\sin AL, \sin AM, bp, bq} = \frac{ij^2 c_2}{i^2 C_2, j^2 C_2} = \sin^2 PQ.$$

In the case of a sphero-conic we obtain analogous expressions for the distance of a point or line (great circle) from the conic, but the value depends on the pair of focal or cyclic planes selected; the ratio of such different values is however the same for all points and lines. Moreover, the ratio of the two distances of a point or of a line is a quantity independent of the point or line, but I have as yet obtained no geometrical definition of it. For this reason I have not treated separately the formulae for a sphero-conic, which are of course like the preceding included in the general formulae of curves*.

I.

All magnitudes which are concerned in plane geometry may be expressed in terms of three, which on this account are of primary importance. These are the distance of two points, the distance of a point from a line, and the sine of the angle between two lines. We obtain the most simple expressions for these magnitudes by employing rectangular Cartesian co-ordinates for the points, and the coordinates of Plücker for the lines; but it is convenient from the first to make these expressions homogeneous by the introduction of a third co-ordinate which may be put ± 1 or $\pm i$ in the two cases respectively. Thus if a_1, a_2, a_3 are the coordinates of the point a , $\frac{a_2}{a_1}$ and $\frac{a_3}{a_1}$ are its distances from the axes, and if A_1, A_2, A_3 are the coordinates of a line A , $\frac{A_2}{A_1}$ and $-\frac{A_3}{A_1}$ are the intercepts it cuts off from the axes. This being so, the expressions for our primary magnitudes are

$$\text{Dist. } ab = \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2}}{a_1b_1},$$

$$\text{Dist. } aB = \frac{a_1B_1 + a_2B_2 + a_3B_3}{a_1\sqrt{(B_1^2 + B_2^2)}},$$

$$\sin AB = \frac{A_1B_2 - A_2B_1}{\sqrt{(A_1^2 + A_2^2)}\sqrt{(B_1^2 + B_2^2)}}.$$

* [cf. however V., p. 132 *infra*, of this paper.]

and these are clearly not invariant in regard to the points and lines. Let us now ask what is the locus of the points b which are at zero distance from a . We find that it consists of the two straight lines

$$\begin{aligned} a_1 b_1 + c a_2 b_2 + (a_1 + c a_2) b_3 &= 0, \\ a_2 b_1 - c a_1 b_2 + (a_1 - c a_2) b_3 &= 0. \end{aligned}$$

Each of these lines passes through the point a ; thus we learn that *the lines of null length are straight lines, and two of them pass through every point of the plane.*

If the point a moves about, each of the lines of null length remains parallel to the same direction, or, which is the same thing, passes through a fixed point at an infinite distance, let these two points be called i, j ; their coordinates may be taken to be

$$i_1 : i_2 : i_3 = \frac{1}{\sqrt{a_2^2}} : \frac{c}{\sqrt{a_2^2}} : 0,$$

$$j_1 : j_2 : j_3 = \frac{c}{\sqrt{a_2^2}} : \frac{1}{\sqrt{a_2^2}} : 0,$$

so that the line ij has coordinates $(0, 0, 1)$. Then we have

$$ab = \frac{1}{\sqrt{a_2^2}} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & c & 0 \end{vmatrix} = \frac{1}{\sqrt{a_2^2}} \{c(a_2 b_2 - a_3 b_3) - (a_1 b_2 - a_3 b_1)\},$$

$$ab = \frac{1}{\sqrt{a_2^2}} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c & 1 & 0 \end{vmatrix} = \frac{1}{\sqrt{a_2^2}} \{c(a_2 b_2 - a_3 b_3) + (a_1 b_2 - a_3 b_1)\},$$

$$ab = a_2, \quad bc = b_2,$$

$$ab, bc = \frac{1}{a_2^2} \{ (a_2 b_2 - a_3 b_3)^2 + (a_1 b_2 - a_3 b_1)^2 \}$$

and therefore

$$\kappa \cdot \text{Dist. } ab = \frac{\kappa^2 (ab, bc)}{ab \cdot bc}, \quad \text{where } \kappa^2 = \frac{1}{a_2^2}.$$

Thus we learn that *all the lines of null length pass through one or other of the points i, j ; and the distance ab may be expressed in terms of the invariants of a, b, i, j to a numerical*

factor $\mu\kappa\delta s$. This factor κ is the same for all distances, depending only upon the absolute value given to the coordinates of i, j . To reduce κ to the value unity, we have only to multiply these coordinates throughout by $\kappa^{-1/2}$.

Similar expressions may now be found for the other two primary magnitudes. We have in fact

$$Ai = \frac{1}{\sqrt{2}}(A_1 + iA_2), \quad Aj = \frac{1}{\sqrt{2}}(A_1 + iA_2),$$

$$Ai, Aj = \frac{1}{2}(A_1^2 + A_2^2),$$

and thence

$$\left[\frac{1}{\kappa}\right] \text{Dist. } AB = \frac{AB}{\cos \theta \sqrt{(Bi, Bj)}} \\ = 2 \epsilon \sin AB = \frac{ABij}{\sqrt{(Ai, Aj)(Bi, Bj)}}.$$

(The last expression may be simplified as follows.)

We have

$$ABij = Ai, Bj + Aj, Bi$$

by a well-known theorem of determinants; but also

$$Ai, Bj + Aj, Bi$$

$$= \frac{1}{2} \{(A_1 + iA_2)(iB_1 + B_2) + (iA_1 + A_2)(B_1 + iB_2)\} \\ = i(A_1B_1 + A_2B_2)$$

and therefore

$$2 \cos AB = 2 \frac{A_1B_1 + A_2B_2}{\sqrt{(A_1^2 + A_2^2)} \sqrt{(B_1^2 + B_2^2)}} = \frac{Ai, Bj + Aj, Bi}{\sqrt{(Ai, Aj)(Bi, Bj)}}.$$

If therefore we write θ for the angle AB , we have

$$\cos \theta + i \sin \theta = e^{i\theta} = \frac{Aj, Bi}{\sqrt{(Ai, Aj)(Bi, Bj)}} \\ \cos \theta - i \sin \theta = e^{-i\theta} = \frac{Ai, Bj}{\sqrt{(Ai, Aj)(Bi, Bj)}},$$

and thence

$$e^{i\theta} = \frac{Aj, Bi}{Ai, Bj}.$$

This last is an anharmonic ratio in which the lines AB divide the segment ij , and so is an absolute invariant.)

By analogy and for convenience of expression we shall call the sine of the angle between two lines the *distance* of the lines; we may then derive from these formulae the following theorems:

If a line pass through either of the points i, j , it is at an infinite distance from all other lines and points, and the distance between any two points on it is zero.

If a point lie upon the line ij , it is at an infinite distance from all other lines and points, and the distance between any two lines through it is zero.

II

A conic may be given as a curve of the second order C_2 or as a curve of the second class c_2 , but either of these expressions may be derived from the other by means of the formula

$$\begin{aligned} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} - 2abcd, \\ c_2^2 &= C_2^2, \end{aligned}$$

and its reciprocal

$$\begin{vmatrix} AX_1 & BX_1 \\ AX_2 & BX_2 \end{vmatrix} = 2ABXYC_2.$$

Namely, if we make x, b identical with x, y , and A, B with X, Y , we have

$$2AX_1Y_2 - 2Y_1X_2 = 2XYC_2^2,$$

and

$$2XYC_2^2 = X^2C_2^2 - Y^2C_2^2 = 2AX_1Y_2.$$

The discriminant C_2^2 may be given in a similar manner. Namely, we have

$$2a^2b^2 - 4b^2c^2 - 4c^2d^2 = 4abcdC_2^2.$$

The relations of a conic to the points i, j are most conveniently expressed in terms of the asymptotes and the foci. The asymptotes P, Q are tangents at the points where the conic

is met by the line ij . The equation to the pair of tangents at the points where a line X meets the conic C_2 must be of the form

$$C_2 - \lambda X^2 = 0.$$

Equating to zero the discriminant of this, we find that

$$C_2 c_2 - 3\lambda X^2 c_2 = 0,$$

whence the equation to the tangents is

$$3C_2 \cdot X^2 c_2 - X^2 \cdot C_2 c_2 = 0.$$

Substituting herein for X the line ij , we obtain for the asymptotes the expression

$$P \cdot Q = 3C_2 \cdot \bar{ij}^2 c_2 - \bar{ij}^2 \cdot C_2 c_2.$$

Thus the product $P \cdot Q$ is of the third order in the coefficients of the conic, whether it is given by its point- or by its line-equation.

To determine the angle between the asymptotes, we observe that the point PQ is the centre of the conic, which is the pole of the line ij , and is therefore represented by $\bar{ij} c_2$. Consequently $PQij$ must be proportional to some power of $\bar{ij}^2 c_2$. If we regard C_2 as initially given, we shall in fact have

$$(PQij)^2 = \kappa (\bar{ij}^2 c_2)^3,$$

each side being of the sixth order in the coefficients. If however c_2 is initially given, the formula becomes

$$(PQij)^2 = \kappa' (\bar{ij}^2 c_2)^3 \cdot C_2 c_2.$$

Next we find by direct substitution

$$Pi \cdot Qi = 3i^2 C_2 \cdot \bar{ij}^2 c_2,$$

$$Pj \cdot Qj = 3j^2 C_2 \cdot \bar{ij}^2 c_2,$$

and therefore

$$\sin^2 PQ = \frac{\bar{ij}^2 c_2}{i^2 C_2 \cdot j^2 C_2} \text{ to a factor } p \text{ res.}$$

But we have also

$$\bar{ij}^2 c_2 = i^2 C_2 \cdot j^2 C_2 - (ij C_2)^2;$$

now it is clear that ηC_2^2 vanishes when the points \bar{p} are conjugate in regard to the conic, or when the asymptotes are at right angles, that is, when $\sin 2\theta = 1$. It follows therefore that the factor in the last equation is unity, and that

$$\cos 2\theta = \frac{\bar{p}C_2^2}{\bar{p}C_2^2 - \bar{p}C_2^2}.$$

We find in this case that the constant formula is (36), so that we may write

$$C^2 \bar{p} \bar{q}^2 = -36 \alpha_2^2 \bar{p}^2 \bar{q}^2.$$

This constant might also have been determined by means of the particular case in which the conic breaks up into two straight lines, in which case these lines are themselves the asymptotes. If $C_1^2 = X, Y_1 = Y_2 = XY^2$, and $\bar{p}C_2^2 = XY\bar{q}^2$, by these formulas the value just obtained may be compared with the known values of $\cos 2\theta$, $\cos 2\theta$.

If the conic \bar{c}_2 touches the four lines joining p, q to i, j , we must have

$$\bar{c}_2(p, q) = 0, \quad \bar{c}_2(i, j).$$

But the equation

$$0 = \bar{c}_2(p, q) = X \cdot \bar{q}^2 \bar{p}^2 = C_1^2 \bar{p}^2 + \frac{36}{5} \bar{p}^2 \bar{q}^2 + \frac{36}{5} X \bar{p}^2 \bar{q}^2$$

gives two values for X , one belonging to p, q and the other to p', q' . Hence substituting X , we may write

$$p, q, p', q' = C_1^2 \bar{p}^2 + \bar{q}^2 \bar{p}^2 + \frac{36}{5} \bar{p}^2 \bar{q}^2 + \frac{36}{5} X \bar{p}^2 \bar{q}^2,$$

a result of the third order in the coefficients of \bar{c}_2 .

If the conic touch the line \bar{ij} , that is, if $\bar{q}C_2^2 = 0$, one of the four, say q , coincides with the point of contact, and two others, p' and q' , coincide with the points i and j respectively; we have therefore in this case $\bar{p}\bar{q} = 0, \bar{p}'\bar{q}' = 0, \bar{p}'\bar{q}\bar{j} = 0$. If the conic pass through the point i , which happens when $\bar{p}'C_2^2 = 0$, the last coincide first and last, p with q' , $suppose$, and q with p' ; we have then $\bar{p}\bar{q} = 0, \bar{p}'\bar{q}' = 0$. Hence the product

$$\bar{p}\bar{p}' \cdot \bar{q}\bar{q}' \cdot \bar{p}'\bar{q}\bar{i} \cdot \bar{p}\bar{q}\bar{j}$$

will vanish in three cases, (1) when $\bar{p}'C_2^2 = 0$, (2) when $\bar{p}C_2^2 = 0$,

(3) when $\bar{ij}^2 c_2 = 0$; and it is easy to see that it cannot vanish in any other case. Consequently we must have

$$pq\bar{i}.pq\bar{j}.p'q'\bar{i}.p'q'\bar{j} = \kappa (i^2 C_2 . j^2 C_2)^x (\bar{ij}^2 c_2)^y,$$

where a comparison of dimensions gives us the equations

$$4x + y = 6, \quad 2x + 2y = 6, \quad x = 1, \quad y = 2.$$

We have also by direct substitution

$$p\bar{ij}.q\bar{ij}.p'\bar{ij}.q'\bar{ij} = \bar{ij}^4 (p . q . p' . q') = -\frac{1}{4} (\bar{ij}^2 c_2)^3,$$

and hence the following expression for the product of the squares of the distances $pq, p'q'$,

$$\overline{pq}^2 . \overline{p'q'}^2 = \kappa' \frac{(i^2 C_2 . j^2 C_2)^x (\bar{ij}^2 c_2)^y}{(\bar{ij}^2 c_2)^6}.$$

Since $\overline{pq}^2 + \overline{p'q'}^2 = 0$, this product is the same thing as $-pq^4$. But we get another equation between x and y by supposing the conic c_2 to break up into two points u, v , which are then themselves the foci. In that case $-4C_2 = \overline{uv}^2$, $\bar{ij}^2 c_2 = \overline{uij} . \overline{vij}$, and the expression for the fourth power of the distance is

$$\overline{uv}^4 = \kappa' \frac{(\overline{uv}^2 . \overline{uij} . \overline{vij})^2}{(\overline{uij} . \overline{vij})^4} = \frac{i^2 C_2 . j^2 C_2}{(\bar{ij}^2 c_2)^4},$$

whence $x = 1, y = 2$, and the general formula is

$$\overline{pq}^2 = \frac{\sqrt{(i^2 C_2 . j^2 C_2)}}{(\bar{ij}^2 c_2)^2}.$$

In this it is clear that c_2 is primarily given; if C_2 is given, the process of calculating it back from the coefficients of c_2 introduces the factor $C_2 c_2$, and we have

$$\overline{pq}^2 = \frac{\sqrt{(i^2 C_2 . j^2 C_2)} C_2 c_2}{(\bar{ij}^2 c_2)^2}.$$

With the aid of this formula and the angle between the asymptotes we may now determine the axes of the conic.

If h and k be the axes, we have

$$\begin{aligned} \overline{pq}^2 &= h^2 - k^2, \\ \cos(PQ) &= \frac{h^2 + k^2}{h^2 - k^2}, \end{aligned}$$

The denominator may be expressed by direct operation on LM ; we should find in fact

$$\begin{aligned} Li. Lj. Mi. Mj &= \{x^2 C_2. i^2 C_2 - (xi C_2)^2\} \{x^2 C_2. j^2 C_2 - (xj C_2)^2\} \\ &= \overline{xi^2 c_2}. \overline{xj^2 c_2}, \end{aligned}$$

and thus finally

$$\sin^2 LM = \kappa' \frac{x^2 C_2. (xij)^2. C_2 c_2}{\overline{xi^2 c_2}. \overline{xj^2 c_2}}.$$

But the denominator may be put into another form which is more useful. Let us assume that the absolute values of the coefficients of p, q, p', q' are so chosen that $pi = p'i, qi = q'i, pj = q'j, qj = p'j$. We have by direct substitution

$$pix. qix. p'ix. q'ix = -\frac{3}{4} \overline{ij^2 c_2}. (\overline{xi^2 c_2})^2,$$

or

$$(pix. qix)^2 = -\frac{3}{4} \overline{ij^2 c_2}. (\overline{xi^2 c_2})^2.$$

Hence we have

$$xpi. xqi. xpj. xqj = -\frac{3}{4} \overline{ij^2 c_2}. \overline{xi^2 c_2}. \overline{xj^2 c_2},$$

and consequently, since

$$\overline{xp^2}. \overline{xq^2} = \kappa \frac{xpi. xqi. xpj. xqj}{(xij)^4 (\overline{pj}. \overline{qij})^2},$$

and

$$(pij. qij)^2 = -\frac{3}{4} (\overline{ij^2 c_2})^2,$$

it follows that

$$xp^2. xq^2 = \kappa \frac{\overline{xi^2 c_2}. \overline{xj^2 c_2}}{(xij)^4 (\overline{ij^2 c_2})^2} = xp'^2. xq'^2.$$

Combining these two results, we find

$$xp^2. xq^2. \sin^2 LM = \kappa' \frac{x^2 C_2. C_2 c_2}{(xij)^2. (\overline{ij^2 c_2})^2},$$

a result which may be further simplified by help of the formula for the distance between the foci. Namely, we have

$$\frac{xp^2. xq^2. \sin^2 LM}{pq^2} = \frac{x^2 C_2}{(xij)^2 \sqrt{(i^2 C_2. j^2 C_2)}} \text{ to a factor } p'q'.$$

We shall call this quantity the *distance** of the point from the conic; it vanishes when the point is on the conic, and is infinite if either the point or the conic has contact with the absolute.

* [Second distance; cf. p. 138.]

It is to be noted that the conic is given as a curve of the second order, in the form C_q ; if it were given in the form c_q , the formula preceding would become

$$xp^2, xq^2, \sin^2 LM : : \kappa'' \frac{x^2 C_q}{(xij)^2 \cdot (ij^2 c_q)^2},^*$$

and this quantity might be taken as the distance of the point from a conic given as a curve of the second class. If the conic breaks up into two lines, the former expression becomes the product of the perpendicular distances of the point from the two lines; if the conic breaks up into two points, the latter expression becomes four times the squared area which they include with the given point. The former expression, however, in which the conic is given as of the second order, admits of a further interpretation, to which we now proceed.*

Through the point x (fig. 13) let a line X be drawn, meeting the conic in l, m . For the product of the segments xl, xm we have the formula

$$xl \cdot xm : : \kappa^2 \frac{x(li \cdot xlj \cdot xmi \cdot xmj)}{(xij)^2 \cdot lij \cdot mij}.$$

The numerator of this expression clearly vanishes if x is on the conic, when one of the lines xl, xm becomes indeterminate; otherwise each of these lines is simply the line X . We must have therefore

$$xli \cdot xlj \cdot xmi \cdot xmj : : (x^2 C_q, Xi, Xj)^2 \text{ to a factor } pr^2s.$$

For the denominator we observe that if U is any arbitrary line,

$$lU \cdot mU : : XU^2 C_q,$$

and taking the co-ordinates of U for the current line co-ordinates, this gives the tangential equation to l, m . From this we get

$$lij \cdot mij : : Xij^2 C_q,$$

and our expression is therefore transformed into

$$xl \cdot xm : : \frac{x^2 C_q, Xi, Xj}{(xij)^2 \cdot Xij^2 C_q}.$$

* [First distance; cf. p. 133.]

The denominator of this will clearly vanish if X is parallel to one or other of the asymptotes P, Q , or if $XPij \cdot XQij = 0$. Let us, therefore, now seek the product of the distances of X from the asymptotes.

We have

$$-4 \sin XP \cdot \sin XQ = \frac{XPij \cdot XQij}{(Xi \cdot Xj) \sqrt{(Pi \cdot Pj \cdot Qi \cdot Qj)}}.$$

But we find by operating with $(Xij)^2$ upon P, Q that

$$XPij \cdot XQij = 3\bar{Xij}^2 C_2 \cdot \bar{ij}^2 c_2,$$

and moreover

$$Pi \cdot Pj \cdot Qi \cdot Qj = 9i^2 C_2 \cdot j^2 C_2 \cdot (\bar{ij}^2 c_2)^2.$$

Hence we have

$$-4 \sin XP \cdot \sin XQ = \frac{\bar{Xij}^2 C_2}{3Xi \cdot Xj \sqrt{(i^2 C_2 \cdot j^2 C_2)}}.$$

Multiplying this by $xl \cdot xm$, the line X disappears from the result and we find

$$xl \cdot xm \cdot \sin XP \cdot \sin XQ = \frac{x^2 C_2}{(xij)^2 \sqrt{(i^2 C_2 \cdot j^2 C_2)}} \text{ to a factor.}$$

But the expression on the right is the same that we previously obtained for the distance of the point from the conic. Hence the quantity $xl \cdot xm \cdot \sin XP \sin XQ$ must be proportional to the quantity $\frac{xp^3 \cdot xq^3}{pq^3} \sin^2 LM$. To determine the constant factor, suppose the conic to break up into a pair of points; these may be taken to be the points p, q , and the asymptotes will both coincide with the line pq (fig. 14). Here it is clear that $xl \cdot xm \cdot \sin XP \sin XQ = xl^2 \cdot \sin^2 XP = \text{squared distance of } x \text{ from line } pq$; while $\frac{xp \cdot xq \sin p x q}{pq} = \frac{\text{twice area } p x q}{pq} = \text{distance of } x \text{ from } pq$. Thus the factor is unity, and we have always

$$xl \cdot xm \cdot \sin XP \cdot \sin XQ = \frac{xp^3 \cdot xq^3}{pq^3} \sin^2 LM.*$$

There is no difficulty in investigating the correlative formulæ. First to find the length of the chord cut off a line X

* [Second distance, p. 133.]

by the conic; let lm be the points of intersection, then we have

$$lm^2 = \left[\frac{1}{\kappa^2} \right] \frac{lmij}{(lij)^2 \cdot (mij)^2}.$$

The numerator vanishes if the line X pass through either of the points ij , or if l, m coincide, that is to say, if X touch the conic c_2 , or if the conic break up into a pair of points. Moreover, since we have

$$l, m = X^2 c_2 \cdot c_2 = (Xc_2)^2,$$

the expression $(lm)^2$ must be of the fourth order in the coefficients of X and of the conic, and therefore

$$(lm)^2 = (Xi)^2 \cdot X^2 c_2 \cdot C_2^2 c_2 \text{ to a factor.}$$

Again, we have by direct substitution

$$lij \cdot mij = X^2 c_2 \cdot ij^2 c_2 = (Xij^2 c_2)^2 = Xij^2 C_2,$$

and thence

$$lm^2 = \frac{Xi \cdot Xj \cdot X^2 c_2 \cdot C_2^2 c_2}{(Xij^2 C_2)^2} \text{ to a factor.}$$

But we have already found that

$$\sin XP \cdot \sin XQ = \frac{Xij^2 C_2}{Xi \cdot Xj \sqrt{(i^2 C_2^2 \cdot j^2 C_2^2)}},$$

consequently

$$[i] \quad lm^2 \cdot \sin^2 XP \cdot \sin^2 XQ = \frac{X^2 c_2 \cdot C_2^2 c_2}{Xi \cdot Xj \cdot i^2 C_2^2 \cdot j^2 C_2^2}.*$$

Lastly, c_2 being primarily given, we have

$$\sin^2 PQ = \frac{ij^2 c_2 \cdot C_2^2 c_2}{i^2 C_2^2 \cdot j^2 C_2^2},$$

and so

$$[ii] \quad \frac{lm^2 \sin^2 XP \cdot \sin^2 XQ}{\sin^2 PQ} = \frac{X^2 c_2}{Xi \cdot Xj \cdot ij^2 c_2}.$$

Let now x be a variable point on the line X (fig. 15), and draw the tangents L, M from x to the conic. Then

$$\sin XL \cdot \sin XM = \frac{XLij \cdot XMij}{Xi \cdot Xj \sqrt{(Li \cdot Lj \cdot Mi \cdot Mj)}}.$$

* [That distance of line X from Conic C_2 ; cf. p. 133.]

In this fraction the numerator vanishes when X touches the conic, and when x is on the line ij . It must be of the first order in the coefficients of c_2 and of the second in those of x . Hence we have, to a factor *près*,

$$XLij \cdot XMij = X^2 c_2 \cdot (xij)^2.$$

Moreover by a previous formula

$$Li \cdot Lj \cdot Mi \cdot Mj = \bar{x}i^2 c_2 \cdot \bar{x}j^2 c_2.$$

Thus

$$\sin XL \cdot \sin XM = \frac{X^2 c_2 \cdot (xij)^2}{Xi \cdot Xj \sqrt{(\bar{x}i^2 c_2 \cdot \bar{x}j^2 c_2)}}.$$

But, also by a previous formula,

$$xp \cdot xq = \frac{\sqrt{(\bar{x}i^2 c_2 \cdot \bar{x}j^2 c_2)}}{(xij)^2 \cdot \bar{i}j^2 c_2};$$

therefore

$$\begin{aligned} \text{[iii]} \quad xp \cdot xq \sin XL \sin XM &= \frac{X^2 c_2}{Xi \cdot Xj \cdot \bar{i}j^2 c_2} \\ &= lm^2 \frac{\sin^2 XP \cdot \sin^2 XQ}{\sin^2 PQ}.* \end{aligned}$$

To verify this, suppose the conic to break up into two lines P, Q (fig. 16), which are themselves the asymptotes; the two foci will then coincide at the point PQ , and the tangents LM will pass through the same point. Then $xp \cdot xq \cdot \sin XL \cdot \sin XM = xp^2 \sin^2 XL =$ squared perpendicular from PQ on X . And

$$\begin{aligned} \frac{lm}{\sin PQ} \cdot \sin XP \sin XQ &= lp \cdot mp \cdot \frac{lm}{\sin PQ} \cdot \frac{\sin XP}{lp} \cdot \frac{\sin XQ}{mp} \\ &= \frac{lp \cdot mp \cdot \sin PQ}{lm} = \frac{2 \text{ area } lmp}{lm} \\ &= \text{perpendicular from } PQ \text{ on } X. \end{aligned}$$

This quantity†, of which three expressions are given in our last equation, may be called the *distance* of the line X from the conic c_2 .

* [Second distance of line X from Conic c_2 .]

† [i. e. iii. *supra*.]

IV.

We now consider a curve C_n of the n^{th} order, which may also be given as a curve $c_{n(n-1)}$ of class $n(n-1)$. To this number $n(n-1)$ there are no reductions in virtue of any singularities that C_n may have; its nodes will enter as double factors and its cusps as triple factors in $c_{n(n-1)}$. This being so, we may write

$$\text{Disct.}_\lambda (x + \lambda y)^n C_n = \frac{\{n\}^{2(n-1)}}{n(n-1)} \cdot \overline{xy}^{n(n-1)} c_{n(n-1)}.$$

Conversely, if we are given a curve c_m of class m , this is also a curve $C_{m(m-1)}$ of order $m(m-1)$; but each double tangent is now a double factor and each stationary tangent a triple factor in $C_{m(m-1)}$. We may gain shortness without introducing confusion, if when C_n is primarily given, we denote $n(n-1)$ by m ; and if when c_m is primarily given, we denote $m(m-1)$ by n . Thus in the latter case we shall have

$$\text{Disct.}_\lambda (X + \lambda Y)^m c_m = \frac{\{m\}^{2(m-1)}}{n} \cdot \overline{XY}^n C_n.$$

The curve c_m has m^2 foci, which are the intersections of the m tangents from the point i with the m tangents from the point j . But for every tangent from i there is *one* tangent from j which meets it in a real point; thus there are m real foci. The foci may be arranged in various ways into m sets, such that no two points of the same set are collinear with i or j . By joining the points of any one set with i and j we obtain all the tangents. The real foci constitute a set; and from them we may pass to any other set by successively substituting for each pair pq their *antipoints* $p'q'$, that is to say, the remaining intersections of pi, pj with qi, qj . Now for every such substitution the following equations hold good:

$$p'q'^2 = -pq^2, \\ xp' \cdot xq' = xp \cdot xq,$$

where x is any point in the plane. Hence if we form the product Πpq^2 of the squared distances of the real foci from one

another, this product can differ only in sign from a similar product formed with any other set; and the same is true of Πxp . The number of possible sets is clearly the same as the number of terms in a determinant of the m^{th} order, viz.: $|m|$. If we then raise Πpq^2 to the power $|m|$, we must have some power of $\Pi \Pi pq^2$, the product of the squared distances of all pairs of foci from one another, excepting of course those pairs which are collinear with i or j . But there are $m^2(m-1)^2$ such pairs; thus if k is the power in question

$$|m| \cdot m(m-1) = km^2(m-1)^2,$$

or

$$k = \frac{|m|}{m(m-1)},$$

whence

$$(\Pi pq^2)^{m(m-1)} = \Pi \Pi pq^2.$$

In a similar way we find

$$(\Pi xp)^m = \Pi \Pi xp.$$

Now

$$\Pi \Pi pq^2 = \Pi \Pi \frac{pq^i \cdot pqj}{pij^2 \cdot qij^2} = \frac{\Pi \Pi pq^i \cdot pqj}{(\Pi pij)^{4m^2(m-1)^2}} \text{ [to factor } m^2 \text{es]}.$$

To obtain the equation of the foci we may proceed as follows. The equation of the tangents from i, j to c_m is $I_m = 0, J_m = 0$,

where

$$x^m I_m = \overline{j x^m c_m}, \quad x^m J_m = \overline{j x^m c_m}$$

and if we then form the equation of the points of intersection of I_m, J_m (which may be written $(XIJ)_{mm} = 0$) it is of the order m in each of them and must contain the equation of the foci. But also it must contain the factor $\overline{ij^m c_m}$, and this to the degree $m-1$; for the factor is involved as the condition that I_m should pass through a double point of J_m . Thus the equation of the foci is of the order $2m - m - 1 = m + 1$ in c_m and $m^2 - m(m-1) = m$ in i and j . In fact one term in it is a numerical multiple of $\overline{ij^m c_m} \cdot (c_m)^m$.

As in the case of the conic, the product $\Pi \Pi pq^i \cdot pqj$ will vanish when $i^n C_n = 0$, or when $j^n C_n = 0$, or when $\overline{ij^m c_m} = 0$.

Thus

$$\Pi \Pi pq^i \cdot pqj = \kappa (i^n C_n \cdot j^n C_n)^w (\overline{ij^m c_m})^v.$$

Now the left-hand side is of order $2(m-1)^2$ in the foci and besides of order $\frac{1}{2}m^2(m-1)^2$ in i and j ; that is, of order $2(m+1)(m-1)^2$ in e_m and $2m(m-1)^2 + \frac{1}{2}m^2(m-1)^2 = \frac{1}{2}m(m+4)(m-1)^2$ in i and j . First regard e_m as given; then \mathcal{O}_n is of order $2(m-1)$ in the coefficient of e_m , and we have

$$2(m+1)(m-1)^2 = 4x(m-1) + y,$$

$$\frac{1}{2}m(m+4)(m-1)^2 = xm(m-1) + my,$$

whence $x = \frac{1}{2}m(m-1)$, $y = 2(m-1)^2$, and consequently

$$111pq i, pq j = \kappa (i^n \mathcal{O}_n, j^n \mathcal{O}_n)^{\frac{1}{2}m(m-1)} (\bar{j}^m e_m)^{2(m-1)^2}.$$

Next we have

$$111pij = (\bar{j}^m e_m)^{m+1} \text{ to a factor,}$$

and therefore

$$111pq^2 = \frac{111pqi, pqj}{(111pij)^{2(m-1)^2}} = \kappa \frac{(i^n \mathcal{O}_n, j^n \mathcal{O}_n)^{\frac{1}{2}m(m-1)}}{(\bar{j}^m e_m)^{2m(m-1)^2}},$$

but

$$(11pq^2)^{m(m-1)} = 111pq^2;$$

therefore

$$11pq^2 = \frac{(i^n \mathcal{O}_n, j^n \mathcal{O}_n)^{\frac{1}{2}}}{(\bar{j}^m e_m)^{m(m-1)}}.$$

We now proceed to determine $11xp$, where x is any point in the plane. We have

$$\begin{aligned} 111xp^2 &= \frac{111xpi, xpj}{(xij)^{2m^2} 111(pij)^2} = \frac{(xi^m e_m, xj^m e_m)^m (ij^m e_m)^2}{(xij)^{2m^2} (\bar{j}^m e_m)^{2m(m+1)}} \\ &= \frac{(xi^m e_m, xj^m e_m)^m}{(xij)^{2m^2} (\bar{j}^m e_m)^{2m}}, \end{aligned}$$

and therefore

$$11xp^2 = \frac{xi^m e_m \cdot xj^m e_m}{(xij)^{2m} (\bar{j}^m e_m)^2}.$$

The curve \mathcal{O}_n has n asymptotes, which are the tangents at the points where it is met by the line ij . If a point x lie on one of the asymptotes, its first polar $x\mathcal{O}_n$ must meet \mathcal{O}_n on the line ij . The condition for this is of the order n in $x\mathcal{O}_n$, $n-1$ in \mathcal{O}_n , and $n(n-1)$ in ij ; that is, of the order n in x , $2n-1$ in \mathcal{O}_n , and $n(n-1)$ in ij . It must also be of the form

$$Ax^n \mathcal{O}_n + x^{n-2} B_{n-2} (xij)^2 = 0,$$

since the n asymptotes form a curve of the n^{th} order touching C_n where it is met by ij . If Δ vanishes, the line ij becomes a double factor; now this can only happen when $\bar{ij}^m c_m = 0$, and a comparison of dimensions shews that Δ differs from this only by a numerical factor. We may therefore write for the equation of the asymptotes

$$\bar{ij}^m c_m \cdot C_n + B_{n-2} \cdot \bar{ij}^2 \equiv \Pi P.$$

We may now find the product of the sines of the angles between them, $\Pi \sin PQ$, and of the angles they make with a line X , $\Pi \sin XP$. Namely, we have

$$\Pi \sin^2 PQ = \frac{\Pi (PQij)^2}{(\Pi Pi \cdot \Pi Pj)^{n-1}} \quad [\text{to factor } p^m \text{es}].$$

Now

$$(\Pi PQij)^2 = (\bar{ij}^m c_m)^{2n-1},$$

and therefore

$$\Pi \sin^2 PQ = \frac{(\bar{ij}^m c_m)^{2n-1}}{(\bar{ij}^m c_m)^{2(n-1)} (i^2 C_n \cdot j^2 C_n)^{n-1}} = \frac{\bar{ij}^m c_m}{(i^2 C_n \cdot j^2 C_n)^{n-1}}.$$

In the next place

$$\Pi \sin^2 XP = \frac{\Pi (XFij)^2}{(Xi \cdot Xj)^n \cdot \Pi Pi \cdot \Pi Pj} = \frac{\{(Xij)^n C_n\}^2}{(Xi \cdot Xj)^n \cdot i^n C_n \cdot j^n C_n}.$$

Let us now suppose a variable line X to be drawn through the fixed point x , meeting the curve C_n in the points l, m, n, \dots then

$$\Pi x l^2 = \frac{\Pi x l i \cdot \Pi x l j}{(xij)^{2n} \Pi (lij)^2} = \frac{(x^n C_n)^2 \cdot (Xi)^n \cdot (Xj)^n}{(xij)^{2n} \cdot [(Xij)^n C_n]^2},$$

or

$$\Pi x l = \frac{x^n C_n \cdot (Xi \cdot Xj)^{\frac{n}{2}}}{(xij)^n \cdot (Xij)^n C_n}.$$

Hence

$$\Pi x l \cdot \Pi \sin XP = \frac{x^n C_n}{(xij)^n \cdot \sqrt{(i^n C_n \cdot j^n C_n)}};$$

this product is therefore independent of the position of the line X , and may be called the *distance* of the point x from the curve C_n .

V.

The elliptic geometry of two dimensions as Dr Klein calls it, or, which is the same thing, geometry on the sphere in which two opposite points are regarded as identical, differs from plane geometry in that instead of the two points \dot{y} we have the proper conic O_2 or o_2 . Lines touching this conic, and points lying on it, are at an infinite distance from all other lines and points; distances measured on them are zero. Using the ordinary co-ordinates we may write

$$\begin{aligned} O_2 &= x_1^2 + x_2^2 + x_3^2, \\ o_2 &= \xi_1^2 + \xi_2^2 + \xi_3^2, \\ a &= a_1\xi_1 + a_2\xi_2 + a_3\xi_3, \\ A &= A_1x_1 + A_2x_2 + A_3x_3, \end{aligned}$$

and then

$$\begin{aligned} \sin^2 ab &= \frac{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} \\ &= \frac{2\overline{ab}^2 o_2}{a^2 O_2 \cdot b^2 O_2} \quad [\text{when } O_2 \text{ given}], \\ \sin^2 AB &= \frac{(A_2B_3 - A_3B_2)^2 + (A_3B_1 - A_1B_3)^2 + (A_1B_2 - A_2B_1)^2}{(A_1^2 + A_2^2 + A_3^2)(B_1^2 + B_2^2 + B_3^2)} \\ &= \frac{2\overline{AB}^2 O_2}{A^2 o_2 \cdot B^2 o_2}. \end{aligned}$$

The distance of a point a from a line B may be derived from these two as follows (fig. 17): through a draw a variable line A meeting B in b ; then we have

$$\sin^2 ab = \sin^2 aAB = \frac{2(aB)^2 \cdot A^2 o_2}{a^2 O_2 \cdot \overline{AB}^2 O_2},$$

and

$$\sin^2 AB = \frac{2\overline{AB}^2 O_2}{A^2 o_2 \cdot B^2 o_2};$$

$$\therefore \sin^2 ab \cdot \sin^2 AB = \frac{4(aB)^2}{a^2 O_2 \cdot B^2 o_2};$$

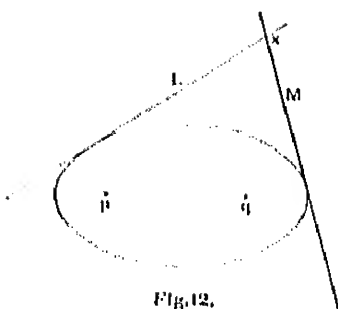


Fig. 12.

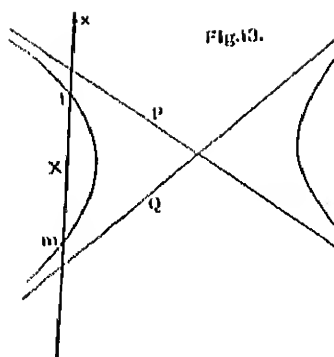


Fig. 13.

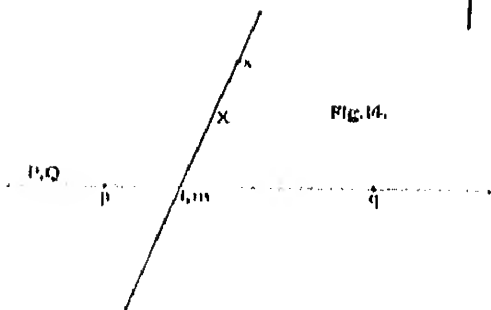


Fig. 14.

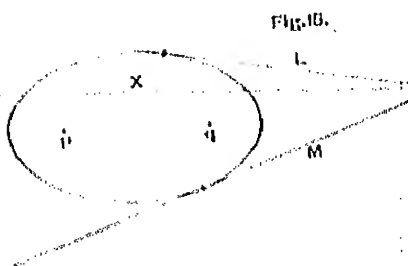


Fig. 15.

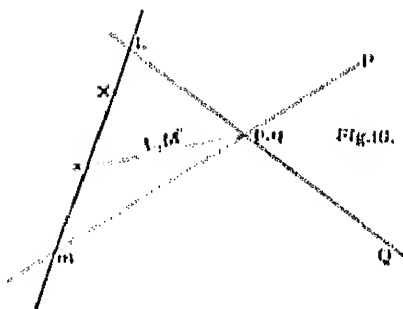


Fig. 16.

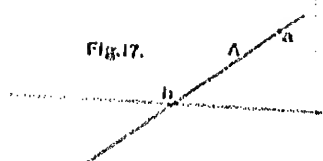


Fig. 17.

a result independent of the position of the line A , which when A is at right angles to B becomes $\sin^2 aB$.

A conic O_3 or c_3 has with the absolute O_2 or o_2 four common points and four common tangents. The lines joining the common points form three pairs of common chords, $P, Q; P', Q'; P'', Q''$; the intersections of the common tangents form three pairs of foci, $p, q; p', q'; p'', q''$. The theories of the foci and the common chords are identical, and it will be sufficient to consider one of them; we shall choose for this purpose the foci.

We obtain a pair of foci by determining λ so that

$$c_2 + \lambda o_2$$

shall break up into factors; the condition for this is

$$0 = 6 \text{ Discr. } (c_2 + \lambda o_2) = O_3 o_2 + 3\lambda O_2 o_2 + 3\lambda^2 c_2 O_2 + \lambda^3 O_2 o_2,$$

and substituting here for $\lambda, -\frac{o_2}{O_2}$, we obtain the equation of the three pairs of foci:—

$$0 = O_2 o_2 \cdot o_2^3 - 3(O_2 o_2 \cdot o_2^2 \cdot c_2 + 3c_2 O_2 \cdot o_2 \cdot c_2^2 - O_2 o_2 \cdot c_2^3).$$

If $c_2 + \lambda o_2$ break up into factors p, q , its reciprocal will be $= \frac{1}{4} p q^3$. Then we have

$$= \frac{1}{4} p q^3 = (O_2 + 2\lambda (co)_2 + \lambda^2 O_2,$$

where $(co)_2$ is the locus of points at which c_2 subtends a right angle. Consequently

$$= \frac{1}{4} p q^3 o_2 = O_3 o_2 + 2\lambda c_2 O_2 + \lambda^2 O_2 o_2 = 2\partial_\lambda \text{ Discr. } (c_2 + \lambda o_2).$$

Moreover

$$pq(O_2 + c_2 O_2 + \lambda O_2 o_2 = \partial_\lambda^2 \text{ Discr. } (c_2 + \lambda o_2),$$

$$\text{but } p^3 O_2 \cdot q^2 O_2 = (pq O_2)^3 = \frac{1}{4} p q^3 o_2 \cdot O_2 o_2,$$

hence

$$\begin{aligned} 3p^2 O_2 \cdot q^2 O_2 &= 3(c_2 O_2 + \lambda O_2 o_2)^2 = 4(O_2 o_2 + 2\lambda c_2 O_2 + \lambda^2 O_2 o_2) O_2 o_2 \\ &= 2O_2 O_2^2 + 4O_2 o_2 \cdot O_2 o_2 - 2\lambda c_2 O_2 \cdot O_2 o_2 - \lambda^2 O_2 o_2^2 \\ &= 3(c_2 O_2)^2 - 3O_2 o_2 \cdot O_2 o_2 = (O_2 o_2 + 2\lambda c_2 O_2 + \lambda^2 O_2 o_2) O_2 o_2. \end{aligned}$$

To simplify these formulae, let us write

$$6 \text{ Disc. } (c_2 + \lambda a_2) = f = a\lambda^3 + 3b\lambda^2 + 3c\lambda + d$$

$$\frac{1}{3} \partial_\lambda f = w = a\lambda^2 + 2b\lambda + c$$

$$y = 3(b^2 - ac) + ax;$$

then whenever λ has a value which makes f vanish, so that $c_2 + \lambda a_2$ breaks up into the factors p, q , we must have

$$- \frac{1}{3} p q^2 a_2 = x,$$

$$3p^2 O_2, q^2 O_2 = y.$$

We shall now prove that if x_1, x_2, x_3 are the values of x , and y_1, y_2, y_3 the values of y , corresponding to the three values of λ given by $f=0$, then

$$x_1^2 y_1 = x_2^2 y_2 = x_3^2 y_3 = -R_F, \text{ the discriminant of } f.$$

We get an equation for determining w by eliminating λ between $f=0$ and $\frac{1}{3} \partial_\lambda f = w = 0$. Namely, the resultant of these two equations is

$$\begin{vmatrix} a & 3b & 3c & d & . \\ . & a & 3b & 3c & d \\ a & 2b & c-w & . & . \\ . & a & 2b & c-w & . \\ . & . & a & 2b & c-w \end{vmatrix} = -a^2 w^3 + 3(b^2 - ac) a w^2 + a R_F, \\ = -a(R_F + w^2 y),$$

which vanishes if

$$w^2 y = -R_F;$$

this equation is therefore true for each of the corresponding pairs of values of w and y . Substituting for these their values, we have

$$(p q^2 a_2)^2 \cdot p^2 O_2 \cdot q^2 O_2 = (p' q'^2 a_2)^2 \cdot p'^2 O_2 \cdot q'^2 O_2 \\ = (p'' q''^2 a_2)^2 \cdot p''^2 O_2 \cdot q''^2 O_2 = -\frac{1}{16} R_F.$$

Now R_F is the resultant of c_2 and a_2 , that is, the invariant whose vanishing is the condition that the two conics shall touch.

It follows further from the cubic equation for x that the product of its three values is $\frac{R_f}{a}$. Hence we have

$$pq^2o_a, p'q^2o_a, p''q^2o_a = -64\frac{R_f}{a},$$

and therefore

$$p^3O_a, q^3O_a, p'^3O_a, q'^3O_a, p''^3O_a, q''^3O_a = -\frac{a^3R_f}{27};$$

whence by division

$$\sin^3 pq, \sin^3 p'q', \sin^3 p''q'' = +\frac{12^3 [2^3]}{6^3} = +[64],^*$$

since $\sin^3 pq = \frac{[2]pq^2o_a}{p^3O_a, q^3O_a} \cdot \frac{O_a o_a}{6}$ when o_a is given.

From the same equation we learn that the sum of the reciprocals of the x is zero. Now

$$-\frac{R_f}{x^3} = \frac{y}{x} = [-2O_a o_a] (\sin pq)^{-2};$$

therefore

$$(\sin pq)^{-3} + (\sin p'q')^{-3} + (\sin p''q'')^{-3} = 0.$$

$$[\text{Also } (\sin pq)^{-2} + (\sin p'q')^{-2} + (\sin p''q'')^{-2} = \frac{1}{3}.]^*$$

We have thus two equations connecting the quantities $\sin pq, \sin p'q', \sin p''q''$, by which when one is given the other two may be determined.

Now let x be any point of the plane (or sphere). Then we have

$$\sin^3 xp, \sin^3 xq = \frac{\tilde{x}p^2o_a, \tilde{x}q^2o_a}{(x^3O_a)^3, p^3O_a, q^3O_a} \left[\left(\frac{O_a o_a}{6} \right)^3 \right].$$

But the numerator of this fraction is clearly the result of operating with x on the common tangents of pq and o_a , that is of $e_a + \lambda o_a$ and o_a , a result which is clearly independent of λ . Hence we have

$$xp^2o_a, xq^2o_a = \tilde{x}p^2o_a, \tilde{x}q^2o_a = \tilde{x}p'^2o_a, \tilde{x}q'^2o_a.$$

Moreover, since

$$(pq^2o_a)^3 p^3O_a, q^3O_a = -\frac{1}{3} R_f,$$

* [The introduced factor, cf. p. 152, brings the results into accordance with Prof. Cayley's equations. See Note to this paper, p. 159 (3).]

it follows that

$$\sin^4 pq = \frac{1}{32} \left(\frac{pq^2 \alpha_2 \cdot O_2 \alpha_2}{p^2 \alpha_2 \cdot q^2 \alpha_2} \right)^2 = \frac{1}{32} \frac{R_F (O_2 \alpha_2)^2}{(p^2 \alpha_2 \cdot q^2 \alpha_2)^2},$$

and consequently

$$\begin{aligned} \frac{\sin xp \cdot \sin xq}{(\sin pq)^3} &= \frac{\sin xp' \cdot \sin xq'}{(\sin p'q')^3} = \frac{\sin xp'' \cdot \sin xq''}{(\sin p''q'')^3} \\ &= \left[\frac{\sqrt{xp^2 \alpha_2 \cdot xq^2 \alpha_2}}{s^2 \alpha_2} \cdot \frac{1}{\sqrt{16} \sqrt{3} \sqrt{2}} \cdot \frac{(O_2 \alpha_2)^2}{R_F} \right] \\ &= \left[\frac{1}{\sqrt{16} \sqrt{3}} \right] \frac{\sqrt{X'(\cos) (O_2 \alpha_2)^2}}{s^2 \alpha_2 \sqrt{2} - R_F}. \end{aligned}$$

The three equations which we have just proved, the theory of three pairs of antipoints on a sphere; three pairs of intersections of four tangents to the sphere; and the theory of three pairs of antipoints on a plane; they take the place of the *two* equations which already used in regard to the two pairs of antipoints on a plane; namely,

$$pq^2 = p'q'^2,$$

and

$$xp \cdot xq = xp' \cdot xq',$$

We now proceed to use them in connection with the theory of the conic c_p .

From the point x let the tangents L, M be drawn to the conic; then

$$\sin^2 LM = \left[\frac{LM^2 O_2}{L^2 \alpha_2 \cdot M^2 \alpha_2} = \frac{s^2 \alpha_2 \cdot x^2 \alpha_2}{xp^2 \alpha_2 \cdot xq^2 \alpha_2} \right] \text{ (no factor } p \text{)}$$

where pq are any set of foci. But also

$$\frac{\sin^2 xp \cdot \sin^2 xq}{(\sin^2 pq)^3} = \left[\frac{1}{12 \sqrt{2}} \right] \frac{s^2 \alpha_2 \cdot x^2 \alpha_2 (O_2 \alpha_2)^2}{(s^2 \alpha_2 \cdot \sqrt{2} - R_F)^3},$$

and therefore

$$\begin{aligned} \frac{\sin^2 xp \cdot \sin^2 xq}{(\sin^2 pq)^3} \sin^2 LM &= \left[\frac{1}{12 \sqrt{2}} \right] \frac{s^2 \alpha_2 \cdot (O_2 \alpha_2)^3}{s^2 \alpha_2 \cdot \sqrt{2} - R_F} \\ &= \left[\frac{1}{12 \sqrt{2}} \right] \frac{s^2 \alpha_2 \cdot (O_2 \alpha_2)^3 (O_2 \alpha_2)^2}{s^2 \alpha_2 \cdot \sqrt{2} - R_F}. \end{aligned}$$

Again, if we draw through x a variable line X , meeting the conic in l, m , we shall have

$$\begin{aligned}\sin^2 xl, \sin^2 xm &= \frac{x l^2 O_1, x m^2 O_2}{(x^2 O_1)^2, l^2 O_2, m^2 O_2} = \frac{(x^2 O_1)^2, (X^2 O_2)^2}{(x^2 O_1)^2, X^2 (CO)_2} \\ &= \frac{(x^2 O_2)^2, (X^2 O_2)^2}{(x^2 O_2)^2, X l^2 O_1, X m^2 O_2},\end{aligned}$$

$$\begin{aligned}\text{and } \frac{\sin^2 XP, \sin^2 XQ}{(\sin^2 PQ)^{\frac{1}{2}}} &= \left[\frac{1}{6\frac{1}{2}} \frac{X l^2 O_1, X m^2 O_2 (O_1 O_2)^{\frac{1}{2}}}{(X^2 O_2)^2 \{l^2 O_2, m^2 O_2 (PQ^2 O_2)^{\frac{1}{2}}\}^{\frac{1}{2}}} \right] \\ &= \frac{1}{6\frac{1}{2}} \frac{X l^2 O_1, X m^2 O_2, (O_1 O_2)^{\frac{1}{2}}}{(X^2 O_2)^2 \sqrt{R_P}},\end{aligned}$$

where R_P signifies the osculant of O_2 and O_1 ; $R_P = (O_2 O_1)^2 (O_2 O_2)^2 R_P$.

Therefore

$$\sin xl, \sin xm = \frac{\sin XP \sin XQ}{(\sin PQ)^{\frac{1}{2}}} = \left[\frac{1}{\sqrt{12^{\frac{1}{2}} \cdot 4}} \right] \frac{x^2 O_2, (O_1 O_2)^{\frac{1}{2}}}{x^2 O_2, \sqrt{R_P}}.$$

[The MS. here ends abruptly. In the results of the last two pages I have introduced a few additions, all, or nearly all, of which agree with results obtained by Prof. Henriet, who most kindly gave me his valuable help in revising the proof sheets.

The subject of V. has been treated from a different point of view by Prof. Cayley, who has allowed me to append the following note.

The results obtained in No. V. of the paper On the Theory of Distances may be worked out in greater detail, and in some measure in a more complete form.

Using line-coordinates, we have $x^2 + y^2 + z^2 = 0$, the conic called the absolute; a conic $(a, b, c, f, g, h, Q, x, y, z)^2 = 0$, which will be called simply the conic; and a point (x) the equation of which is $l x + m y + n z = 0$. The common tangents of the conic and the absolute intersect in pairs in six points p, q, p', q', p'', q'' , which are the foci of the conic; or if we regard the four lines simply as any four tangents of the absolute, then the six points are a system of foci; and we obtain in the first instance formulae relating to such a system, alone or in connection with the point x ; afterwards, taking them to be the foci of the conic, we further consider the two tangents l, m from the point (x) to the conic; and an arbitrary line X through the point (x) .

1. The coordinates of a tangent of the absolute are (x_1, y_1, z_1) , where these are any values such that $x_1^2 + y_1^2 + z_1^2 = 0$; and we consider the four tangents

$$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4).$$

Writing for a moment

$$X_1 = p_1 - p = p_1^2 x_1 - p_1^2 x_2 - p_1^2 x_3 - p_1^2 x_4 - p_1^2 x_5,$$

$$X_2 = p_2 - p = p_2^2 x_1 - p_2^2 x_2 - p_2^2 x_3 - p_2^2 x_4 - p_2^2 x_5,$$

we have

$$X_1 + p_1 q_1 + p_2 = 0, \quad X_2 + p_2 q_2 + p_1 = 0$$

for the equations of the points p and q respectively, and the expression for the distance is given by

$$\cos^2 P/Q = \frac{(X_1 + p_1 q_1 + p_2)^2}{(X_1 + p_1 q_1 + p_2)^2 + (X_2 + p_2 q_2 + p_1)^2},$$

and if for brevity we write

$$D_1 = p_1^2 x_1^2 + p_1^2 x_2^2 + p_1^2 x_3^2 + p_1^2 x_4^2 + p_1^2 x_5^2,$$

then the values of X_1, p_1 , give

$$X_1^2 + p_1^2 + p_2^2 = (D_1)^2,$$

$$X_2^2 + p_2^2 + p_1^2 = (D_2)^2,$$

$$X_1^2 + p_1^2 + p_2^2 = (D_1 + D_2 + 1)^2,$$

I write $\sqrt{D_1 + D_2 + 1} = L, \quad \sqrt{D_1 + D_2 + 1} = M, \quad \sqrt{D_1 + D_2 + 1} = N,$

and I say that we have $L + M + N = 0$. The function becomes

$$\cos^2 P/Q = \frac{L^2 + M^2 + N^2}{L^2 + M^2 + N^2},$$

and we thence have

$$\sin^2 P/Q = \frac{(L^2 + M^2 + N^2)(L^2 + M^2 + N^2) - (L^2 + M^2 + N^2)(L^2 + M^2 + N^2)}{(L^2 + M^2 + N^2)(L^2 + M^2 + N^2)},$$

and consequently for the three pairs of factors we have

$$\sin^2 P/Q = \frac{L^2 + M^2 + N^2}{L^2 + M^2 + N^2} = \frac{L^2 + M^2 + N^2}{L^2 + M^2 + N^2} = \frac{L^2 + M^2 + N^2}{L^2 + M^2 + N^2} \quad *$$

3. The assumed relation $L + M + N = 0$ is of course verified by the equations

$$\begin{cases} L_1 + M_1 + N_1 = 0, \\ L_2 + M_2 + N_2 = 0, \\ L_3 + M_3 + N_3 = 0, \\ L_4 + M_4 + N_4 = 0, \end{cases}$$

which is identically true if $L_1, M_1, N_1, L_2, M_2, N_2, L_3, M_3, N_3, L_4, M_4, N_4$ satisfy the equations $x_1^2 + x_2^2 + x_3^2 = 0$, and L_1, M_1, N_1

$$\begin{cases} L_1^2 + M_1^2 + N_1^2 = 0, \\ L_2^2 + M_2^2 + N_2^2 = 0, \\ L_3^2 + M_3^2 + N_3^2 = 0, \\ L_4^2 + M_4^2 + N_4^2 = 0, \end{cases}$$

which is in fact the rationalized form of $L + M + N = 0$.

* Results are marked with an asterisk.

B. The foregoing values give

$$\left. \begin{aligned} \sin^2 pq, \sin^2 p'q', \sin^2 p''q'' &= 01, \\ \sin^2 pq + \sin^2 p'q' + \sin^2 p''q'' &= \frac{1}{4fgh} \frac{1}{h^3}, \\ \sin^2 pq + \sin^2 p'q' + \sin^2 p''q'' &= \frac{1}{4fgh} \frac{1}{h^3}, \end{aligned} \right\} \quad 3$$

4. Considering now, in connection with the foci, the point (x) determined by the equation $lx + my + nz = 0$, we have

$$\cos^2 xp = \frac{(lx + my + nz)^2}{(l^2 + m^2 + n^2)(x^2 + y^2 + z^2)},$$

λ, μ, ν as before, and therefore

$$\lambda^2 + \mu^2 + \nu^2 = (14)^2.$$

Moreover

$$(lx + my + nz)^2 = \begin{vmatrix} l & m & n \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}^2 = \begin{vmatrix} l^2 + m^2 + n^2 & 01 & 01 \\ 01 & & 14 \\ 01 & 14 & \end{vmatrix},$$

(if for shortness $01 = lx_1 + my_1 + nz_1$, &c.)

$$= (l^2 + m^2 + n^2)(14)^2 + 2 \cdot 01 \cdot 01 \cdot 14.$$

The formula thus is

$$\cos^2 xp = \frac{(l^2 + m^2 + n^2)(14)^2 + 2 \cdot 01 \cdot 01 \cdot 14}{(l^2 + m^2 + n^2)(14)^2} = \frac{14 + 2 \cdot 01}{14};$$

or passing to $\sin^2 xp$, and then writing down the analogous value of $\sin^2 xq$, we have

$$\sin^2 xp = \frac{2 \cdot 01 \cdot 01}{14 \cdot (l^2 + m^2 + n^2)}; \quad \sin^2 xq = \frac{2 \cdot 03 \cdot 03}{23 \cdot (l^2 + m^2 + n^2)};$$

and in like manner for the other two pairs of foci

$$\begin{aligned} \sin^2 xp' &= \frac{2 \cdot 02 \cdot 01}{21 \cdot (l^2 + m^2 + n^2)}; \quad \sin^2 xq' = \frac{2 \cdot 03 \cdot 01}{31 \cdot (l^2 + m^2 + n^2)}, \\ \sin^2 xp'' &= \frac{2 \cdot 03 \cdot 01}{31 \cdot (l^2 + m^2 + n^2)}; \quad \sin^2 xq'' = \frac{2 \cdot 01 \cdot 03}{13 \cdot (l^2 + m^2 + n^2)}. \end{aligned}$$

6. These formulae give

$$\sin^2 xp \sin^2 xq \sin^2 \frac{1}{2} p'q' + \sin^2 xp' \sin^2 xq' \sin^2 \frac{1}{2} p'q' + \sin^2 xp'' \sin^2 xq'' \sin^2 \frac{1}{2} p''q'' \\ = \frac{4 \cdot 01 \cdot 02 \cdot 03 \cdot 01}{(4fgh)^{\frac{1}{2}}} \cdot (l^2 + m^2 + n^2)^{-2};$$

or as this may also be written

$$= 4^{\frac{1}{2}} (01 \cdot 02 \cdot 03 \cdot 01) (12 \cdot 13 \cdot 14 \cdot 23 \cdot 24 \cdot 34)^{-\frac{1}{2}} (l^2 + m^2 + n^2)^{-2}, \quad 3$$

where it will be recollected that 01 denotes $lx_1 + my_1 + nz_1$, &c., and 12 denotes $x_1x_2 + y_1y_2 + z_1z_2$, &c. ...

6. Taking now (x, y, z) , &c., as the common tangents of the absolute and the conic, or say as the roots of the equations

$$x^2 + y^2 + z^2 = 0, \quad (a, b, c, f, g, h) \chi(x, y, z)^2 = 0,$$

the expression on the right hand side, qua symmetrical function, homogeneous of the degree zero in the roots, and also homogeneous of the degree zero in the coefficients l, m, n , will be expressible as an absolute invariant of the two quadric functions and of the linear function $lx + my + nz$: and I say that the value is

$$= -4^{-\frac{1}{2}} \cdot \square \cdot \Omega^{-\frac{1}{2}} (l^2 + m^2 + n^2)^{-\frac{1}{2}},$$

\square being the Resultant of the three functions, and Ω the Factinvariant of the two quadric functions, as presently appearing. It is to be observed that $l^2 + m^2 + n^2$ is in fact the Reciprocant of $lx + my + nz$ and $x^2 + y^2 + z^2$, viz. the Reciprocant of $(a, \dots) \chi(x, y, z)^2$ and $lx + my + nz$ is

$$(bc - f^2, ca - g^2, ab - h^2, gh - af, hf - bg, fg - ch) \chi(l, m, n)^2,$$

and for the quadric function $x^2 + y^2 + z^2$ this becomes $= l^2 + m^2 + n^2$.

7. Considering the three functions

$$\begin{aligned} &x^2 + y^2 + z^2, \\ &(a, b, c, f, g, h) \chi(x, y, z)^2, \\ &lx + my + nz, \end{aligned}$$

it will be sufficient as regards the resultant to write down those terms which are independent of f, g, h ; these are at once obtained by writing f, g, h each $= 0$, and the resultant then presents itself as the norm of $l\sqrt{b-c} + m\sqrt{c-a} + n\sqrt{a-b}$; and we thus obtain (attending only to the terms in question)

$$\begin{aligned} \square &= l^4 (b-c)^2 + m^4 (c-a)^2 + n^4 (a-b)^2 \\ &\quad - 2m^2 n^2 (c-a)(a-b) - 2n^2 l^2 (a-b)(b-c) - 2l^2 m^2 (b-c)(c-a). \end{aligned}$$

The Resultant is at once expressed in terms of the roots (x_1, y_1, z_1) , &c., by the formula

$$\square = G (lx_1 + my_1 + nz_1) (lx_2 + my_2 + nz_2) (lx_3 + my_3 + nz_3) (lx_4 + my_4 + nz_4),$$

or according to the foregoing notation

$$\square = G \cdot 01 \cdot 02 \cdot 03 \cdot 04,$$

where, and in what follows, G is written to denote an essentially indeterminate constant, having (it may be) different values in different equations.

8. Moreover writing as usual

$$K = abc - af^2 - bg^2 - ch^2 + 2fgh,$$

$$\Theta = bc - f^2 + ca - g^2 + ab - h^2,$$

$$\Theta' = a + b + c,$$

$$K' = 1,$$

the Tactinvariant is taken to be

$$\Omega = 27K^2K'^2 + 4KK'^3 + 4K'O^3 - 18KK'O'O' - O^2O'^2$$

(which for f, g, h each $= 0$, reduces itself to

$$\Omega = -(b-c)^2(a-a)^2(a-b)^2).$$

The Tactinvariant vanishes if, and only if, a pair of roots (x_1, y_1, z_1) , (x_2, y_2, z_2) become identical, say $x_1 : y_1 : z_1 = x_2 : y_2 : z_2$. But we have $(y_1z_2 - y_2z_1)^2$
 $(y_1^3 + z_1^3)(y_2^3 + z_2^3) - (y_1y_2 + z_1z_2)^3 = 0$, if $x_1x_2 + y_1y_2 + z_1z_2 = 0$, that is if $12 = 0$;
 and similarly $(z_1x_2 - z_2x_1)^2 = 0$, and $(x_1y_2 - x_2y_1)^2 = 0$, if $12 = 0$. And we are thus
 led to the equation

$$\Omega = O.12.13.14.23.24.34.$$

9. The combination $(1^3\Omega)^{-1}$ contains the roots homogeneously in the degree zero, and it will therefore have a determinate value, which is in fact found by the process which I present as a verification. The result is

$$(1^3\Omega)^{-1} = -64(01.02.03.04)^2(12.13.14.23.24.34)^{-1}.$$

In verification, take the function $(a, \dots \mathfrak{Q}x, y, z)^2$ to be $= x^2 + \omega y^2 + \omega^2 z^2$, ω an imaginary cube root of unity: the roots may be taken to be $(1, \omega^2, \omega)$, $(1, -\omega^2, \omega)$, $(1, \omega^2, -\omega)$, $(1, -\omega^2, -\omega)$. Attending only to the terms in 1 , we have

$$(1^3\Omega)^{-1} = -3!^2; \Omega = 27; 01.02.03.04 = 1!^4; 1^2.13.14.23.24.14$$

(is a product of factors each as $1 - \omega + \omega^2 = -2\omega$, and is) $= -64$; or the equation becomes $(-3!^2)^2(27)^{-1} = -64!^2(04)^{-1}$, which is right. We have thus

$$= -4114\Omega^{-\frac{1}{2}} = (01.02.03.04)(12.13.14.23.24.34)^{-\frac{1}{2}},$$

and hence the foregoing value of $\sin^2 \alpha p \sin^2 \alpha q \sin^{-\frac{1}{2}} \beta q$: say

$$\sin^2 \alpha p \sin^2 \alpha q \sin^{-\frac{1}{2}} \beta q = -4^{-\frac{1}{2}} \Omega^{-\frac{1}{2}} (l^2 + m^2 + n^2)^{-\frac{1}{2}}. \quad *$$

10. From the point (x) we draw to the conic tangents L, M : taking their coordinates to be (x_1, y_1, z_1) , (x_2, y_2, z_2) , these are the roots of

$$lx + my + nz = 0,$$

$$(a, \dots \mathfrak{Q}x, y, z)^2 = 0,$$

and we have

$$\sin^2 L.M = \frac{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)}.$$

11. We have $x_1^2 + y_1^2 + z_1^2 = 0$, or $x_2^2 + y_2^2 + z_2^2 = 0$, as the condition in order that the resultant (1) may vanish, and consequently

$$(1) = O(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2).$$

It is easy to see that the function in the numerator will vanish if $l^2 + m^2 + n^2 = 0$, or if the Reciprocant $(l, m, n)^2$ of the function $(a, \dots \mathfrak{Q}x, y, z)^2$ and $lx + my + nz$ be $= 0$; or calling this reciprocal P we have

$$(l^2 + m^2 + n^2)P = O\{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2\}.$$

The values of O in these two equations have a determinate ratio, and we find

$$\frac{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)} = -4P. (l^2 + m^2 + n^2)^{-1}.$$

and by reason hereof the foregoing numerator becomes

$$V(Vr^2 - 2\sigma_1\sigma_2r) + \sigma_1^2\sigma_2^2 = (Vr - \sigma_1\sigma_2)^2.$$

We thus have

$$\sin^2 XL \cdot \sin^2 XM = \frac{(Vr - \sigma_1\sigma_2)^2}{V^2\rho_1\rho_2},$$

$$= \frac{\{(a^2 + \beta^2 + \gamma^2)(x_1x_2 + y_1y_2 + z_1z_2) - (ax_1 + \beta y_1 + \gamma z_1)(ax_2 + \beta y_2 + \gamma z_2)\}^2}{(a^2 + \beta^2 + \gamma^2)^2 \cdot (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)}.$$

14. The numerator-function

$$(a^2 + \beta^2 + \gamma^2)(x_1x_2 + y_1y_2 + z_1z_2) - (ax_1 + \beta y_1 + \gamma z_1)(ax_2 + \beta y_2 + \gamma z_2)$$

$$\text{is } = (\beta z_1 - \gamma y_1)(\beta z_2 - \gamma y_2) + (\gamma x_1 - \alpha z_1)(\gamma x_2 - \alpha z_2) + (\alpha y_1 - \beta x_1)(\alpha y_2 - \beta x_2),$$

which vanishes if $\alpha : \beta : \gamma = x_1 : y_1 : z_1$ or $= x_2 : y_2 : z_2$, that is if $(\alpha, \dots, \gamma)^2 = 0$.

Moreover observing that $l : m : n = \beta z_1 - \gamma y_1 : \gamma x_1 - \alpha z_1 : \alpha y_1 - \beta x_1 = \beta z_2 - \gamma y_2 : \gamma x_2 - \alpha z_2 : \alpha y_2 - \beta x_2$, it also vanishes if $l^2 + m^2 + n^2 = 0$; and we hence have

$$U \{(a^2 + \beta^2 + \gamma^2)(x_1x_2 + y_1y_2 + z_1z_2) - (ax_1 + \beta y_1 + \gamma z_1)(ax_2 + \beta y_2 + \gamma z_2)\} \\ = (l^2 + m^2 + n^2) \cdot (\alpha, \dots, \gamma)^2$$

(viz. this equation is true when $la + m\beta + n\gamma = 0$; it is a particular case of a more general formula where α, β, γ are arbitrary, and there are on the right hand side terms containing the factor $la + m\beta + n\gamma$). And we have as before

$$U(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) = U^2.$$

Squaring each side of the first equation, and dividing by the two sides of the second equation, we obtain a determinate result which is

$$\frac{\{(a^2 + \beta^2 + \gamma^2)(x_1x_2 + y_1y_2 + z_1z_2) - (ax_1 + \beta y_1 + \gamma z_1)(ax_2 + \beta y_2 + \gamma z_2)\}^2}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)} \\ = \frac{(l^2 + m^2 + n^2)^2 \{(\alpha, \dots, \gamma)^2\}^2}{U^2},$$

viz. if to verify we assume as before $la + m\beta + n\gamma = x - z$; and $(\alpha, \dots, \gamma)^2 = x^2 + \omega y^2 + \omega^2 z^2$; consequently

$$\gamma : \alpha \text{ and } (\alpha, \dots, \gamma)^2 = \alpha^2 + \omega\beta^2 + \omega^2\alpha^2 = \omega(\alpha^2 - \beta^2);$$

$$\text{also } (x_1, y_1, z_1), (x_2, y_2, z_2) = (1, 1, 1), (1, -1, 1), \square = 0\omega^2,$$

then the equation becomes

$$\frac{\{2\alpha^2 + \beta^2 - (2\alpha + \beta)(2\alpha - \beta)\}^2}{U} = \frac{4\{\omega(\beta^2 - \alpha^2)\}^2}{0\omega^2},$$

which is right.

15. We hence have

$$\sin^2 XL \cdot \sin^2 XM = (l^2 + m^2 + n^2)^2 \{(\alpha, \dots, \gamma)^2\}^2 \cdot \square^{-1} \cdot (\alpha^2 + \beta^2 + \gamma^2)^{-1}, \quad *$$

and thence also

$$\sin^2 XL \cdot \sin^2 XM \sin^2 xp \sin^2 xq \sin^{-\frac{1}{2}} pq \\ = \dots \cdot \square^{-\frac{1}{2}} \cdot \{(\alpha, \dots, \gamma)^2\}^2 \cdot \square^{-\frac{1}{2}} \cdot (\alpha^2 + \beta^2 + \gamma^2)^{-1}, \quad *$$

which is the reciprocal of Clifford's formula, p. 157.]

ON THE THEORY OF DISTANCES*.

THIS communication relates to the following two theorems on the foci and asymptotes of curves.

Theorem i. L, M, N, \dots are the m tangents from a point a to a curve C_m of the m^{th} class; B is any line through a , meeting the curve in $m(m-1)$ points; $l, m, n, \dots P, Q, R, \dots$ are the $m(m-1)$ asymptotes of the curve, and p, q, r, \dots are a set of m foci.

$$\frac{\sin^2 LM \cdot \sin^2 LN \cdot \sin^2 MN \dots (\overline{ap}^2 \cdot \overline{aq}^2 \cdot \overline{ar}^2 \dots)^{m-1}}{al \cdot am \cdot an \dots \sin BP \cdot \sin BQ \cdot \sin BR \dots} = \overline{pq}^2 \cdot \overline{qr}^2 \cdot \overline{pr}^2 \dots$$

Theorem ii. l, m, n, \dots are the n intersections of a line A with a curve C_n of the n^{th} order; b is any point on A from which are drawn the $n(n-1)$ tangents; $L, M, N, \dots p, q, r, \dots$ are a set of $n(n-1)$ foci, and P, Q, R, \dots are the n asymptotes.

$$\frac{\overline{lm}^2 \cdot \overline{ln}^2 \cdot \overline{mn}^2 \dots (\sin^2 AP \cdot \sin^2 AQ \cdot \sin^2 AR \dots)^{n-1}}{\sin AL \cdot \sin AM \cdot \sin AN \dots lp \cdot lq \cdot lr \dots} \\ = \sin^2 PQ \cdot \sin^2 QR \cdot \sin^2 PR \dots$$

The numerator and denominator of the fraction on the left-hand side of the Equation in Theorem i. are quantities either of which I call the distance of the point a from the curve C_m . The corresponding quantities in Theorem ii. I call the Distance of the line A from the curve C_n . The reason of this is in the similarity of the analytical expressions for the distance of two geometrical forms in all cases, viz. the distance vanishes when the two forms have contact, and is infinite when either of them has contact with the "absolute." The "absolute" in plane geometry (so called by Professor Cayley) is the two circular points at infinity.

I also consider the modifications undergone by these theorems in the case of spherical curves. The method of investigation employed is an extension of the "geometric analysis" of Grassmann, itself a development of a remark of Leibnitz.

* [Notices and Abstracts... from Report of the thirty-ninth meeting of the British Association for the Advancement of Science, held at Exeter, August, 1869, p. 9.]

XVII.

ON A CASE OF EVAPORATION IN THE ORDER OF A RESULTANT*.

A PARTICULAR case of the following Theorem was required in the course of my proof that every rational equation has a root†; but I have thought that the theorem itself (though indeed a mere obvious remark) was worthy of being placed on record, because of the extremely small number of results of this kind that have yet been arrived at, and of their great importance in analysis.

Theorem. Let it be required to eliminate w between two equations homogeneous in w and certain other variables y, z, \dots , in which equations, however, w only occurs in virtue of the occurrence of a quantity

$$w = w^\alpha y^\beta z^\gamma \dots, \dots,$$

where

$$\alpha + \beta + \gamma + \dots = \mu;$$

let also m, n be the orders of the equations, and h, k the remainders after division of m, n respectively by μ ; then the order of the resultant is

$$\frac{mn - hk}{\mu}.$$

Demonstration. Suppose that p, q are the quotients of the division of m, n respectively by μ ; that is to say, let

$$m = p\mu + h, \quad n = q\mu + k,$$

* [From the *Proceedings of the London Mathematical Society*, Vol. III. Nov. 26, 20, pp. 801-823.]

† [See p. 22, *supra*.]

then the two equations may be written

$$a_m + a_{m-\mu} \cdot w + a_{m-2\mu} \cdot w^2 + \dots + a_n w^p = 0,$$

$$b_n + b_{n-\mu} \cdot w + b_{n-2\mu} \cdot w^2 + \dots + b_k w^q = 0,$$

where the suffixes of the several coefficients indicate their orders in the variables y, z, \dots . Instead of directly eliminating x from these equations, we may eliminate w ; and the result of this may be written down at once by Professor Sylvester's dialytic method. It is in fact

$$\begin{array}{l} \alpha_m \quad \alpha_{m-\mu} \dots\dots\dots \\ , \quad \alpha_m \dots\dots\dots \\ , \quad \quad \dots\dots\dots \\ \dots\dots\dots \\ \dots\dots\dots b_k \dots \\ \dots\dots\dots b_{k+\mu} \quad b_k \end{array}$$

where in the principal diagonal of the determinant the constituent a_m occurs q times, and the constituent b_k occurs p times. The order of the resulting term $(a_m)^q.(b_k)^p$ is $mq+kp$, and this therefore (since the determinant must be homogeneous) is the order of the resultant itself. If we had written the b coefficients before the a coefficients, we should have obtained $np+hq$ as the value of the same quantity. These two values are identical, since by hypothesis

$$q(m-l) = p(n-l) = pq\mu,$$

and therefore $qm + pl = pn + qh = r$, suppose.

$$\begin{aligned} \text{Now } mn &= (p\mu + h)(q\mu + k) \\ &= pq\mu^2 + pk\mu + hq\mu + hk \\ &= \mu r + hk; \\ \therefore r &= \frac{mn - hk}{\mu}, \end{aligned}$$

's was to be proved.

The following extension is brought to light by a different method of proving the original theorem.

Let it be required to eliminate $k-1$ variables x, y, \dots from k equations, homogeneous in these, and certain other vari-

ables, in which equations, however, x, y, \dots only occur in virtue of the occurrence of $k-1$ quantities u, v, \dots all of the same order μ ; let also m_1, m_2, \dots, m_k be the orders of the equations, and

$$m_i = p_i \mu + h_i, \quad h_i < \mu;$$

then the order of the resultant is

$$\Pi p \left(\sum \frac{h}{p} + \mu \right).$$

For the equations may be written

$$a_h(u, v, \dots)^p + \theta a_{h_1 \mu}(u, v, \dots)^{p-1} + \dots + \theta^p a_{h+p\mu} = 0,$$

where the h, p are to be affected successively with the suffixes $1, 2, \dots, k$, and θ may be considered $= 1$. Now these equations may be regarded as having coefficients of the constant order h , but the weight of every coefficient of θ^r equal to $r\mu$. This being so, the degree of the resultant in the uneliminated variables will be the sum of its order and weight calculated on these suppositions. But its order is $h_1 p_1 p_2 \dots p_k$, or $\frac{h_1}{p_1} \Pi p$, due to the coefficients of the first equation, $\frac{h_2}{p_2} \Pi p$ due to the coefficients of the second, and so on; while its weight is $\mu \Pi p$. Hence the entire order of the resultant is

$$\Pi p \left(\sum \frac{h}{p} + \mu \right),$$

as stated above.

XVIII.

ON A THEOREM RELATING TO POLYHEDRA, ANALOGOUS TO MR COTTERILL'S THEOREM ON PLANE POLYGONS*.

MR COTTERILL'S theorem, presented last year to the Society, is as follows: For every plane polygon of n vertices there is a curve of class $n - 3$ touching all the diagonals; the number of diagonals is such as to exactly determine this curve and no more; and when the curve touches the line at infinity, the area of the polygon is zero.

The proof of this depends essentially upon the fact that if we join the vertices of the polygon to any point in its plane, the area of the polygon is equal to the sum of the triangles so formed, taken of course with their proper signs according to the rule of Möbius.

The analogous theorem in space should therefore apply in the first instance to those solids whose volume can be expressed as the sum of tetrahedra, having one vertex at an arbitrary point of space, and the other three at three vertices of the figure; that is to say, it should apply to solids having *triangular faces*.

For such solids I find accordingly that the analogy is very complete and exact. It is convenient to define a plane containing three vertices but not being a face, as a diagonal plane; and

* [From the *Proceedings of the London Mathematical Society*, Vol. iv. No. 51, pp. 178—185. Mr Cotterill's paper is given, in part, in Vol. iv. No. 49.]

a line joining two vortices but not being an edge, as a diagonal line. This being so, the theorems which I shall prove are the following:

For every polyhedron of n summits having only triangular faces (Δ -faced n -acron (AYLEY) there is a surface of class $n-4$ touching all the diagonal planes.

This surface contains all the diagonal lines.

The diagonal planes and lines are so situated, however, that the conditions of touching the planes and containing the lines are precisely sufficient to determine a surface of class $n-4$.

When this surface touches the plane at infinity, the volume of the solid is zero.

To apply these propositions to polyhedra having other than triangular faces, we must consider such polygonal faces as *singularities*. Each of them, in fact, may by a small deformation of the polyhedron be resolved into a certain number of triangles; and we may thus regard a quadrangular face, for example, as the special case of two adjacent triangular faces being in one plane. Thus the quadrangular face $abcd$ [fig. 18] may be regarded as produced by coplanarity of the triangles abd , cbd . The effect of this is also to unite together the two diagonal planes abc , adc , and to make the diagonal line ac lie in the face. Thus the surface of class $n-4$ must touch the face $abcd$; but it does not in general contain the lines ac , bd . It touches the face at their point of intersection. And, in general, it is not necessary to consider the diagonals of a polygonal face as diagonals of the polyhedron, and they do not in fact lie upon the surface d_{n-4} . But a polygonal face with m vertices is a multiple tangent plane of order $m-3$, and the curve of contact is Mr Cotterill's curve appertaining to the polygon.

It is interesting to consider from this point of view the correlative propositions. Just as we have regarded a solid with a given number of summits, or *polyacron*, as having normally or in the most general case only triangular faces, while polygonal faces present themselves as singularities, and polyacra possessing them as degenerate forms; so we must regard a *polyhedron*, or

a, b, c, \dots relatively to the plane X , but absolutely independent of the position of p . If, then, we make the equation integral, by multiplying both sides into $pX, lX, mX, nX, \Pi.fX$, we see that the expression

$$\Sigma (p f g h) \frac{\Pi.fX}{fX.gX.hX}$$

must be divisible by pX ; because its equivalent on the other side is so divisible, and the equation is an identity so far as p is concerned. The result of the division is of the order $n-4$ in X ; or, which is the same thing, if X be regarded as a variable plane, the equation

$$\Sigma (p f g h) \frac{\Pi.fX}{pX.fX.gX.hX} = 0, \dots\dots\dots(2)$$

represents a surface of class $n-4$.

Two things are now clear from our previous equation and from the form of this one.

1°. If the equation is satisfied when X is the plane at infinity, then $V=0$; or, if the surface (2) touch the plane at infinity, the volume of the solid is zero.

2°. The equation (2) is satisfied if $lX=0, mX=0, nX=0$, where l, m, n are any three vertices not in the same face. Therefore the surface (2) touches all the diagonal planes.

The investigation, so far, is a mere reproduction of that of Mr Clutterbill, with the addition of an extra letter to apply it to three dimensions instead of two. I shall take the liberty of calling the surface thus arrived at the *index-surface* of the polyhedron, and shall denote it by the symbol v_{n-4} .

The index-surface contains all the diagonal lines. For let ab be a diagonal line, and o any other summit of the solid; then abo is a diagonal plane. For if it were a face, ab would be an edge, contrary to the supposition. Consequently $n-2$ diagonal planes can be drawn through every diagonal line; now all these are touched by the index-surface. But if more than $n-4$ tangent planes can be drawn through a straight line to a surface of class $n-4$, the line must lie altogether in the surface.

Through an edge of the solid, on the other hand, two faces and $n-4$ diagonal planes can be drawn; which latter are, of course, the tangent planes from it to the surface.

There are certain diagonal planes which it is convenient to consider separately. They are those which contain three edges of the solid; and I shall call them *single planes*. A diagonal plane may, of course, contain three diagonal lines, or two, or one, or none; but if it contains any diagonal line, the condition of touching it is already involved in the condition of containing that line. So that the facts we know about the index-surface may be summed up in saying that it passes through all the diagonal lines and touches all the single planes.

I now go on to prove that in general these conditions are precisely sufficient to determine a surface of class $n-4$. In order to do this, it will be necessary to make use of the researches of Prof. Cayley upon the Δ -faced polyacra, contained in the 1st volume of the 3rd series of the *Manchester Memoirs*, p. 248; particularly of the following passage:—

“An n -acron has n summits, $3n-6$ edges, $2n-4$ faces; and it is easy to see that there are the following three cases only, viz.:

1. The polyacron has at least one tripleural summit.
2. The polyacron, having no tripleural summit, has at least one tetrapleural summit.
3. The polyacron, having no tripleural or tetrapleural summit, has at least twelve pentipleural summits.

In fact, if the polyacron has c tripleural summits, d tetrapleural summits, e pentipleural summits, and so on, then we have

$$n = c + d + e + f + g + h + \&c.,$$

$$6n - 12 = 3c + 4d + 5e + 6f + 7g + 8h + \&c.;$$

$$\text{and therefore } 12 = 3c + 2d + e + 0f - g - 2h - \&c.,$$

$$\text{or } 3c + 2d + e = 12 + g + 2h + \&c.;$$

whence, if $c = 0$ and $d = 0$, then $e = 12$ at least.”

Upon this theorem Prof. Cayley founds a method of deriving all polyacra with $n+1$ summits from those with n summits. If we remove from a polyacron a triploural summit, as a in the figure [fig. 19], we may derive from it a new polyacron with one summit less by regarding the diagonal plane bed as a face of the new solid. Conversely, we may add one summit to any polyacron by crowning any one of its faces with a triploural summit, and then regarding this face as a diagonal plane. This process is called by Prof. Cayley the First Process. In a similar manner, the skew quadrilateral $bead$ [fig. 20] formed by two adjacent faces may be crowned by a tetraploural summit a , with the edges ab, ac, ad , the faces bed, ead becoming diagonal planes of the new solid; this is called the Second Process. Again, the skew pentagon $befed$ [fig. 21] formed by three adjacent faces may be crowned by a pentiploural summit a , with the edges ab, ac, ad, ae, af , the faces bed, ead, eaf becoming diagonal planes; this is called the Third Process. And it appears from the theorem quoted above, that every $(n+1)$ -acron can be made out of an n -acron by one or other of these processes, according as it belongs to the first, second, or third case of the theorem.

I shall now show, then, that if the conditions of containing the diagonal lines and touching the single planes are precisely sufficient to determine the index-surface of an n -acron, then the same thing will be true for any $(n+1)$ -acron derived from it by either of these processes. This will prove that the theorem is true for all N -faced polyacra, provided we can show that it is true for all pentacra. Now there is only one pentacron, the figure formed of two tetrahedra with a common face, $abeda$ [fig. 22]. This figure has the diagonal line ac and the single plane bed ; and the index surface is the point v , which is precisely determined as the intersection of these.

In determining the number of conditions involved in passing through a system of lines, we must remember that every intersection of two lines diminishes the number by one, except where three or more lines are in one plane. We have only to deal with the case of three lines in one plane; the number of conditions is then reduced by two for the three intersections.

First Process.—Let D be the number of diagonal lines of the n -acron; then when we pass to the $(n+1)$ -acron, the following is the increase in the number of conditions:

| | |
|---|-----------------------------|
| The D -lines are on a surface of class $n-3$ | |
| instead of $n-4$; this makes an increase | $+ D$ |
| There are $n-3$ new diagonals joining the | |
| new summit a to all the old summits | |
| except b, c, d | $+ (n-2) (n-3)$ |
| One or other of these, however, meets each | |
| of the old ones at least once; which is | |
| all that need be counted, because if any | |
| old diagonal meets two new ones a | |
| triangle is formed | $- D$ |
| The new diagonals all meet in a point, count- | |
| ing as $\frac{1}{2} (n-3) (n-4)$ intersections..... | $- \frac{1}{2} (n-3) (n-4)$ |
| There is a new single plane bcd | $+ 1$ |

The total increase is therefore $\frac{1}{2} (n-1) (n-2)$,
 which is the difference between the number of conditions re-
 quired to determine a surface of class $n-3$ and the number
 required for class $n-4$.

Second Process.—Let D be the number of old diagonal lines less be ; then we have the following increase:

| | |
|---|-----------------------------|
| The D lines on surface of higher class..... | $+ D$ |
| There are $n-4$ new diagonals..... | $+ (n-2) (n-4)$ |
| One or other of these, however, meets each | |
| of the D lines at least once; and, as | |
| before, this is all that need be counted... | $- D$ |
| The new diagonals all meet in a point, count- | |
| ing as $\frac{1}{2} (n-4) (n-5)$ intersections..... | $- \frac{1}{2} (n-4) (n-5)$ |
| The edge cd becomes a diagonal line | $+ (n-2)$ |
| If, however, we join this edge to the $n-4$ | |
| summits different from bcd , there is a | |
| reduction 1 in the case of each; for | |
| either the plane was a single plane, or | |
| it contained one or two diagonals..... | $- (n-4)$ |
| The diagonal be is on surface of higher class | $+ 1$ |
| Total, as before | $\frac{1}{2} (n-1) (n-2)$. |

Third Process.—Let D be the number of old diagonal lines, less df, fb, bc ; then we have the following increase :—

| | |
|---|---------------------------|
| The D lines on surface of higher class..... | + D |
| There are $n-5$ new diagonals..... | + $(n-2)(n-5)$ |
| Intersections of these with D lines..... | - D |
| Now diagonals meet in a point..... | - $\frac{1}{2}(n-5)(n-6)$ |
| The edges cd, ce become diagonal lines..... | + $2(n-2)$ |
| If we join these to the $n-5$ summits different from $bedcf$, there is a reduction 1 for each plane..... | - $2(n-5)$ |
| The diagonals df, fb, bc on surface of higher class..... | + 3 |
| Their intersections with cd, ce , and of these with one another..... | - 3 |
| Total, as before..... | $\frac{1}{2}(n-1)(n-2)$. |

Passing now to the consideration of polygonal faces, I remark first that, by direct application of Mr Cotterill's theorem, we have an expression for the volume of the pyramid standing on a plane polygon. For let p be the vertex of the pyramid, q any point in the plane of the polygon; then we have

$$\sum' \frac{(pqab)}{pX \cdot qX \cdot aX \cdot bX} = U \frac{(1234)}{1X \cdot 2X \cdot 3X \cdot 4X},$$

in which U is the ratio of the volume of the pyramid to that of the fundamental tetrahedron, and Σ' refers to a summation going round the m sides of the polygon in order. From this it appears that the expression

$$\frac{1}{qX} \cdot \sum' \{(pqab) \cdot \Pi' \cdot cX\}$$

(in which $\Pi' \cdot cX$ means a product involving all the vertices of the polygon except a and b) is integral, independent of q and of the order $m-3$ in X . If we equate it to zero, we in fact obtain the equation of Mr Cotterill's curve belonging to the polygon.

Now if this polygon form a face P of a polyacron, we obtain, as before, the following expression for the volume of the solid ;

$$\begin{aligned} \Sigma' \frac{(pqub)}{pX \cdot qX \cdot aX \cdot bX} + \Sigma' \frac{(pfgh)}{pX \cdot fX \cdot gX \cdot hX} \\ = V \cdot (1X \cdot 2X \cdot 3X \cdot 4X) \end{aligned} \quad (1234)$$

Thus the equation of the index-surface may be written

$$\Pi \cdot fX \cdot \Sigma' [(pqub) \cdot 1X \cdot cX] + \frac{1}{pX} \cdot \Sigma [(pfgh) \cdot \Pi \cdot aX] = 0.$$

Here $\Pi \cdot aX$ must in every case contain $m - 2$ factors at least belonging to points on the plane P ; or it vanishes in the order $m - 2$ when the coordinates of P are substituted in it. The term Σ' vanishes as we have seen in the order $m - 3$ in the same case. Thus P is a multiple tangent plane of order $m - 3$, and the curve of contact is determined by the term Σ' ; that is to say, it is Mr Clutterbill's curve belonging to the polygon.

Fig. 18.

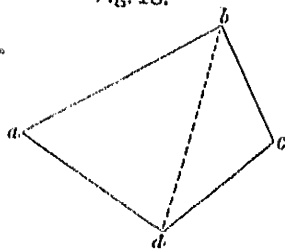


Fig. 19.

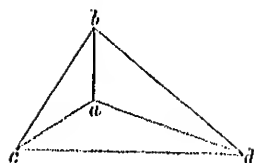


Fig. 20.

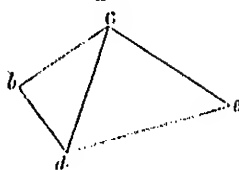


Fig. 21.

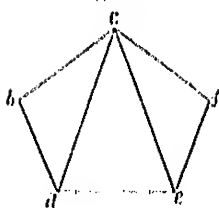
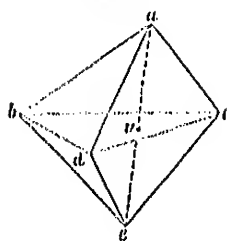


Fig. 22.



XIX.

GEOMETRY ON AN ELLIPSOID*.

THE metric properties of an ellipsoid are entirely determined by the four points in which it is met by the imaginary circle at infinity. I shall start, therefore, by assuming the existence of these four coplanar points a_1, a_2, a_3, a_4 , which, taken all together, I call the absolute.

1. To represent the ellipsoid on a plane we require also two fixed points i, j ; the plane sections of the ellipsoid are then represented by conics through these points, and the generating lines by lines on the plane through them. In fact, if we take a fixed point a on the ellipsoid E_2 , and draw a line through a and a variable point x on the ellipsoid, this line will meet a plane L in one point y , which is the representative of x ; the points i, j will then represent the generators through a . If we take the points i, j to be the absolute of the plane L , then all the plane sections will be represented by circles, the lines through i will represent one system of generating lines, and the lines through j the other. We shall have then, in addition, to consider the four points a ; and the geometry of the ellipsoid will be merely the geometry of the plane considered in relation to these four points, which are conyclic.

2. We know, then, that the antipoints of the a lie upon three new circles, orthotomic of each other and of the first. These correspond to the principal sections of the ellipsoid. The antipoints themselves represent umbilici, four of which are real;

* [From the *Proceedings of the London Mathematical Society*, Vol. iv. No. 54, pp. 215-217.]

I call these u_1, u_2, u_3, u_4 , and we may now take the point o absolute instead of the points o .

3. What now are the directions of the lines of curvature at any point x of the ellipsoid? First, the indicatrix at x is represented by the point-circle at its corresponding point y . Conjugate directions at x correspond to rectangular directions at y . Next, the tangent plane at x meets the plane at infinity in a line, say β . Through the points o can be drawn two lines to touch β , say at p, q . The lines xp, xq are tangents to the lines of curvature at x , since the points p, q are conjugate points of the section of the ellipsoid at infinity and of the indicatrix circle. But now let $o_1 o_2$ and $o_3 o_4$ meet β in r, s respectively. Then the involution made by rs and the points where the imaginary circle have p, q for double points. The interpretation of this on the plane is, that the directions corresponding to xp, xq make equal angles with the directions $yo_1 o_2, yo_3 o_4$. Hence—

The lines of curvature of the ellipsoid are represented by confocal anallagmatics having the u for foci.

Sections made by two conjugate planes of the ellipsoid are represented by orthotomic circles.

4. A straight line γ in space may be denoted by points c_1, c_2 , where it meets the ellipsoid. The section through this line will be represented by the series of circles through c_1, c_2 . The sections through δ , the polar of γ , will therefore be represented by the series of coaxial circles through d_1, d_2 , the antipoints of c_1, c_2 . Thus, a straight line being represented by a pair of points, its polar is represented by their antipoints; as is otherwise obvious.

I denote further the principal sections and the sections at infinity by $XYZU$, which notation will serve also for the lines which represent them. Now, in general, a section of the ellipsoid through a fixed point of space is represented by a circle orthotomic of a fixed circle. In particular, the points $o_1, o_2, o_3, o_4, o_5, o_6, o_7, o_8$, correspond in this way to the lines $XYZU$. I want now to find the interpretation on the

the rectangularity of the lines γ and δ . The planes joining them to the point a_1a_3, a_2a_4 are harmonic of the lines a_1a_3, a_2a_4 . Hence the circles c_1c_3X, d_1d_3X are harmonic of the circles coaxial with them and passing through a_1a_3, a_2a_4 respectively. This is to be true when X and Y are interchanged: the conditions may finally be written

$$\frac{c_1c_3YZ}{d_1d_3YZ} = \frac{c_1c_3ZX}{d_1d_3ZX} = \frac{c_1c_3XY}{d_1d_3XY}.$$

If now for d_1, d_3 we may substitute c_1, c'_1 where c'_1 is indefinitely near to c_1 in any direction, c_1c_3 represents the normal at c_1 .

5. A circle P , orthotomic of U , represents a diametral section. Let the pole of this section be called p ; p is a point at infinity. We know that it is always possible to find another point q at infinity, which is conjugate to p with respect both to the ellipsoid and to the imaginary circle. We may then endeavour to find the circle Q , of which q is the pole. Further, lines λ, μ can be drawn through p, q respectively, which are at right angles, and also conjugate polars of the ellipsoid. To represent these we must find a pair of points on P which have their antipoints on Q . These circles cut orthogonally; on each of them, then, there is a singly infinite number of point-pairs representing axes of the quadric, viz., the point-pairs determined by diameters of the other circle. That is to say, any circle P , orthotomic of U , being given, there can always be found a point q^0 , such that the lines through q^0 determine on P point-pairs representing axes of the quadric.

The determination of q^0 depends on the position of the projecting point a . The generators through a meet the diametral section Q in two points; the remaining generators through these intersect on the representative of q^0 .

6. I now proceed to construct Q when P is given. In the first place, Q has to be orthotomic of P and U . Next, if we draw through P and Q two new circles, one of which has a_1, a_2 for harmonics, and the other a_3, a_4 , these must be harmonics of

P and Q . But a circle orthotomic of U and having o_1, o_2 for harmonics, must have them for inverse points, and therefore have its centre on $o_1 o_2$. Hence the line joining the centres of P and Q is cut harmonically by the lines $o_1 o_2, o_3 o_4$. Similarly, it is cut harmonically by $o_1 o_3, o_4 o_2$, and by $o_1 o_4, o_2 o_3$. Hence the centres of P and Q are polar opposites in regard to the quadrangle $o_1 o_2 o_3 o_4$. They are therefore conjugate points in regard to the circle U .

7. Intersections of the ellipsoid by spheres are represented by anallagmatics *passing through* the four points o_1, o_2, o_3, o_4 . There are two systems of real circles passing through pairs of them; these represent the circular sections. Sphero-conics are represented by such of these anallagmatics as have $XYZU$ for focal circles. To find the axes of any circle P of the U system we must then draw two such anallagmatics having double contact with P ; the point of contact in pairs will represent the axes of the corresponding section.

XX.

PRELIMINARY SKETCH OF BICQUATERNIONS*.

I.

THE *vectors* of Hamilton are quantities having magnitude and direction, but no particular position; the vector AB being regarded as identical with the vector CD when AB is equal and parallel to CD and in the same sense. The translation of a rigid body is an example of such a quantity; for since all particles of the body move through equal distances along parallel straight lines in the same sense, the motion is entirely specified by a straight line of the given length and direction drawn through any point whatever. A couple, again, may be adequately represented by a vector; since the axis of a couple is any line of length proportional to its moment drawn perpendicular from a given face of its plane.

For many purposes, however, it is necessary to consider quantities which have not only magnitude and direction, but *position* also. The rotational velocity of a rigid body is about a certain definite axis, and equal rotations about two parallel axes are not equivalent to one another. A force acting upon a solid has a definite line of action, and equal forces acting along parallel lines differ by a certain couple. The difference between the two kinds of quantities is clearly seen when we consider the geometric calculus which is used for the study of each. In

* [From the *Proceedings of the London Mathematical Society*, Vol. iv Non. 61, 65, pp. 381—395.]

studying the motions of a particle or the composition of couples, the only construction required is that of the "line polygon," and the theory involved is that of the addition of vectors, but in the static or kinematic of solids we require, in addition, the construction of the "link polygon," and there is needed the theory of the involution of lines in space, or of the linear complex.

The name *vector* may be conveniently associated with a velocity of *translation*, as the simplest type of the quantity denoted by it. In analogy with this, I propose to use the term *rotor* (short for *rotator*) to mean a quantity having magnitude, direction, and position, of which the simplest type is a velocity of *rotation* about a certain axis. A rotor will be geometrically represented by a length proportional to its magnitude, measured upon its axis in a certain sense. The rotor AB will be identical with CD if they are in the same straight line, of the same length, and in the same sense, or a vector may be considered parallel to itself, but a rotor only in its own line.

The *addition* of rotors will be governed by the rules of both of these, the composition of forces and rotations. Here, however, we come upon a very important break in the analogy between rotors and vectors. While the sum of any number of vectors is always a vector, it will only happen in special cases that the sum of a number of rotors is a rotor. In fact, the composition of two forces whose lines of action do not intersect, is a rotation-velocity whose axis is the most interesting point of the system of forces on the one hand, and the most general velocity of a rigid body on the other. These still proper couple systems have been studied, and the theory of three rotations in a certain position completely worked out, by Dr Hall.

A system of forces may be reduced to one force P , and a couple C whose plane is perpendicular to the line of action of the force, or several axes. If I think of either of the system of forces as a screw about a certain axis, the axis of the screw being the central axis, and the pitch being the ratio $\frac{C}{P}$ of the couple to the force. Similarly the velocity of a

rigid body may be represented in one way only as a rotation-velocity ω about a certain axis combined with a translation-velocity v along that axis. Dr Ball speaks of this velocity as a *twist*-velocity about a certain screw; the axis of this screw being the axis of rotation, and its pitch the ratio $\frac{v}{\omega}$ of the translation to the rotation. A *screw* is here a geometrical form resulting from the combination of an *axis* or straight line given in position with a *pitch* which is a linear magnitude. A *wrench* is the association with this geometrical form of a magnitude whose dimensions are those of a force; a *twist*-velocity the association of a magnitude whose dimensions are those of an angular velocity. The extreme convenience of this nomenclature is well exemplified in the remarkable memoir above referred to.

Just as a vector (translation velocity or couple) is magnitude associated with direction, and as a rotor (rotation velocity or force) is magnitude associated with an axis; so this new quantity, which is the sum of two or more rotors (twist-velocity or wrench), is magnitude associated with a screw. Following up the analogy thus indicated, I propose to call this quantity a *motor*; the simplest type of it being the general motion of a rigid body. And we shall say that in general the sum of rotors is a motor, but that in particular cases it may degenerate into a rotor or a vector.

II.

A *quaternion* is the ratio of two vectors, or the operation necessary to make one into the other. Let the vectors be [fig. 23] AB and AC , as they may both be made to start from any arbitrary point A . Then AB is made into AC by turning it round an axis through A perpendicular to the plane BAC until its direction coincides with that of AC , and then magnifying or diminishing it until it is of the same length as AC . The ratio of two vectors then is the combination of an ordinary numerical ratio with a rotation; or, as Hamilton expresses it, a quaternion is the product of a tensor and a versor. Since the point A is perfectly arbitrary, this rotation is not about a definite axis;

numerical ratio with a *twist*. This twist is associated with a perfectly definite screw, and is only specified when its angular magnitude and the screw (involving direction, position, and pitch) are given. We may say also that just as the rotation (versor) involved in a quaternion is the ratio of two directions, so the twist involved in the ratio of two rotors is really the ratio of their axes.

Here again a remark must be made about the range of this operation. Using the expression *tensor-twist* to mean the ratio of two rotors (which is in fact a twist multiplied by a tensor), we may say that a tensor-twist can operate upon any rotor which meets its axis at right angles. Let t denote the operation which converts AB into CD , so that $t = \frac{CD}{AB}$, and $t \cdot AB = CD$; then if EF be any other rotor which meets AC at right angles, the expression $t \cdot EF$ will have a definite meaning, viz., it will mean a rotor obtained by sliding EF along a distance equal to AC , turning it about AC as axis through an angle equal to BAF , and altering its length in the ratio $AB : CD$. But if EF be a rotor not meeting AC , or meeting it at any other than a right angle, the expression $t \cdot EF$ will have no meaning whatever.

We have now defined the ratio of two rotors, and shown that like a quaternion it has a restricted range of operation. The question naturally arises, What now is the operation which converts one *motor* into another? We can answer this question very easily in the case in which the two motors have the same pitch; for in this case their ratio is a tensor-twist whose tensor is the ratio of their magnitudes and whose twist is the ratio of their axes. We are led to this by considering each motor as the sum of two rotors which do not intersect. Let α and β be two such rotors, t a tensor-twist whose axis meets them both at right angles; then $t\alpha$ is a rotor, say γ , and $t\beta$ is another rotor, say δ . If therefore we assume the distributive law, we have

$$t(mx + n\beta) = m\gamma + n\delta,$$

$$\text{or} \quad t = \frac{m\gamma + n\delta}{mx + n\beta}.$$

It is a mere translation of known theorems to say that the axis of t meets at right angles the axes of the motors $mx + ny$ and $my + nz$, and that one of these axes is converted into the other by the same twist that makes α into γ or β into δ .

The solution of this problem in the general case in which the pitches are different, is not so easy. In the first place, we must remember that every motor consists of a rotor part and a vector part, and that its pitch is determined by the ratio of these two parts. By combining a suitable vector with a motor, therefore, we may make the pitch of it anything we like, without altering the rotor part. Now let it be required to find the operation which will convert a motor A into a motor B . Let B' be a motor having the same rotor part as B , and the same pitch as A ; and let $B = B' + \beta t$, where βt is a vector parallel to the axis of B . Then the ratio $\frac{B}{A} = \frac{B'}{A} + \frac{\beta t}{A}$; but $\frac{B'}{A}$ is a tensor-twist, say t , and we may write

$$\frac{B}{A} = t + \frac{\beta t}{A},$$

where it now only remains to find an operation which will convert a motor A into a vector βt .

In order to do this, we must introduce a symbol whose nature and operation will at first sight appear completely arbitrary, but will be justified in the sequel. *The symbol ω , applied to any motor, changes it into a vector parallel to its axis and proportional to the rotor part of it.* That is to say, it changes rotation about any axis into translation parallel to that axis and a force into a couple in a plane perpendicular to its line of action. But if the rotation is accompanied by translation or the force by a couple, this symbol takes no account whatever of these accompaniments; and if made to operate directly on a vector, reduces it to zero. It follows from this that if it be made to operate twice upon a motor, it reduces it to zero; or $\omega^2 A = 0$ always. The portion of any expression which involves ω must therefore be treated as an infinitesimal of the first order; all higher orders being uniformly neglected.

Since then $\omega A = \alpha$, a vector, and the ratio $\frac{\beta}{\alpha}$ is a quaternion q so that $q\alpha = \beta$, we may write successively

$$\beta = q\alpha = q\omega A,$$

$$\frac{\beta}{A} = q\omega,$$

and then

$$\frac{\beta}{A} = l + q\omega,$$

or the ratio of two motors may be expressed as the sum of two parts, one of which is a tensor-twist, and the other is ω multiplied by a quaternion.

The same ratio may be expressed in another form. Let an arbitrary point O be assumed as the origin; then every motor may be expressed in one way as the sum of a rotor passing through O and a vector. Now the theory of rotors passing through a fixed point is exactly the same as that of vectors in general, and the ratio of any two of them is a tensor-twist whose pitch is zero, or what is the same thing, a quaternion whose axis is constrained to pass through the fixed point. If we use cursive (Greek) letters (as α, β) in general to represent rotors through the origin, we may distinguish vectors from them by prefixing the symbol ω ; thus $\omega\alpha$ denotes a vector parallel and proportional to the rotor α . The ratio $\frac{\beta}{\alpha}$ will then be a

quaternion q , which is also the ratio $\frac{\omega\beta}{\omega\alpha}$ *. The general expression for a motor is then $\alpha + \omega\beta$. Let it now be required to find the ratio of two motors $\alpha + \omega\beta, \gamma + \omega\delta$; or the value of the expression

$$\frac{\gamma + \omega\delta}{\alpha + \omega\beta}.$$

First, let $\frac{\gamma}{\alpha} = q$; then $q(\alpha + \omega\beta) = \gamma + q\omega\beta = \gamma + \omega q\beta$.

The symbol $q\beta$ has at present no geometrical meaning; for in general the rotors α, β, γ will not be coplanar, and cannot

* It follows from this that $\omega q = q\omega$, or ω is commutative with quaternions.

therefore be operated on by the same quaternion q . If however (as in the Calculus of Quaternions) we consider all these quantities as expressed in terms of three rectangular unit vectors through the origin, $\frac{\tilde{h} - q\tilde{l}}{a}$ will be a perfectly definite quaternion x . The equation

$$ra + \tilde{h} = q\tilde{l}$$

is, like the equation $q(1 + a\tilde{l}) = q(1 + aq\tilde{l})$,

at present purely literal and devoid of meaning. Yet if (remembering the properties of the symbol a) we add to three the first equation to the second and assume the distributive law, we obtain

$$(q + 1)a\tilde{l} = (1 + a\tilde{l})\tilde{h} = q + 1a\tilde{h}$$

In this way the ratio $\frac{q + 1a\tilde{h}}{1 + a\tilde{l}}$ is expressed in the form $q + 1a\tilde{h}$,

which expression may conveniently be called a *biquaternion**, The final equation, however, is not susceptible of interpretation in the same sense as the equation $q\tilde{l} = q$. The expression $q + 1a\tilde{h}$ does not denote the sum of geometrical operations which can be applied to the vector $a + 1a\tilde{h}$ as a whole, and the ratio of two vectors is only expressed by a symbol as the sum of two parts, each of which separately has a distinct meaning in certain other cases, but not in the case in point. In following sections this difficulty will be partly overcome by showing that the system here sketched is the limit of another in which it does not occur.

The preceding remarks may however explain, and be illustrated by, the following table:

| Geometrical Form | Quaternion | Biquaternion | Remark |
|-----------------------|--------------------|--|-------------------|
| Direction on st. line | Vector on st. line | Addition and Subtraction | Directed Motion |
| Direction in plane | Vector in plane | Composition, Resolution | Rotational Motion |
| Direction in space | Vector in space | Translation, Rotation | Translation |
| Axis | Motor | Rotation Velocity, Power | Axis |
| Scalar | Motor | Translation Velocity, Rotation of Rotors | Biquaternion |

* Hamilton's *biquaternion* is a quaternion with complex coefficients; but it is convenient (as Prof. Peirce remarks) to suppose from the beginning that a

III.

That geometry of three-dimensional space which assumes the Euclidian postulates has been called by Dr Klein the *parabolic* geometry of space, to distinguish it from two other varieties, which assume uniform positive and negative curvature respectively, and which he calls the *elliptic* and *hyperbolic* geometry of space. The investigations which follow involve the postulates of elliptic geometry. As, however, the postulate of uniform positive curvature is not sufficient to define this, it may be worth while to devote a short space to an explanation of its nature.

Space of three dimensions is that the points of which may be associated with systems of values of three variables x, y, z . It is not in general possible, however, so to make this association that to every system of values there shall correspond in general one point, and to every point in general one system of values. When this is the case, the space is called *unicursal*. An *algebraic* space is one in which the position of a point may be uniquely defined by a set of values of periodic algebraic integrals, without exceptions which form a part of the space. Thus, unicursal spaces are a particular case of algebraic. Attending now to unicursal spaces only, we must observe that there are in general exceptions to the unique correspondence of points and value-systems; namely, there are certain points to each of which correspond an infinite number of values of the coordinates satisfying a certain equation or equations; and there are certain value-systems to which correspond, not points, but loci in the space. The assignment of these point-equations and loci-values and of their relations with one another serves to determine the *projective-connection* of the space; and when once these are known, the whole of its projective geometry may be worked out. The point equations and loci-values may or may not involve imaginary values of the variables or their coefficients; but in all cases they must be taken into account. The

regulars may be complex. As the word is thus no longer wanted in its old meaning, I have made bold to use it in a new one.

points which correspond to real systems of values are called *real points*; those which correspond to imaginary systems, *imaginary points*; the study of these latter, which does not enter the belonging to that of three-dimensional space, is instead their study for the sake of the former.

Locii which correspond to linear equations between the coordinates may at present be called *planes*, and those systems of linear equations *lines*; this is a purely projective definition, and these loci are not necessarily *flat* planes and straight lines in the usual sense. Points, lines, and planes are included as the *active elements*.

The *metric geometry of space** is the theory of the projective relations of certain fixed geometrical forms with all other geometrical forms, or of the invariant relations of certain fixed algebraic forms with all other algebraic forms. The word *metric* will be explained as much as is possible in this chapter, though while it may be said that these fixed forms are called *the absolute* (the *absolute*) are given when we know the points, the lines, and the planes of the absolute, or say the elements of the absolute, and that the power of an element of the absolute as against an arbitrary element is infinite. In other words, the equations are general equations of the absolute in point, line, and plane coordinates respectively.

A *univalent space* is the points of which every loci is represented uniquely by value systems of the coordinates, e.g. x, y, z , and the exception of any point equations or loci relations is called a *linear space*. This is merely a projective definition, and because the absolute, therefore the whole of metric geometry, can be termed.

There is a particular determination of the other loci in a linear space which is of the utmost importance. It is that in which the points of the absolute are themselves on a quadric surface, while the lines and planes of the absolute are the lines which touch this surface, or in which the planes represent the

* This theory of metric geometry is given in the 2d chapter of "Metric Geometry and Quaternions," Phil. Trans., 1869.

the absolute are of the second degree. There are three cases* to be considered, no being the only ones of which observed space can form a part :

- (1) *Elliptic geometry*; all the elements of the absolute are imaginary.
- (2) *Hyperbolic geometry*; the absolute contains no real straight lines, and surrounds us. In this case, real points situate on the other side of the surface are called *ideal*.
- (3) *Parabolic geometry*; the surface degenerates into an imaginary conic in a real plane. The points of the absolute are points in the (real) plane of this conic; the lines and planes are the imaginary lines and planes which meet and touch the conic respectively.

The *first* of these suppositions will be made in what follows. It may be well here to set down in what it consists.

(1) The space to be considered is such that there is one point of it for every set of values of the coordinates x, y, z , and one set of values for every point, without any exception whatever.

(2) There is a certain quadric surface, called the absolute, all whose points and tangent planes are imaginary. If the line joining two points a, b meet the absolute in i, j , the quantity

$$\frac{ab \cdot ij}{Q(a, aj, bi, bj)} = ab,$$

(which is a function of anharmonic ratio, and therefore an invariant,) is called the *power* of the points a, b in regard to one another, or of either in regard to the other. The *distance* of these two points is an angle θ such that

$$\sin \theta = ab.$$

Similarly, if through the line of intersection of the planes A, B there be drawn the tangent planes I, J to the absolute,

* On this division see Dr Klein, "Ueber die so genannte Nicht-Euklidische Geometrie," *Math. Annalen*, Bd. 4. The second case is the geometry of Lobachevsky and Bolyai.

the power of the planes A, B in regard to one another is the quantity

$$\frac{AB, II}{\sqrt{(AI, AJ, BI, BF)}} = AR,$$

and the angle between them is an angle ϕ such that

$$\sin \phi = AR.$$

(3) If two points are conjugate in regard to the absolute, they are distant a *quadrant* from one another; if two lines on planes are conjugate in regard to the absolute, they are at right angles. Through all the points at a quadrant distance from a given point are situate on its polar plane in regard to the absolute, and every plane through it cuts this polar plane at right angles. Every line has a polar line in regard to the absolute, so that every point on the polar line is distant a quadrant from every point on the line; and every line which is at right angles to either meets the other. Through an arbitrary point can in general be drawn one line perpendicular to a given plane, namely, the line joining the point to the pole of the plane. If, however, the point is the pole of the plane, every line through it is perpendicular to the plane. Similarly, from a point not on the polar of a given line can be drawn one and only one perpendicular to the line; namely, the line through the point which meets the given line and its polar.

(4) In general, two lines can be drawn so that each meets two given lines at right angles, and these are poles of one another. One line may therefore be converted into another by rotation about two polar axes. These axes are determined as the lines which meet the two given lines and their polars. If we travel continuously along one of these lines and draw perpendiculars on the other, one of these axes determines the shortest distance between the lines, and the other the longest. If then these two are equal, the lines are equidistant along their whole length. Thus there is a case of exception in which two lines and their polars belong to the same set of generators of a hyperboloid; the lines are then equidistant along their whole length, and meet the same two generators of one system of the

absolute. I shall use the word *parallel* to denote two lines so situated; and they shall be called *right* parallel or *left* parallel according as one is converted into the other by a right-handed or left-handed twist. Through an arbitrary point can be drawn one right parallel and one left parallel to a given line; the angle between them is twice the distance of the point from the line. There are many points of analogy between the *parallels* here defined and those of parabolic geometry. Thus, if a line meet two parallel lines, it makes equal angles with them; and a series of parallel lines meeting a given line constitute a ruled surface of zero curvature. The geometry of this surface is the same as that of a finite parallelogram whose opposite sides are regarded as identical.

(5) A twist-velocity of a rigid body must be regarded as having *two* axes. For a motion of translation along any axis is the same thing as a rotation about the polar axis, and *vice versa*. Hence a twist-velocity is compounded of rotation-velocities about two polar axes; say these are θ , ϕ . Then the motion may be regarded either as a twist-velocity about a screw whose pitch is $\frac{\phi}{\theta}$ and whose axis is the first axis, or about a screw

whose pitch is $\frac{\theta}{\phi}$ and whose axis is the polar axis. In general, then, a motor has two axes, and is expressible in one way only as the sum of two polar rotors. There is, however, one case of exception in which the axes of a motor are indeterminate; that, namely, in which the magnitudes of the two polar rotors are equal*. If a rigid body receive at the same time a rotation about an axis and an equal translation along it, all the points of the body will describe parallel straight lines; and the motion of the body is at the same time a rotation about any one of these lines combined with an equal translation along it. Such a motion may be adequately represented by a line of given length drawn through any point whatever parallel to a given line. A motor of pitch unity, or which is its own polar, may therefore

* This motion is described in another connection by DR. KLEIN and I *Math. Annalen*, Vol. 4; it is a transformation of the absolute into itself in which two generators remain unaltered.

be regarded as having the nature of a *vector*, and shall in future be denoted by that name. For we may define a vector as a motor whose axes are indeterminate; and the case we are now considering is the only case of such indetermination which occurs in elliptic geometry. Vectors will be called *right* or *left* according as the twist of them is right- or left-handed.

Prop.: *Every motor is the sum of a right and a left vector.* For let A be a motor, and A' the polar motor; then we have $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$. Now $A + A'$ and $A - A'$ are both motors of pitch unity, but one right-handed and the other left-handed.

IV.

A fixed point being chosen as origin, let three lines perpendicular to one another be drawn through it, and let three unit-rotors having these lines as axes be denoted by the symbols i, j, k . Then every rotor through the origin will be denoted by an expression of the form $ix + jy + kz$, where x, y, z are scalar quantities, or the ratios of magnitudes. The symbols i, j, k shall have also another meaning; viz., each shall signify the rotation through a right angle about its axis of any rotor which meets that axis at right angles. When they are performed on rotors passing through the origin, these operations satisfy the equations $i^2 = j^2 = k^2 = ijk = -1$, by the ordinary rules of quaternions; and it is easy to see that the same equations hold good when the operations are performed on rotors not passing through the origin. The compound symbol $ix + jy + kz$ is also to have an analogous secondary meaning; viz., a rectangular rotation about the axis of the rotor which it previously denoted, combined with a tensor $\sqrt{(x^2 + y^2 + z^2)}$. It can operate only on a rotor which meets its axis at right angles. This being so, the ratio of any two rotors through the origin is a *quaternion* of the form $q \equiv w + ix + jy + kz \equiv w + \rho$, say. The axis ρ of this quaternion is perpendicular to the plane of the two rotors. If a be a rotor through the origin and q a quaternion, the product qz can be formed according to the Hamiltonian rules of multiplication, and is in general a quaternion r . In this general case

the equation $qx = r$ can only be interpreted by giving to α its *secondary* meaning; and the translation of this statement into words is as follows:—If a rotor be capable of being successively operated upon by the rectangular versor α and the quaternion q , the final result will be the same as if it had been originally operated upon by the quaternion r . If, however, the axes of q and α are at right angles, the scalar part of r will be wanting, and we may write the equation $qx = p$. This equation is now susceptible of a *primary* interpretation; viz., the quaternion q operating on the rotor α produces the rotor p ; although the *secondary* interpretation does not cease to be true.

With such conventions, the two sides of the equation

$$(q + r)s = qs + rs$$

(in which q, r, s are quaternions) have always the same meaning when both are interpretable; which is what is meant by saying that the distributive law holds good for these symbols.

The ratio of two rotors which do not meet is a twist which in general has perfectly definite axes. But when the rotors are polars of one another, the axes of the twist are indeterminate; for any line meeting both meets them at right angles, and will serve for an axis. It is therefore always possible to find a twist which shall simultaneously convert two given rotors into their polars; and any two rectangular twists with pitch 1 or -1 have a pair of common rotors on which they can operate, and which they convert into one another. Hence we may consider that

All rectangular twists of pitch 1 are equivalent to one another; and all rectangular twists of pitch -1 are equivalent to one another.

The rectangular twist of pitch 1 shall be denoted by the symbol ω ; the expression $\omega\alpha$ will denote the rotor polar to α and equal to it in magnitude, obtained from it by a left-handed twist. During the operation of this twist, every point of the rotor describes a straight line; if therefore the twist be continued through two right angles, the rotor will be replaced in its original position, *not* reversed; we have therefore

$$\omega^2 = 1.$$

Every motor can be expressed as the sum of two rotors, one passing through the origin and the other being polar to a rotor through the origin. The general expression for a motor is therefore

$$a + \omega\beta.$$

This will represent a *rotor* if the two rotor constituents intersect, or if each is perpendicular to the polar of the other; or if $S\alpha\beta = 0$.

Let now
$$\xi = \frac{1 + \omega}{2}, \quad \eta = \frac{1 - \omega}{2};$$

then
$$\xi^2 = \frac{1 + 2\omega + \omega^2}{4} = \frac{2 + 2\omega}{4} = \xi,$$

$$\eta^2 = \frac{1 - 2\omega + \omega^2}{4} = \frac{2 - 2\omega}{4} = \eta,$$

$$\xi\eta = \frac{1 - \omega^2}{4} = 0.$$

Any motor $a + \omega\beta$ can also be expressed in the form $\xi\gamma + \eta\delta$. It is clear that $\xi\gamma$ is the right vector part of this motor, and that $\eta\delta$ is the left vector part. If we multiply $\xi\gamma + \eta\delta$ by ξ , the result is merely $\xi\gamma$; so the effect of multiplying a motor by ξ is merely to pick out the right vector part of it. The symbols ξ, η are thus in a certain sense *selective* symbols, and are analogous to the S and V of quaternions.

Ratio of two motors.—We can find immediately now the operation which converts a motor $\xi\gamma + \eta\delta$ into a motor $\xi\alpha + \eta\beta$. For if we perform the operation

$$\left(\xi \frac{\alpha}{\gamma} + \eta \frac{\beta}{\delta} \right) (\xi\gamma + \eta\delta),$$

remembering the laws of multiplication of ξ, η , we obtain the result $\xi\alpha + \eta\beta$. If then $\frac{\alpha}{\gamma} = q, \quad \frac{\beta}{\delta} = r$, we may write

$$\frac{\xi\alpha + \eta\beta}{\xi\gamma + \eta\delta} = \xi \frac{\alpha}{\gamma} + \eta \frac{\beta}{\delta} = \xi q + \eta r,$$

and the latter may be written in the form

$$\frac{q+r}{2} + \omega \cdot \frac{q-r}{2} = s + \omega t,$$

showing that *the ratio of two motors is a biquaternion*.

The motor $\xi\alpha + \eta\beta$ will be a *rotor* if

$$S(\alpha + \beta)(\alpha - \beta) = 0,$$

or if

$$T\alpha = T\beta;$$

and it is easy to see from this that the biquaternion $\xi q + \eta r$ will be a *twist*, or the ratio of two rotors, if $Tq = Tr$.

V.

1. *Position-Rotor of a Point*.—The coordinates of a point in regard to a quadrantal tetrahedron 1234 being w_1, w_2, w_3, w_4 , the equation to the absolute is $\Sigma w^2 = 0$. The rotor from the origin (the point 4) to the point x is represented by

$$i_1 \frac{w_1}{w_4} + i_2 \frac{w_2}{w_4} + i_3 \frac{w_3}{w_4}, \text{ or } \Sigma i_k \frac{w_k}{w_4} (k = 1, 2, 3),$$

where i_1, i_2, i_3 are rotors along the edges of the tetrahedron from the origin to the middle points of the edges. The tensor of this rotor is the tangent of the angular distance from the origin to the point it represents. For if

$$\rho = i_1 \frac{w_1}{w_4} + i_2 \frac{w_2}{w_4} + i_3 \frac{w_3}{w_4},$$

$$(T\rho)^2 = \frac{w_1^2 + w_2^2 + w_3^2}{w_4^2} = \tan^2 \widehat{ox}, \text{ where } o \text{ is the origin.}$$

The angular distance from the origin to a point has an infinite number of values, which differ by multiples of π . If therefore a rotor be considered to have this angular distance as its length, the rotor of a point can only be defined by such an equation as $\check{p} \equiv \check{\alpha} \pmod{\check{\pi}_0}$. To obviate this indetermination, there is required a one-valued unicursal function having the period π ; the tangent of the angular distance is hereby completely singled out.

2. *Equation of a Straight Line*.—Let OM [fig. 9] be perpendicular from the origin O upon the straight

and let ON be a line perpendicular to OM in the plane MOP . Then from the triangle MOP we have

$$\frac{\tan OM}{\tan OP} = \cos MOP;$$

or if $OM = \alpha$, $OP = \rho$, $ON = \beta$, $T\alpha = T\rho \cos MOP$; so that α is the component of ρ in the direction OM , and we have $\rho = \alpha + \beta x$, where x is some scalar.

By varying x , then, we get all the points in the line MP . But if α_1 is any particular value of ρ , the equation may just as well be written

$$\rho = \alpha_1 + \beta x,$$

where now α_1 is not necessarily perpendicular to β .

This form may be reduced to the preceding as follows:

To find the perpendicular from O , put $\delta T\rho = 0$; this gives

$$S\alpha_1\beta + \beta^2 x = 0,$$

and the equation becomes

$$\rho = \alpha_1 - \beta S \frac{\alpha_1}{\beta} - \beta x,$$

where $\alpha_1 - \beta S \frac{\alpha_1}{\beta} = \alpha$ of the former equation.

3. Rotor along Straight Line whose Equation is given.

Let OR [fig. 26] be the rotor through the origin which has right parallelism with MP . Then $\angle NOR = OM$. Let OK be perpendicular to ON and OM , and of such length that

$$\frac{\tan OK}{\tan ON} = \tan NOR.$$

Then, if $\gamma = OK$, $OR = \beta + \gamma$.

Now $\frac{T\gamma}{T\beta} = T\alpha$, and $U\gamma = U\alpha\beta$, since γ is perpendicular to α and β . Hence $\gamma = \alpha\beta$; and if R be a rotor along MP , m a scalar,

$$\text{right vector of } R = \xi R = m\xi(\beta + \gamma) = m\xi(\beta + \alpha\beta),$$

so left vector of $R = \eta R = m\eta(\beta - \gamma) = m\eta(\beta - \alpha\beta)$;

therefore $R = m(\beta + \omega\alpha\beta)$.

Now if R have the same length as β , we have

$$\beta^2 = R^2 = m^2 (\beta^2 + \overline{\alpha\beta^2}) = m^2 \beta^2 (1 - \alpha^2);$$

therefore

$$R = \frac{\beta + \omega \alpha \beta}{\sqrt{1 - \alpha^2}},$$

Conversely, equation to axis of rotor $\gamma + \omega \delta$ is

$$\rho = \frac{\delta}{\gamma} + \gamma x.$$

This finds the rotor in the case in which $\rho = \alpha + \beta x$, where $S_2 \beta = 0$. But in the general case we have only to write the equation in the form

$$\rho = \alpha - \beta S \frac{\alpha}{\beta} + \beta x,$$

whence

$$\begin{aligned} R &= \frac{\beta + \omega \left(\alpha - \beta S \frac{\alpha}{\beta} \right) \beta}{\sqrt{\left(1 - \alpha^2 - \beta^2 S^2 \frac{\alpha}{\beta} + 2 S_2 \beta S \frac{\alpha}{\beta} \right)}} \\ &= \frac{\beta + \omega V \alpha \beta}{\sqrt{\left(1 + S_2 \beta S \frac{\alpha}{\beta} - \alpha^2 \right)}}. \end{aligned}$$

4. Rotor joining Points whose Position-Rotors are α, β .

The equation of this rotor is

$$\rho = \alpha + (\beta - \alpha) x,$$

whence

$$mR = \beta - \alpha + \omega V \alpha \beta.$$

Now if $a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4$ are the coordinates of the points, we have

$$(TR)^2 = \tan^2 ab = \frac{\sum (a_k b_k - a_k b_k)^2}{(\sum a_k b_k)^2} = - \frac{(\alpha - \beta)^2 + (V \alpha \beta)^2}{(1 - S_2 \beta)^2},$$

therefore

$$R = \frac{\beta - \alpha + \omega V \alpha \beta}{1 - S_2 \beta}.$$

Con.—If ρ [fig. 27] be the rotor of a variable point on a curve, $d\lambda$ a rotor along the tangent of length equal to the arc of the curve between ρ and $\rho + d\rho$, we have

$$d\lambda = \frac{d\rho + \omega V \rho d\rho}{1 - \rho^2}.$$

5. *Rotor parallel to pl through Point whose Position is x .*

The general equation to a line through the point $p = a + Ax$, where A is any rotor through the origin, along this line is $A + aFA$; if this is right parallel to pl ,²

$$\frac{1}{2}(A + aFA) = \frac{1}{2}pl, \quad (Fa = F_l)$$

or

$$A + aFA = pl$$

Operating by S_l , we have, since $S_l aFA = 0$,

$$SA = -SApl,$$

whence, by addition, $A + SA = pl + SApl$,

and $A = (1 + a^{-1})\frac{1}{2}(pl + SApl) = pl + (1 + a)^{-1}FApl$.

The rotor required is

$$A + aFA, \text{ or } A + a(pl + AX)$$

This becomes, then,

$$pl + (1 + a)^{-1}FApl + a(1 + a)^{-1}FApl = pl + 2a(1 + a)^{-1}F$$

Instead of operating by S_l on the equation

$$A + aFA = pl,$$

we might have operated with F_l , and got

$$FA + aFA = FApl, \text{ since } F_l aFA = aFA,$$

therefore

$$FA = (1 + a)^{-1}FApl,$$

and

$$A = pl - FA = pl - (1 + a)^{-1}FApl$$

Similarly, we have for the rotor lr parallel to pl ,

$$A = pl + (1 + a)^{-1}FApl,$$

and the rotor lr

$$\begin{aligned} A + a(A + pl) &= pl + (1 + a)^{-1}FApl + a(1 + a)^{-1}FApl \\ &= pl + 2F_l(1 + a)^{-1}FApl \end{aligned}$$

XXI.

GRAPHIC REPRESENTATION OF THE HARMONIC COMPONENTS OF A PERIODIC MOTION*.

FOURIER'S theorem asserts that any motion having the period P may be decomposed into simple harmonic motions having periods $P, \frac{1}{2}P, \frac{1}{3}P, \&c.$; and assigns the amplitudes and phases of these motions by means of definite integrals. In fact, if $\phi(x+2\pi) = \phi(x)$ for all values of x ,

$$\text{then } \phi(x) = \frac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + \dots + b_m \cos mx + \dots \\ + a_1 \sin x + a_2 \sin 2x + \dots + a_m \sin mx + \dots,$$

where
$$\pi b_m = \int_{-\pi}^{+\pi} \phi(x) \cos mx \, dx,$$

$$\pi a_m = \int_{-\pi}^{+\pi} \phi(x) \sin mx \, dx;$$

and this is made applicable to the general case of periodic motion by putting $\frac{x}{2\pi} = \frac{t}{P}$, where t is the time elapsed since the era of reckoning.

The terms $b_m \cos mx + a_m \sin mx$ constitute a simple harmonic motion of period $\frac{P}{m}$; the object of the present communication is to represent this motion by a graphical construction.

If a right circular cylinder be made to revolve uniformly about its axis, while a pencil in contact with its surface has

* [From the *Proceedings of the London Mathematical Society*, Vol. v, No. 67, pp. 41-44.]

plane perpendicular to the axis (*i.e.* a horizontal plane) is merely the circular section of the cylinder, which is the same as the projection of any ellipse traced on it. If therefore we cut the cylinder by a plane parallel to the maximum plane of the contour C_1 , the elliptic section E_1 will have the same area as that contour in magnitude and direction.

The contour C_n goes twice round the cylinder; therefore its projection on a plane perpendicular to the axis is the circle described twice in the same direction, and its area is twice that of the projection of any ellipse. If therefore we cut the cylinder by a plane parallel to the maximum plane of C_n , we shall obtain an ellipse E_n whose area is half that of the contour C_n and parallel to it. Similarly, the contour C_m goes m times round the cylinder, and a plane parallel to its maximum plane will determine an ellipse E_m whose area is $\frac{1}{m}$ th of that of C_m .

Now first let a circle be drawn on the cylinder whose height is the mean height of C_1 . On this circle is the middle point of all the component oscillations.

Next, while the cylinder goes round once in the period P , let the pencil follow the ellipse E_1 ; it will then have a simple harmonic motion of period P , which is, in fact, the first or fundamental component. Then, while the cylinder goes round twice in the time P let the pencil follow the ellipse E_2 ; the resulting simple harmonic motion of period $\frac{P}{2}$ is the second component. Analogously, while the cylinder goes round m times in the time P , let the pencil follow the ellipse E_m ; this simple harmonic motion is the m th component.

The demonstration of this result is very simple. The values of a_n and b_n may be written as follows:

$$\pi a_n = \int_{-\pi}^{+\pi} \phi x \cos nx dx = \frac{1}{n} \int_{-\pi}^{+\pi} \phi x d(\sin nx),$$

$$\pi b_n = - \int_{-\pi}^{+\pi} \phi x \sin nx dx = - \frac{1}{n} \int_{-\pi}^{+\pi} \phi x d(\cos nx).$$

Suppose now that ϕx is set up vertically at P (fig. 28), when $TCd = mx$, then $d \cos mx$ is the element of CA , and $d \sin mx$ is the element of CB ; so that the differentials under the integral signs are respectively elements of the areas projected on vertical planes through AA' and BB' . If in these integrals we write $b_m \cos \alpha + a_m \sin \alpha$ in place of ϕx , we get the areas of the corresponding projections of the ellipse E_m ; these are mb_m and ma_m respectively. Thus the area of E_m , projected on these planes at right angles is m times that of the ellipse E_1 ; i.e. the areas of the two curves have the same aspect and are in the ratio $m : 1$; which was to be proved.

XXII.

ON THE TRANSFORMATION OF ELLIPTIC
FUNCTIONS*.

THE following communication is an attempt to apply Jacob's geometrical representation of the addition theorem in elliptic functions to the theory of their transformation. For this purpose I use the said representation in the following form.

Consider two circles, one of which is wholly within the other, but which are not concentric, as in the figure (fig. 20). The points of the outer circle may be uniquely represented by a parameter x , such that if U and U' are the points represented by these values respectively, and OU is the tangent at the former, x is proportional to the ratio of the sines of the angles which the radius with the lines OU and OU' . Let the angle $OUO' = \phi$, then, if we make $x = i \tanh (\frac{1}{2}(\theta - \phi))$, the values $1, -1$ of the parameter will belong to the circular points at infinity. Let then $k', -k'$ be the values belonging to the imaginary points of intersection of the two circles. Through the points U, U' let tangents be drawn to the inner circle, meeting the outer circle in y, y' ; these being so chosen that, when x moves continuously to U, U' will move continuously to y, y' . Then Jacob's theorem is that, if

$$u \rightarrow \sin (u, k), \quad y \rightarrow \sin (y, k'), \quad x \rightarrow \sin (x, k), \quad \text{then } u \rightarrow \sin (y, k).$$

The extension to any two circles, made by Prof. Cayley, may be put into the same form. The representation of each point

* (From the *Proceedings of the London Mathematical Society*, Vol. vii. (1895-96), 91, pp. 29-30.)

of a conic by a parameter is determinate when we know the parameters of any three points. Now the four intersections of two conics U, V may be divided into pairs in three ways, and will so determine three involutions upon the conic U . Let one of these be chosen, and let the parameters 0 and ∞ be assigned to its united points. Then, if the value 1 be assigned to one of the intersections, -1 will belong to another of them; and the remaining two will have parameters equal in magnitude but of contrary signs. Call these $\pm k^{-1}$, and draw the tangents as before; then Jacobi's theorem remains true, except that we must write $\pm \operatorname{sn}^{-1} \xi = \operatorname{sn}^{-1} x + \operatorname{sn}^{-1} c$; the sign of $\operatorname{sn}^{-1} \xi$ depending on the reality of the common tangents.

The proof of it depends on the symmetrical (2, 2) correspondence, considered by Euler, in relation to the addition of elliptic integrals. Given that the relation between the points x and ξ is that the line $x\xi$ touches the conic V , it is clear that to one position of x correspond two positions of ξ ; and to one position of ξ , two positions of x ; or, the points have a (2, 2) correspondence. Hence the equation connecting the quantities x, ξ must be of the second order in each; and it must obviously be symmetrical. Let the equation be

$$(ax^2 + 2bx + c)\xi^2 + 2(bx^2 + 2b'x + c')\xi + (cx^2 + 2c'x + c'') = 0;$$

or, which is the same thing,

$$(a\xi^2 + 2b\xi + c)x^2 + 2(b\xi^2 + 2b'\xi + c')x + (c\xi^2 + 2c'\xi + c'') = 0.$$

The values of x which make the two corresponding values of ξ coincide are given by the equation

$$X = (ax^2 + 2bx + c)(cx^2 + 2c'x + c'') - (bx^2 + 2b'x + c')^2 = 0,$$

and similarly the values of ξ which make the two corresponding values of x coincide are given by $\Xi = 0$, where Ξ is the same function of ξ that X is of x . Now, by differentiating the original equation, we easily find $dx : \sqrt{X} = d\xi : \sqrt{\Xi}$.

The roots of the equation $X = 0$ are clearly the parameters of the points of intersection of the two conics; for these are the only points on U from which two coincident tangents can be drawn to V . If, then, the parameters of these points have

been made equal to ± 1 , $\pm k^2$, X must be proportional to $(1-x^2)(1-k^2x^2)$, and the differential equation becomes

$$\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}};$$

whence, since, when $x=0$, $\xi=c$,

$$\pm \sin^{-1}x \pm \sin^{-1}c = \sin^{-1}\xi,$$

which is the theorem in question.

If we change Γ into $U \pm \sigma \Gamma$, and allow σ to vary, this varying conic will always have the same intersections with U , and therefore k will be constant; but c will depend upon the value of σ . It is clear that c^2 is given uniquely when σ is given, but, when c is given, there are two values of σ . When $c=0$, the two conics must touch at 0, and therefore must coincide; thus both values of σ vanish. When $c=\infty$, the conic $U \pm \sigma \Gamma$ becomes a pair of lines intersecting on the line $0=\infty$; let a and β be the values of σ which belong to these, and which are, of course, roots of the equation $[\Gamma](U \pm \sigma \Gamma) = 0$.

$$\text{Then we must have } c^2 = \frac{m\sigma^2}{(\sigma-a)(\sigma-\beta)^2},$$

where m is an undetermined constant.

Suppose, now, that a polygon is inscribed in U by the following process: a tangent is drawn from x to a conic σ_1 , meeting U again in σ_2 ; then from σ_2 to a conic σ_2 , meeting U in σ_3 , and so on; finally, let σ_{n+1} be joined to σ_1 . If c_1, c_2, \dots, c_{n+1} be the constants belonging to these conics respectively, we shall have

$$\pm \sin^{-1}\sigma_1 \pm \sin^{-1}x = \pm \sin^{-1}c_1,$$

$$\pm \sin^{-1}\sigma_2 \pm \sin^{-1}\sigma_1 = \pm \sin^{-1}c_2;$$

$$\&c. \qquad \&c.$$

$$\pm \sin^{-1}\sigma_{n+1} \pm \sin^{-1}\sigma_n \pm \sin^{-1}c_{n+1};$$

whence, by addition, with proper changes of sign,

$$\pm \sin^{-1}\sigma_{n+1} \pm \sin^{-1}x \pm \sin^{-1}c_1 \pm \sin^{-1}c_2 \pm \dots \pm \sin^{-1}c_{n+1} \\ = \pm \sin^{-1}x \pm \sin^{-1}c_n, \text{ suppose,}$$

From this equation it appears that the last side of the polygon will always touch the same conic of the series $U + \sigma V$, wherever the starting-point x is taken. Or, if a polygon be inscribed in U , and move so that all but one of its sides touch conics of the series $U + \sigma V$, then the remaining side will also touch a conic of the series. This theorem of Poncelet's is proved in this way by Jacobi. It is to be noticed that the signs of the quantities c depend upon which tangent is drawn from the corresponding point x ; the two tangents belong in a definite way to the two tangents from 0. The final value of c_n being thus determined, one of the two conics belonging to it is singled out by the sign given to $\text{sn}^{-1} x_{n-1}$. In fact, the whole series of conics $U + \sigma V$ is divided into three parts by the three line-pairs it contains. For two conics in the same part of the series the signs of $\text{sn}^{-1} \xi$ are certainly the same; for conics in different parts they may be different.

And it appears that the conics mentioned in the theorem may belong to any part of the series if the signs be properly chosen in each of the equations. It is remarked by Jacobi that with these restrictions the position of x_{n-1} does not depend upon the order in which the conics are taken.

Now suppose two such polygons to be drawn with the same system of conics, x being continuously moved to ξ , and at the same time x_1 to ξ_1 , &c. We shall then have the equations

$$\pm \text{sn}^{-1} x_1 = \text{sn}^{-1} x + \text{sn}^{-1} c_1,$$

$$\pm \text{sn}^{-1} \xi_1 = \text{sn}^{-1} \xi + \text{sn}^{-1} c_1,$$

the signs being the same in both. Consequently

$$\pm (\text{sn}^{-1} x_1 - \text{sn}^{-1} \xi_1) = \text{sn}^{-1} x - \text{sn}^{-1} \xi,$$

or the lines $x_1 \xi_1$, $x \xi$ touch the same conic of the series $U + \sigma V$.

Proceeding in this way, we may show that

$$\pm (\text{sn}^{-1} x_r - \text{sn}^{-1} \xi_r) = \text{sn}^{-1} x - \text{sn}^{-1} \xi;$$

whence it appears that the lines joining corresponding vortices of the two polygons all touch the same conic of the series when the n conics touched by the sides belong to the same part; but if they belong to different parts, each joining line touches one

of two conics which harmonically divide the pairs of lines $U + \alpha V$, $U + \beta V$.

Attending now only to the first case, it will be convenient to re-state the two theorems together, as follows:—

If a polygon be inscribed in a conic U so that all its sides but one touch conics of the series $U + \alpha V$, the remaining side will also touch a conic of the series. —(Poncelet's Theorem.)

When all these conics can pass continuously into one another without breaking up into two straight lines, the lines joining corresponding vertices of two such polygons will all touch a conic of the series.

Let us now consider the particular case in which all the sides of the moving polygon touch the same conic. Here the second theorem is true without restriction; the lines joining corresponding vertices of two such polygons will always touch one conic passing through the intersections of the other two. In this case also *the vertices of the variable polygon determine upon the conic U an involution of the n^{th} order*; that is to say, if the parameters of the vertices of one polygon be determined by an equation $p_n = 0$ of the n^{th} order, and those of another polygon by an equation $q_n = 0$, then the vertices of any third polygon will be determined by an equation $p_n + yq_n = 0$, where y is a variable quantity, which we may call the parameter of the polygon. The relation between y , the parameter of the polygon, and x , the parameter of any one of its vertices, is $y = p_n : q_n$, where p_n, q_n are rational integral functions of the n^{th} order in x .

Suppose, then, the relation between U and V to be such that a polygon of n sides may be inscribed in U and circumscribed to V . Let $x, x_1 \dots x_{n-1}$ be the vertices of such a polygon; then, if $x = \alpha n$, we must have $x_1 = \alpha n (n + \gamma)$, $x_2 = \alpha n (n + 2\gamma)$, $x_3 = \alpha n (n + 3\gamma) \dots x_{n-1} = \alpha n [n + (n - 1)\gamma]$, and consequently $x = \alpha n (n + n\gamma)$. Therefore $n\gamma$ is a period of the elliptic function; and the number of conics of the series $U + \alpha V$, which can be inscribed in n -gons inscribed in U is equal to the number of periods whose n^{th} parts are not congruent, that is, for n a prime number it is $n + 1$.

Now let another polygon be drawn having a vertex at E , and let η be its parameter. Then the lines AE, BE, \dots will all touch a conic W . Let this conic be held fixed, and the two polygons moved so that the lines joining corresponding vertices always touch W . Then to every value of y will belong two values of η , and *vice versa*, and this (2, 2) correspondence is symmetrical. Hence a symmetrical (2, 2) correspondence between individual vertices implies a symmetrical (2, 2) correspondence between the polygons.

Now, the parameter of every polygon is determined when we know the parameters of three polygons. Let the parameters of the polygons which have a vertex at 0, 1, and ∞ be made equal to 0, 1, and ∞ respectively; this amounts to saying that $p_n = 0$ has a root 0, $q_n = 0$ has a root ∞ , and $p_n - q_n = 0$ has a root 1. It is clear, then, from the symmetry of the figure, that y must be an odd function of x , so that $p_n + q_n = 0$ will have a root $\infty + 1$. This amounts to saying that p_n is x multiplied by a rational integral function of x^2 , and q_n (which is really only of the order $n-1$) is another rational integral function of x^2 . This being so, let λX^2 be the parameter of those polygons which have vertices at the remaining two points of intersection of the conics. Then the quantities y and η are connected by a symmetrical (2, 2) correspondence such that the values of y which give equal values for η are $\pm 1, \pm \lambda X^2$. Therefore, if $y = \operatorname{sn}(u', \lambda)$, we must have $\eta = \operatorname{sn}(u' + b, \lambda)$, where b is a constant.

We have arranged that y is divisible by x or $\infty + 1$, by making 0 the parameter of that polygon which has a vertex at the point 0. It appears thus that y must also be divisible by w_1, w_2, \dots, w_{n-1} , since, when any one of these is zero, y vanishes. We may write, therefore, $y = m w_1 \dots w_{n-1}$, where m is a constant, since y is only infinite when one or other of the x is infinite. The products $w_1 w_{n-1}, w_2 w_{n-2}, \dots$, are given rationally as ratios of quadratic functions of w by the original equation of (2, 2) correspondence. To determine m , we have

$$y = m \operatorname{sn} u \operatorname{sn}(u + \gamma) \operatorname{sn}(u + 2\gamma) \dots \operatorname{sn}\{u + (n-1)\gamma\} / w\gamma - aK^2 + bK'^2;$$

but, since $y = 1$, when $x = \operatorname{sn} u = 1$, or $u = K$, we have also

$$1 = m \operatorname{sn} K \operatorname{sn} (K + \gamma) \operatorname{sn} (K + 2\gamma) \dots \operatorname{sn} \{K + (n-1)\gamma\},$$

$$\text{and therefore } y = \frac{\operatorname{sn} u \operatorname{sn} (u + \gamma) \dots \operatorname{sn} \{u + (n-1)\gamma\}}{\operatorname{sn} K \operatorname{sn} (K + \gamma) \dots \operatorname{sn} \{K + (n-1)\gamma\}};$$

the denominator may, of course, be simplified, as in Jacobi's formula, but for present purposes it may be left as it stands. Now λ^{-1} is the value of y when $x = k^{-1}$, or when $u = K + iK'$,

$$\lambda^{-1} = \frac{\operatorname{sn} K \operatorname{sn} (K + \gamma) \dots \operatorname{sn} \{K + (n-1)\gamma\}}{k \operatorname{sn} (K + iK') \operatorname{sn} (K + iK' + \gamma) \dots \operatorname{sn} \{K + iK' + (n-1)\gamma\}},$$

which, again, may be easily reduced.

If we now assume that $u = Mu'$, where $y = \operatorname{sn} (u', \lambda)$ and M is a constant, we may determine M by observing that it is the value of $x : y$ when both of them are zero. Namely we have

$$M = \frac{\operatorname{sn} (K + \gamma) \operatorname{sn} (K + 2\gamma) \dots \operatorname{sn} \{K + (n-1)\gamma\}}{\operatorname{sn} \gamma \operatorname{sn} 2\gamma \dots \operatorname{sn} (n-1)\gamma}.$$

And thus, with the exception of the assumption just made, the theory of transformation is established by means of that of the in-and-circumscribed polygon.

Let us now endeavour to generalize the theory of the in-and-circumscribed polygon. In the first place, we may observe that every involution of the third order gives rise to an elliptic transformation; for two triangles inscribed in a conic are always circumscribed to the same conic*. For convenience, we will now consider the involution as determined on the conic V ; namely, the triangles form groups of three tangents which are in involution. And we may now generalize our theorem as follows:—

If a complete n -gram move with its sides touching a conic so

* Every substitution $y = \frac{U}{V}$, where U, V are cubic functions of x , has an elliptic differential which it transforms. The cubic forms U, V are first polars of two points in regard to a single quartic form F (Gundelfinger, *Math. Annalen*). Let $X = 0$ give the four points whose first polars in regard to F have a square factor; then $dx : \sqrt{X}$ is the elliptic differential required. We have $X = jF + H$, where H is the Hessian of F , and i, j the quadriinvariant and cubinvariant, [cf. however p. 231, *infra*.]

that they form groups of n tangents in involution, the locus of the $\frac{1}{2}n(n+1)$ vertices is a curve of order $n+1$.

For, consider the number of points which the curve has in common with any one tangent of the conic. It determines uniquely in the involution the group of n tangents to which it belongs, and can have no other point on the locus of intersections except those $n+1$ in which it meets the other $n+1$ tangents of this group. Therefore &c.

If, in a curve of order $n+1$, it is possible to inscribe one complete n -gram whose sides all touch a conic, then it is possible to inscribe a singly infinite number, and the sides determine upon the conic an involution of the n^2 order.

Let $A, B, C, \dots, X = 0$ be the equations of the sides of the complete n -gram, then the equation of the curve may be written

$$\frac{a}{A+1} + \frac{\beta}{B+1} + \frac{\gamma}{C+1} + \dots + \frac{v}{X+1} = 0, \dots\dots\dots (1),$$

where $a, \beta, \gamma, \dots, v$ are constants. But since A is a tangent to the conic, its equation may be written in the form $X = a + V + \epsilon Z = 0$, where a is the parameter of the tangent and X, V, Z three fixed lines. The condition for three tangents to meet in a point is

$$0 = \begin{vmatrix} 1 & a & a^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y - az)(z - ax) - (y^2 - az^2), \dots\dots\dots (2),$$

where x, y, z are their parameters. Hence the condition that the tangents whose parameters are x, y shall meet on the curve (1) is

$$(a+x)(a+y)^{-1}(b+x)(b+y)^{-1} + \dots + (u+x)(u+y)^{-1} = 0, \dots\dots\dots (3),$$

or, which is the same thing,

$$\sum \frac{a}{a+x} = \sum \frac{1}{a+y},$$

and therefore, if tangents xy and also zx meet on the curve, it follows that yz will meet on the curve. Starting, then, from any arbitrary tangent x , we can find the $n+1$ points in which

$n \rightarrow 1$ in ω . By a slight change of notation, let the n roots be called x_1, x_2, \dots, x_n , and let $\Pi_n(y)$ be defined

$$(y - x_1)(y - x_2) \dots (y - x_n).$$

Then we have

$$(x - x_1)(x - x_2) \dots (x - x_n) = \Pi_n(x - a) + \theta \Pi_n(x - a) \Sigma_1(x - a)^{-1/2}$$

and therefore

$$\Pi_n(y - x) = \Pi_n(y - a) + \theta \Pi_n(y - a) \Sigma_1(y - a)^{-1/2}.$$

Multiplying together two such equations, we obtain

$$\begin{aligned} \Pi_n(y - x) \cdot \Pi_n(z - a) = \\ \Pi_n(y - a) \cdot \Pi_n(z - a) \\ + \theta \Pi_n(y - a) \cdot \Pi_n(z - a) \{ \Sigma_1(y - a)^{-1/2} + \Sigma_1(z - a)^{-1/2} \} \\ + \theta^2 \Pi_n(y - a) \cdot \Pi_n(z - a) \cdot \Sigma_1(y - a)^{-1/2} \cdot \Sigma_1(z - a)^{-1/2}. \end{aligned}$$

Now, if we examine this equation, we shall find that the left-hand member is $4RZ^2 - \Lambda^2$, where $4R \rightarrow 0$, $R \rightarrow 0$ are the tangents which make up the n gram belonging to the parameter θ ; the first term on the right is $4RZ^2 - \Lambda^2$, the n gram for $\theta = 0$, and the last term is $\theta^2 Z^2 Z_4^2 - \Lambda_4^2$, the n gram for $\theta = \infty$, which has the line $Z = 0$ as its ∞ line. There we may write the equation

$$4RZ - 4R_1 + \theta^2 Z^2 + \theta^2 \Pi_n^2(x_1, x_2, \dots, x_n, a) = 0,$$

and it only remains to find the nature of the curve P_n^2 of the n^{th} order. We may see from its equation,

$$0 = \Pi_n(y - a) \Pi_n(z - a) \{ \Sigma_1(y - a)^{-1/2} + \Sigma_1(z - a)^{-1/2} \},$$

that it passes through the points of contact and all the intersections of the n tangents Π_n ; and then it is obvious, from the symmetry, that it must pass through the points of contact and all the intersections of the tangents Π_n^2 . But perhaps the simplest way is to consider the envelope of the n gram Π_n , which we know must consist of the curve K_n once, and the locus of the nodes C_{n-1} twice; thus we shall have

$$4RZ - 4R_1 + \theta^2 Z^2 = P_n^2 = K_n C_{n-1},$$

to a factor θ^2 , and this equation gives at once the intersections of P_n^2 with K_n , and with the n gram

We may now state the following proposition:—

Given any two in-and-circumscribed n -grams UX and WZ , there exists always a curve P_0 of order n which passes through their $n(n+1)$ vertices and their $2n$ points of contact with the conic.

The equation of any other n -gram may be written in the form

$$0 = UX + \lambda P_0 + \lambda^2 WZ.$$

The relation between λ , the parameter of the n -gram, and x the parameter of one of its sides, is

$$(11a + \lambda) \Sigma x (a + x)^{-1} = 11a, \Sigma x a^2,$$

and λ is the product of the roots of this equation.

I have here taken UX , (X_1, X_2, \dots, X_n) for the first n -gram, corresponding to $\lambda = 0$, instead of WZ , corresponding to $\lambda = 11a$ or $\theta = 0$.

We may show, conversely, that if the envelope of

$$0 = P + \lambda Q + \lambda^2 R,$$

where $P = 0$, $Q = 0$, $R = 0$ are three curves of the n^{th} order, in $4PR = Q^2 + K_2 U^2$, U being of order $n+1$; then P and R are each an assemblage of n straight lines. For the curve P has nodes on all its intersections with U , since $4PR = Q^2 + K_2 U^2$; that is, $\frac{1}{2}n(n+1)$ nodes, so that it must consist of n straight lines.

[This point of view immediately suggests the extension of the whole theory to quadric surfaces. If the envelope of $P + \theta Q + \phi R + \theta\phi S$ is $PS = QR = K_2 U^2$, where P , Q , R , S are of order n , and U of order $n+1$, each of the surfaces P , Q , R , S will meet the quadric K_2 in two curves of the n^{th} order, and therefore will have $\frac{1}{2}n^2$ contacts with it; and similarly will meet U_{n+1} in two curves of order $\frac{1}{2}n(n+1)$, which intersect in $\frac{1}{2}n^2(n+1)$ points; these are not contacts, but nodes on the surface P . We thus get a theory of surfaces of order n , having $\frac{1}{2}n^2$ contacts with a quadric surface, and $\frac{1}{2}n^2(n+1)$ nodes on a fixed surface of order $n+1$. It appears that n must be even, and of course the variable surface is subject to other conditions,

Thus, in the case of a conic doubly tangent to the quadric, and having its vertex on a fixed plane, it has also to pass through two fixed points on the plane and four on the quadric.

The application of the identity $4PR = Q^2 + K_1C^2$ to surfaces only reproduces the theory of the plane conic.

At any point of intersection of the two n -grains,

$$0 = HX + xP_n + x^2 HZ,$$

$$0 = HX + yP_n + y^2 HZ,$$

we shall have $xy : x + y : 1 = HX : P_n : HZ$.

Consequently, any symmetrical (m, m) correspondence between the two n -grains expresses that they intersect on a curve of order mn , namely $(HX : P_n : HZ)^n$, if the equation of the (m, m) correspondence is $(xy, x + y, 1)^m$.

If we substitute the values $xy : x + y : 1$ for $HX : P_n : HZ$, in the equation of a third n -grain $0 = HX + zP_n + z^2 HZ$, we shall get simply $(z - x)(z - y) = 0$. Consider, then, m different n -grains $H\Delta_1, H\Delta_2, \dots, H\Delta_m$, and form the curve of order $n(m - 1)$

$$\frac{H_1}{H\Delta_1} + \frac{H_2}{H\Delta_2} + \dots = 0 \quad (7)$$

If the n -grains x, y meet on this curve we shall have

$$\sum \frac{H}{(a - x)(a - y)} = 0;$$

or, what is the same thing,

$$\sum \frac{H}{a - x} = \sum \frac{H}{a - y},$$

where a_1, a_2, \dots, a_m are the parameters of the n -grains. It follows, as before, that a singly infinite number of groups of n -grains can be totally inscribed in the curve (7); a group being totally inscribed when all the intersections of any two n -grains of the group are on the curve.

So far we have dealt only with totally inscribed n -grains; and, as this case is represented only by the triangle when the curve of inscription is a conic, it might seem that there should

be more general theorems corresponding to the inscription of polygons of a greater number of sides in a conic. But, in fact, the case of total inscription is the general case, and all others are cases of decomposition of the curve C_{n-1} . Consider, for example, a hexagon inscribed in a conic. If we produce all the sides, we shall get nine more intersections; three of these lie on a straight line, and the other six on a conic which circumscribes two triangles. These two conics and the straight line make up the curve C_5 of the fifth order. Again, let eight lines A, B, C, D, E, G, H, K (fig. 30) touch a conic, and let the twelve points marked \circ in the figure touch a cubic; then the octagram may be moved round the conic so as to keep these twelve points on the cubic. The points marked \times will lie on a fixed straight line, and a second cubic will pass through the intersections of A, B, C, D among themselves, and of E, G, H, K among themselves. These two cubics and the straight line make up the curve C_7 of the seventh order; and it is easy to see that there is an analogous case for any even number of lines. In order that a porismatic polygon may be inscribed in a curve, it is necessary that either the order or the curve or the number of sides should be even.

XXIII.

NOTES ON THE COMMUNICATION ENTITLED "ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS,"*

SOME of the following notes† would have been incorporated in the paper by the process of revision for the press, if that had not been kindly performed for me during an extended absence from the English climate. As regards all but one of them, I am glad of the opportunity which has thus been afforded me of extension and correction. But it is a matter of great regret to me that I discovered too late the priority of M. Darboux in the principal theorem of the second part of the paper; viz., the porismatic character of a polygon circumscribed to a conic and totally inscribed in a curve of order one less than the number of sides. In one of the notes to a book, which it is almost unnecessary in a geometer not to have read, marked, learned, and inwardly digested‡, M. Darboux has stated and proved the theorem, and has followed it by further investigations of the highest interest and importance. The method even of my investigation in the same as that of M. Darboux (as indeed was inevitable from the nature of the subject), namely, the representation of a point in a plane by means of the parameters of the tangents drawn from it to a fixed conic. It is not the first time

* [From the *Proceedings of the London Mathematical Society*, Vol. xxi. Nov., 1892, 103, pp. 225-230.]

† These remarks apply also to certain developments which I have since thought it better to communicate under separate titles.

‡ *Sur une classe remarquable de courbes et de surfaces algébriques*. Paris, Gauthier-Villars, 1878. Note II., p. 190.

that I have had the honour of following, however distantly, in the footsteps of that eminent geometer; but on other occasions it has been my good fortune to discover the fact in time.

Completion of the Geometric Proof of the Transformation-Formulae.

In my former paper one point was assumed as given by the analytical theory of transformation, namely, that the new argument u' is equal to the old argument u divided by a constant M . Having now found a simple proof of this, I will take the liberty of re-stating in outline, for the sake of clearness, the whole demonstration; availing myself of a remark of M. Darboux.

It is proved by Jacobi's method that if x_1, x_2, \dots, x_n are parameters of the points of contact of the n sides of a polygon circumscribed to a conic U and inscribed in a conic V , and if $\pm 1, \pm k^{-1}$ are the parameters of the points of contact of the common tangents of the two conics, then

$x_1 = \operatorname{sn} u, x_2 = \operatorname{sn} (u + \gamma), x_3 = \operatorname{sn} (u + 2\gamma), \dots, x_n = \operatorname{sn} \{u + (n-1)\gamma\}$, where $n\gamma = 4K + 4iK'$, and the modulus of the elliptic function is k .

This being so, an infinite number of in-and-circumscribed polygons can be drawn.

If $p_n = 0$ be the equation in x whose roots are x_1, x_2, \dots, x_n , and $q_n = 0$ an equation in x whose roots are the parameters of the sides of another such polygon; then $p_n - yq_n = 0$ will have for its roots the parameters of the sides of an in-and-circumscribed polygon, whatever value be given to y .

For the locus of the $\frac{1}{2}n(n-1)$ intersections of the tangents at the n points $p_n - yq_n = 0$, when y is made to vary, is a curve of order $n-1$, which has $2n$ points in common with the conic V , and therefore contains that conic entirely*.

The quantity y being now regarded as the parameter of a varying polygon, let p_n be chosen to represent that polygon

* This is in substance the remark of M. Darboux referred to. *Op. cit.*, p. 190.

which has the parameter of one of its sides equal to zero, and q_n to represent that which has the parameter of one of its sides infinite. Then g , which is $p_n : q_n$, will be an odd function of w , because of the symmetry of the figure and the fact that $q_n = 0$ (having one root infinite) is only of degree $n - 1$.

If then p_n and q_n be affected with such constant multipliers that $g = 1$ when $x = 1$, we must have $g = -1$ when $x = -1$. And we may suppose that $g = \pm \lambda^{-1}$ when $x = \pm k^{-1}$.

Now the intersections of corresponding sides of two inscribed-circumscribed polygons lie on a conic touching the common tangents of U and V .

For the parameters being respectively an , $an + \gamma$, &c., and an , $an + \gamma$, &c., the common difference of the arguments is $n - \gamma$.

If we suppose the two polygons to vary subject to this condition, their parameters g and g' will be connected by a (2, 2) correspondence, such that the values of g which make the two corresponding values of g' coincide are $\pm 1, \pm \lambda^{-1}$.

Therefore, if $g = \sin(u', \lambda)$, we must have $g' = \sin(u' + v', \lambda)$, where v' is a constant. But the relation between corresponding sides x, x' of these polygons is $x = \sin(u, k), x' = \sin(u + v, k)$, since they intersect on the fixed conic V .

Hence the quantities u, u' are so related that a constant difference between two values of u implies a constant difference between the corresponding values of u' . Hence* (by Euclid's definition of proportion) a varying difference between two values of u implies a *proportional* difference between the corresponding values of u' . But $u' = 0$ when $u = 0$; therefore the two quantities are proportional, and $u' = Mu$ where M is constant.

Now the relation between the parameters of two consecutive sides of a polygon being of the second degree in each, the products $x_1 x_n, x_2 x_{n-1}, \&c.$, are given as ratios of quadratic functions of w . Hence the product $x_1 x_2 \dots x_n$ regarded as a function of

* This is Archimedes' proof that a body which passes over equal spaces in equal times will pass over proportional spaces in unequal times.

x_1 , is a rational fraction whose numerator is of order n , and whose denominator is of order $n-1$. But y is a rational fraction whose numerator and denominator are of just these orders; and y vanishes whenever one of the quantities $x_1, x_2 \dots x_n$ vanishes, and becomes infinite when one of them becomes infinite. Therefore

$$y = mx_1 x_2 x_3 \dots x_n,$$

where m is a constant; that is to say,

$$\operatorname{sn} \left(\frac{u}{M}, \lambda \right) = m \operatorname{sn} u \operatorname{sn} (u + \gamma) \operatorname{sn} (u + 2\gamma) \dots \operatorname{sn} \{u + (n-1)\gamma\},$$

when $n\gamma = 4K + 4iK'$. We determine m by remarking that $y=1$, when $x=1$, or when $u=K$; and then λ , by remarking that $y=\lambda^{-1}$, when $x=k^{-1}$, or when $u=K+iK'$. Finally M is determined by differentiating the equation and making $u=0$.

(Cayley's Theorem.)—Every Cubic Transformation has an Elliptic Differential which it transforms.

This theorem was given by Prof. Cayley in the *Philosophical Magazine*, Vol. 15 (Fourth Series), p. 363. I here reproduce his investigation, slightly altered to suit the generalization which follows. On the very beautiful solution of the complete question (Given the elliptic differential, to find the transformation) by Hermite (*Crelle*, vol. 60, p. 304) and Clebsch (*Theorie der binären alg. Formen*, p. 405), I hope to say something at another time*.

Let U, V be any two cubic functions of x , and consider the transformation $y = \frac{U}{V}$.

Suppose that

$$\operatorname{Disct.} (U - Vy) = A + By + Cy^2 + Dy^3 + Ey^4,$$

where, of course, $A = \operatorname{Disct.} U, E = \operatorname{Disct.} V$. And let y_1, y_2, y_3, y_4

* The foot-note in my previous paper gave an erroneous expression for X . The article there referred to (Gundelfinger, *Math. Annalen*, Vol. VII., p. 452) is a simplification of the method and results of Clebsch in regard to the typical representation of two cubics.

be the roots of the equation $\text{Diect. } (U - Vg) = 0$. Then each of the cubics $U - Vg_1, U - Vg_2, U - Vg_3, U - Vg_4$ has a square factor, because its discriminant vanishes. Now, if $U - Vg_1$ has a square factor $(x - a)^2$, then $x - a$ divides $U' - Vg_1'$; that is, for the value $x = a$ we have at the same time

$$\begin{aligned} U - Vg_1 &= 0 \\ U' - Vg_1' &= 0 \end{aligned} \quad \left\{ U'', U''' = \frac{dU}{dx} = \frac{dV}{dx} \right\},$$

and therefore also $VU' - V'U = 0$; that is to say, these four linear factors which are squared in the cubics $U - Vg_i$, &c., occur as single factors in $VU' - V'U$. It follows that

$$A(U - Vg_1)(U - Vg_2)(U - Vg_3)(U - Vg_4) = (VU' - V'U)^2 X,$$

where X represents the product of the single factors of the four cubics. Or, which is the same thing,

$$AU^4 + BU^3V + CU^2V^2 + DVUV^3 + EV^4 = (VU' - V'U)^2 X.$$

Now, since $y = \frac{U}{V}$, we have $dy = \frac{U'V - V'U}{V^2} dx$. Hence, if

we transform the differential $\frac{dy}{\sqrt{\text{Diect.}(U - Vg)}}$ by the substitu-

tion $y = \frac{U}{V}$, we get

$$\frac{U'U' - V'U}{V^2} = \frac{U''}{(U' - V'U)\sqrt{X}} dx \text{ or } \frac{dx}{\sqrt{X}}.$$

That is, we have

$$\frac{dy}{\sqrt{\text{Diect.}(U - Vg)}} = \frac{dx}{\sqrt{X}},$$

where y is connected with x by the equation $y = \frac{U}{V}$, which is the theorem in question.

New Stand-Point for the Algebraic Transformation Theory.

We may generalize this result by applying an analogous treatment to transformations of any order. The problem is considered in the following form: Given a transformation $y = U : V$, it is required to find—

(1) What are the necessary and sufficient conditions to be satisfied by the functions U, V , in order that the transformation $y = U : V$ may be able to transform an elliptic differential;

(2) These conditions being supposed satisfied, what is the differential which can thus be transformed.

We will consider first the case in which U and V are of odd order, say $2m + 1$, or, to speak more correctly, the case in which $U = Vg$ in of order $2m + 1$ in x .

The necessary conditions may at once be derived from consideration of the varying in-and-circumscribed polygon the parameters of whose sides are the roots of the equation $U = Vg = 0$.

Starting with any one side of the polygon, which touches what we may call the inner conic, we find its intersections with the outer conic, and then from these draw new tangents to the inner conic. Proceeding in this way symmetrically on both sides of the original tangent, we find at last that the two tangents to the inner conic meet on the same point of the outer conic. We must clearly end with two tangents, and not with one, because the polygon has an odd number of sides.

We might, however, start with a vertex of the polygon on the outer conic, draw two tangents to the inner conic, then from their intersection with the outer conic two more tangents, and so on; at last we shall reach a pair of vertices such that the line joining them touches the inner conic. In this case we must end with two vertices on one side, and with one vertex on two sides, for the same reason as before, that we have supposed the polygon to have an odd number of sides.

Now suppose that in the first mode of construction we start with a common tangent to the two conics; then its two intersections with the outer conic will coincide, and consequently the tangents from them to the inner conic coincide also. We may, however, go on with the construction; and, after drawing $m + 1$ successive tangents, we shall have an exceptional case of an in-and-circumscribed polygon, in which the side first drawn (the common tangent of the two conics) counts singly, and each

of the m others counts doubly, so that the polygon has altogether $2m+1$ sides. But the last pair of tangents being coincident, must be regarded as intersecting on the inner conic, and therefore their point of contact must be an intersection of the two conics.

So that we cannot by the second mode of construction get a degenerate polygon of an odd number of sides different from those just considered. If we start with a vertex at a point of intersection of the two conics, the tangents drawn from this to the inner conic will of course coincide, and we therefore visit the points in which they meet the outer conic again, and we may continue the process; but we only have a right to stop when the line joining the two last coincident vertices on the outer conic (*i. e.*, the tangent at that point to the outer conic) touches the inner conic; that is to say, when we come upon a common tangent of the two conics.

What actually happens may be illustrated by the case of an ins-and-circumscribed pentagon. Let ab be a common tangent of the two conics, touching the inner conic at a , and the outer at b . From b draw the other tangent bc to the inner conic, meeting the outer conic again at c ; then from c draw the other tangent cd to the inner conic meeting the outer conic again at d . Then, *if pentagons can be drawn inscribed to the outer conic, and circumscribed to the inner, the point d will be an intersection of the two conics.* And the pentagon whose sides are ab, bc, cd, da , and in a degenerate case of an ins-and-circumscribed pentagon, the side bd being single, and the sides da, ad , each double, then single.

Observe that, if d were not an intersection of the two conics, we should still have an improper solution of the problem, to find five points on the outer conic such that the line joining every successive two shall touch the inner conic. But if we started from d as an intersection of the conics, and there found the points a, b, c as before, except that da is now not a tangent to the outer conic at a , we should not have found a solution of that problem, but of this other: The first five tangents to the inner conic, so that the intersection of every successive two shall be upon the outer conic.

For the purpose of our present investigation, the result may be stated thus: no in- and circumscribed polygon of an odd number of sides can only have two sides coincident, when one of its sides is a common tangent of the two conics, and all the others coincide in pairs.

Or, the equation $U - Vy = 0$ can only have coincident roots, when one represents a common tangent, and the others coincide in pairs: so that $U - Vy$ has one single factor, and m square factors. Moreover, there are four values of y , and four only, which bring this about. Now the discriminant of $U - Vy$ is of the order $4m$. Hence we must have

$$\text{Discr. } (U - Vy) = Y^2, \text{ where } Y = (U, y)^2,$$

and if y_1, y_2, y_3, y_4 are the roots of the equation $(U, y)^2 = 0$, then each of the quantities $U - Vy_i$, &c., has one single factor and m square factors.

These conditions are necessary; we shall show that they are sufficient, by solving the second part of the problem.

In the first place, we have

$$(U, y_1)(U - Vy_1)(U - Vy_2)(U - Vy_3)(U - Vy_4) = (U, V)^2,$$

and therefore, as before,

$$(U, V)^2 = (UV - V^2U)^2 X,$$

where $(UV - V^2U)^2$ is the product of all the square factors, and X of the four single factors. From this it follows immediately that

$$\frac{dy}{\sqrt{Y}} = \frac{dx}{\sqrt{X}},$$

if $y = \frac{U}{V}$; which is the transformation required. The result may thus be stated:

The necessary and sufficient conditions that the substitution $y = U : V$ shall be able to transform an elliptic differential, $U - Vy$ being of order $2m + 1$ in x , are that $\text{Discr. } (U - Vy)$ shall be a perfect m th power, and that those forms $U - Vy$ which have a square factor at all shall have m square factors. This

$$\text{CLIP.} \quad \text{UNIVERSITY OF CHICAGO LIBRARY} \quad 1880$$

being so, the differential $\frac{dy}{\sqrt[m]{\text{Disct.}(U - Vy)}}$ will be transformed by the given substitution into $\frac{dx}{\sqrt{X}}$, where X is the product of the four single factors.

It may be observed that, if those forms $U - Vy$ which have a square factor at all have m square factors, it will follow that $\text{Disct.}(U - Vy)$ is a perfect m th power; but the converse is not true.

Passing now to the case of a transformation of even order, we enquire, as before, in what cases the in-and-circumscribed polygon can have two sides coincident. If we start with a tangent to the inner conic, and from its intersections with the outer conic draw two more tangents, and so on; there cannot be an in-and-circumscribed polygon of an even number of sides, unless we come to a pair of intersections such that the line joining them touches the inner conic. Suppose then that the first side is a common tangent, so that its two intersections with the outer conic coincide, and that we draw another tangent from this point, another from its second point of intersection, and so on; we must finally come to a point on the outer conic where the tangent touches the inner conic; that is, we must come to another common tangent. In the case of a quadrilateral, for example, let ab be a common tangent, touching the inner conic at a and the outer at b . From b draw the other tangent to the inner conic, meeting the outer again at c ; then cd must be also a common tangent, touching the outer conic at c and the inner at d . Thus the sides of the degenerate quadrilateral are bab , bc , cdc , cb , the sides bab , cde counting singly, and bc double. And in general the two common tangents will count singly, and all the rest double. It is manifest that there are only two degenerate polygons of this kind, each containing two of the four common tangents.

The second construction also gives us two degenerate polygons, but of quite a different character. Starting with a point on the outer conic, we draw two tangents to the inner, and from their new intersections with the outer, two more, and so on;

we must at last come to two tangents which meet on the outer conic. If then our starting-point is a point of intersection, so that the two tangents coincide all through the process, we must come to a pair whose intersection, that is, their point of contact with the inner conic, is on the outer conic; or, which is the same thing, we must come to another intersection of the conics. To use again the quadrilateral as an illustration, the tangents to the inner conic at two points of intersection α and γ must meet on the outer conic at β , and the sides of the quadrilateral are then $\alpha\beta$, $\beta\gamma$, $\gamma\beta$, $\beta\alpha$, so that *all* of them count double. And generally, in degenerate polygons of this kind, all the sides count double. There are clearly two such degenerate polygons, each having two points of intersection for vertices.

To sum up, then, there are four degenerate polygons of even order $2m$; two of them have each two common tangents as sides, and two of them have each two points of intersection as vertices. The former have the common tangents as single sides, and all the rest double; the latter have all their sides double.

It follows that, if the substitution $y = U : V$ is capable of transforming an elliptic differential, $U - Vy$ being of order $2m$ in x , there are only four values of y which make $U - Vy$ have a square factor; two of these make it have m square factors, and the other two make it have $m - 1$ square factors and two single factors. Consequently the former two are m -fold roots, and the latter two $(m - 1)$ -fold roots, of the equation $\text{Disct. } (U - Vy) = 0$. That is to say, we have

$$\text{Disct. } (U - Vy) = Y^{m-1} : (y - y_1) (y - y_2),$$

where $Y = k (y - y_1) (y - y_2) (y - y_3) (y - y_4) = (1, y)^4$, say.

$$\text{Hence, as before, } (U, V)^4 = (VU' - V'U)^2 \cdot X,$$

where X is the product of the four single factors due to y_3 and y_4 . From these equations it follows directly that

$$\frac{dy}{\sqrt{Y}} = \frac{dx}{\sqrt{X}},$$

which is the transformation required.

In regard to these conditions it is to be observed that in general they imply a special constitution of the quantities U, V ,

as well as a special relation of them to each other. This consideration, however, does not come in until the sixth order of transformation is reached. Thus, in the case of the quartic transformation, the only condition is that U , V shall be simultaneously reducible to the canonical form; which being so, we may find linear combinations of them such that one is the Hessian of the other, thus falling back upon Hermite's very elegant form. In the quintic transformation V may be taken arbitrarily, but the involution $U = Vy$ is then completely determined. In the sextic transformation, however, U and V must each be the product of three quartics in involution (viz. the same involution in the two cases); so that a certain invariant of each must vanish. (Salmon's *Higher Algebra*, p. 219 and Appendix; see Clebsch, *Alg. Formen*, p. 238.)

*ADDITIONS TO PAPER ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

1. Completion of doctrine of in and circumscribed polygon so as to make it a formal proof of the transformation theory. In p. [216] the quantities n , n' are proportional because constant difference in the values of n means constant difference in the values of n' , and they vanish together.

2. Polygonatic representation of spherical harmonics as sums of sectorial harmonics. "Nodal curve" of order n may be expressed as a sum of n^{th} powers, $n + 1$ in number in a singly infinite number of ways.

3. Polyhedra whose faces osculate a skew-cubic. Special case noticed by Erdős. Involution of one variable: curve and ruled surface. Involution of two variables, surfaces of order $n - 2$; (2,2,2) correspondence for quadric surfaces; cases of degeneration: Δ -faced polyhedra. Application of Cauchy's theorem; vanishing area and volume of the polygonatic polygon and polyhedron.

4. Multiplication. Sylvester's theory of derived points on cubic. Do. for quanticubic. Sereda. Arithmetical Theorem.

* [These more heads of an intended paper are printed as they were found: no complete paper seems to embody them. Perhaps I was worked up into *XXIV., and it seems to be connected with XXV.]

Fig. 23.

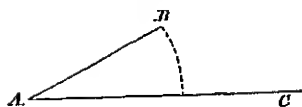


Fig. 24.

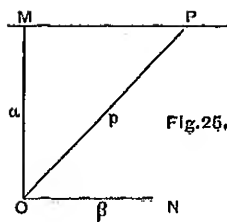
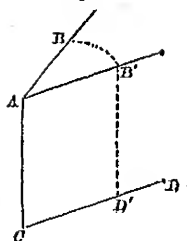


Fig. 25.

Fig. 26.

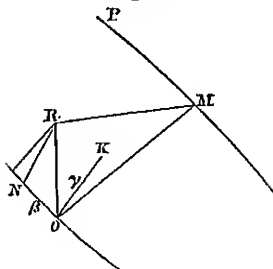


Fig. 27.

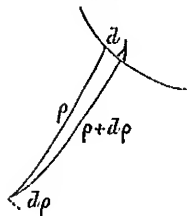


Fig. 28.

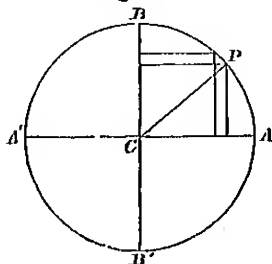


Fig. 29.

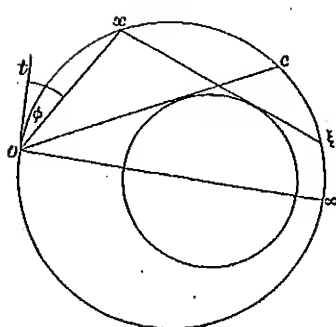
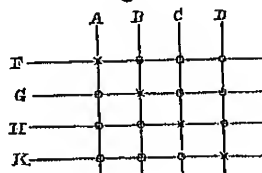


Fig. 30.



ON IN-AND-CIRCUMSCRIBED POLYHEDRA.

THE extension of the theory of the in-and-circumscribed polygon, made in my paper "On the Transformation of Elliptic Functions," was suggested by a particular case of it studied by Dr. Liüroth in the *Math. Annalen*, I. [pp. 37—53]. It had been remarked by Clebsch (*Crelle*, [Bd. 59]) that not every curve of the fourth order can have its equation expressed as the sum of five fourth-powers; but that for this to be possible a certain invariant (the determinant of the six second derived forms) must vanish. Dr. Liüroth proved that in this case the proposed reduction can be effected in a singly infinite number of ways, and that the lines represented by equating the fourth-powers to zero all touch the same conic. All these pentagrams, moreover, are totally inscribed in a covariant quartic of the original curve (locus of points whose covariant cubics are equianharmonic, so that their Hessians break up into three straight lines). We thus have a varying pentagram circumscribed to the conic and totally inscribed in a quartic; and it was this pentagram which suggested the theorems in my paper above referred to.

The starting-point of the present communication is a remark by Dr. Frahm on a question allied to the foregoing (*Math. Ann.* VII. p. 635). It had been assumed by Dr. Salmon that the equations of three quadric surfaces may simultaneously be reduced to the form of the sum of five squares. Now Hesse

established a connection between the theory of three quadric surfaces and that of a plane quartic curve; namely, if $u, v, w = 0$ are the three surfaces, then the equation

$$\text{Disct. } (\lambda u + \mu v + \nu w) = 0$$

is of the fourth order in λ, μ, ν , and taking these as co-ordinates of a point in a plane, the equation is that of a quartic curve which corresponds point for point with the locus of vertices of the cones $\lambda u + \mu v + \nu w = 0$. Dr. Frahm remarked that if the three quadrics could be simultaneously reduced to Salmon's canonical form, then this quartic curve is totally circumscribed to a pentagram, and is therefore not the general quartic, but the special form pointed out by Lüroth; so that the problem of effecting this reduction is porismatic—it can either not be solved at all or be solved in a singly infinite number of ways. In the latter case, then, we have a singly infinite number of pentaplanes in regard to which the reduction can be effected; and each of these is totally inscribed in a curve of order 6 and deficiency 3, locus of the vertices of the cones

$$\lambda u + \mu v + \nu w = 0.$$

On considering the envelope of these pentaplanes, I found it to be a twisted cubic. The road to further generalizations was now clearly open.

I.

The equation of any osculating plane to a twisted cubic may be written

$$X + 3\theta Y + 3\theta^2 Z + \theta^3 W = 0,$$

where $X, Y, Z, W = 0$ are four fixed planes, and θ a parameter determining the particular osculating plane. The developable generated by tangent lines to the cubic is

$$(XZ - Y^2)(YW - Z^2) - 4(XW - YZ)^2 = 0 \dots\dots\dots(1),$$

and from this we see that Y passes through the tangent line of X and the point of contact of W, Z through the tangent line of W and the point of contact of X ; or we may say that XY

and ZH' are two tangent lines, YZ their chord of contact. The equations to the cubic itself are

$$\begin{vmatrix} X & Y & Z \\ Y & Z & W \end{vmatrix} = 0.$$

The co-ordinates of the point of intersection of three planes x, y, z are

$$\begin{vmatrix} 1 & 3x & 3x^2 & x^3 \\ 1 & 3y & 3y^2 & y^3 \\ 1 & 3z & 3z^2 & z^3 \end{vmatrix} = 3xyz : -yz : -zx : -xy : x + y + z : -3,$$

and if we substitute these in the function $X + 3aY + 3a^2Z + a^3W$ belonging to any fourth plane, we get $3(x-a)(y-a)(z-a)$.

If a variable group of n osculating planes of a twisted cubic form an involution of the n^{th} order, the locus of their lines of intersection is a ruled surface R of order $2(n-1)$, and the locus of their points of intersection is a curve of the order of which is $\frac{1}{2}(n-1)(n-2)$, and which is a triple curve on the ruled surface.

Consider the sections of this curve and surface by a fixed osculating plane L of the cubic. There is one group of the involution to which it belongs, and it meets the curve only where it meets the lines of intersection of the remaining $n-1$ planes of this group; that is, in $\frac{1}{2}(n-1)(n-2)$ points. This therefore is the order of the curve. All other osculating planes meet the plane L in lines which touch a fixed conic in that plane; in fact, it meets the developable (1) in this conic and in its tangent line taken twice. The variable group of n planes in involution determines upon L a variable group of n tangents in involution; and the locus of their intersections is a curve of order $n-1$, by what we have already proved. Besides this curve, the plane L meets the ruled surface in $n-1$ straight lines, in which it meets the $n-1$ other planes of the group which contain it; so that the order of the whole intersection is $2(n-1)$, which is therefore the order of the surface. That the curve is a triple curve on the surface is clear from the fact that through any point of it there may be drawn three straight lines in the

surface, these being symmetrically related and not in the same plane.

By means of the in-and-circumscribed polygon determined upon the plane L we may find the equation of the ruled surface. Since the parameters of the several osculating planes of the cubic may be taken as parameters of the tangents to the conic in the plane L in which they are cut by that plane, it follows that the condition for two osculating planes x, y to belong to the same group is the same as the condition for the two tangents x, y to belong to the same group; that is to say, it is a condition of the form

$$\sum_i \frac{\alpha_i}{(x-a_i)(y-a_i)} = 0 \text{ or } \sum_i \frac{\alpha_i}{x-a_i} = \sum_i \frac{\alpha_i}{y-a_i} \dots\dots\dots (2),$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are parameters of some one group, and a_1, a_2, \dots, a_n are constants.

When this condition is fulfilled, the line of intersection of the planes x, y meets the curve γ . Now through any point on the surface R there can be drawn a line which is the intersection of two osculating planes of the cubic, and these two planes will satisfy the condition (2). If therefore the point of intersection of xyz lies on the surface R , the condition (2) must be satisfied either for yz or for xz or for xy . That is, we must have

$$\sum \frac{\alpha_i}{(y-a_i)(z-a_i)} \cdot \sum \frac{\alpha_i}{(z-a_i)(x-a_i)} \cdot \sum \frac{\alpha_i}{(x-a_i)(y-a_i)} = 0 \dots\dots (3).$$

It remains to translate this equation into the ordinary point-co-ordinates. Let $A_1, A_2, \dots, A_n = 0$ be the equations to the n planes whose parameters are $\alpha_1, \alpha_2, \dots, \alpha_n$; that is, let

$$A_i = X + 3a_i Y + 3a_i^2 Z + a_i^3 W;$$

then the result of substituting in A_i the co-ordinates of the point of intersection of x, y, z is $3(x-a_i)(y-a_i)(z-a_i)$. If then we multiply together terms of like suffixes in the factors of (3), we get in the product the sum of n terms

$$9 \sum_i \frac{\alpha_i^3}{A_i^3}.$$

Next, the equation of the plane passing through the tangent line of A_i and the point of contact of A_j is

$$(ij) : X + 2a_i Y + a_i^2 Z + a_i (Y + 2a_i Z + a_i^2 W) = 0,$$

and when we substitute in this the co-ordinates of intersection of xyx_i we obtain

$$(x - a_i)(y - a_j)(z - a_i) + (y - a_j)(z - a_i)(x - a_i) + (z - a_i)(x - a_i)(y - a_i).$$

Selecting then from (3) the products of two like suffixes by one unlike, we get the sum of $n(n-1)$ terms

$$\sum_{i \neq j} a_i^2 a_j (ij),$$

hence the equation of the plane passing through the points of contact of A_i, A_j, A_k is

$$(ijk) : X + (a_i + a_j + a_k) Y + (a_i a_j + a_j a_k + a_i a_k) Z + a_i a_j a_k W = 0,$$

and when we substitute in this the co-ordinates of intersection of x, y, z , we get $\frac{1}{6} \sum (x - a_i)(y - a_j)(z - a_k)$, where the x, y, z are to be permuted in all possible ways. Thus the products in (3) of three unlike suffixes give us the $\frac{1}{6} n(n-1)(n-2)$ terms

$$\frac{1}{6} \sum_{i \neq j \neq k} a_i a_j a_k (ijk),$$

and the equation to the surface R is therefore

$$0 = 1 \frac{1}{6} \sum_i a_i^3 + \frac{n}{2} \sum_{i \neq j} a_i^2 a_j (ij) + \sum_{i \neq j \neq k} a_i a_j a_k (ijk).$$

The equation shows that when cleared of fractions it is an integral of the order $\frac{1}{2} n(n-1)$.

XXV.

ON A CANONICAL FORM OF SPHERICAL HARMONICS*.

THE canonical form in question is an expression of the general harmonic of order n as the sum of a certain number of sectorial harmonics, this number being, when n is even,

$$\frac{5n-10}{2},$$

and when n is odd,

$$\frac{5n-9}{2}.$$

Laplace's operator,

$$\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2},$$

may be obtained from the tangential equation of the imaginary circle $\xi^2 + \eta^2 + \zeta^2 = 0$, by substituting $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ for ξ , η , ζ . If, therefore, a form $U \equiv (x, y, z)^n$ is reduced to zero by this operation, it follows from Prof. Sylvester's theory of contravariants that the curve $U = 0$ is connected by certain invariant relations with the imaginary circle. I find that U can be

* [Notices and Abstracts...from Report of the Forty-first Meeting of the British Association for the Advancement of Science, held at Edinburgh, August, 1871, p. 10. A discussion followed the reading of Prof. Clifford's paper, and a result was that in the same Report, pp. 25, 26, is printed a communication by Sir W. Thomson, On the General Canonical Form of a Spherical Harmonic of the n th order. In this Sir W. Thomson answers the question, Can canonical forms not be found in which the nodal conic of each constituent is not resolvable into circular cones and planes?]

expressed in the form

$$U \equiv A^n + B^n + C^n + \dots$$

where $A = 0, B = 0, \dots$ are great circles touching the imaginary circle, the number of terms being as above. Now if $L = 0, M = 0$ be two such great circles meeting in a real point a , and if ϕ be a longitude and θ latitude referred to a as pole, it is easy to see that

$$L^n + M^n = l \sin^n \theta \sin n\phi + m \sin^n \theta \cos n\phi,$$

a sum of two sectorial harmonics, which is the proposed reduction.

When n is less than 5, exceptions of interest occur. For $n = 3$, if we take a, b , corresponding points on the Hessian of the nodal curve $U = 0$ (Thomson and Tait, *Treatise on Natural Philosophy*, § 780 [first edition]), and if we call ϕ_1, ϕ_2 the longitudes, θ_1, θ_2 the latitudes referred to these poles, we have

$$\begin{aligned} U \equiv & l \sin^3 \theta_1 \sin 3\phi_1 + m \sin^3 \theta_1 \cos 3\phi_1 \\ & + n \sin^3 \theta_2 \sin 3\phi_2 + s \sin^3 \theta_2 \cos 3\phi_2. \end{aligned}$$

For $n = 4$, the nodal curve is of the species first noticed by Clebsch, of which many most beautiful properties have been pointed out by Dr. Lüroth. The form U is expressible as the sum of *five* fourth powers; so that if we take a, b real points of intersection of two pairs of them, c a real point on the fifth, calling $\phi_1, \phi_2, \phi_3, \theta_1, \theta_2, \theta_3$ longitudes and latitudes referred to them, we have

$$\begin{aligned} U \equiv & l \sin^4 \theta_1 \sin 4\phi_1 + m \sin^4 \theta_1 \cos 4\phi_1 \\ & + p \sin^4 \theta_2 \sin 4\phi_2 + q \sin^4 \theta_2 \cos 4\phi_2 \\ & + r \sin^4 \theta_3 \cos 4\phi_3. \end{aligned}$$

XXVI.

ON THE FREE MOTION UNDER NO FORCES OF A RIGID SYSTEM IN AN N -FOLD HOMALOID. - (Provisional Notice.)*

THE problem of the rotation under no forces of a rigid body about a fixed point in ordinary three-dimensional space is the same as the problem of free motion under no forces; for the motion about the centre of inertia takes place as if it were a fixed point. But it is also the same thing as the problem of the free motion of a rigid system on the surface of a sphere, or in elliptic space of two dimensions†. So also the problem of the free motion of a solid in elliptic space of three dimensions is the same as that of the free motion, or motion about a fixed point, in parabolic or homaloidal space of four dimensions. And, in general, the problem of free motion in elliptic space of n dimensions is identical with that of free motion, or motion about a fixed point, in parabolic space of $n + 1$ dimensions.

The form of the problem which is considered in what follows is that which deals with the motion about a fixed point in parabolic space of n dimensions.

* [From the *Proceedings of the London Mathematical Society*, Vol. VII., Nos. 92, 93, pp. 67—70.]

† According to Dr Klein's nomenclature, a space every point of which can be uniquely represented by a set of values of n variables is called elliptic, parabolic, or hyperbolic, when its curvature is uniform and positive, zero, or negative. The geometry of the sphere becomes elliptic when opposite points are regarded as identical.

I.

Let the coordinates of a point, referred to a rectangular system, be x_1, x_2, \dots, x_n . If this point belongs to a rigid system in motion, its velocity is given by the equation

$$\dot{x}_k = \sum p_{k\alpha} \dot{\alpha}_k \quad (k = 1, 2, \dots, n) \quad (1),$$

where $p_{k\alpha} = p_{\alpha k}, p_{k\alpha} = 0$. The $\frac{1}{2}n(n+1)$ quantities $p_{\alpha k}$ of the nature of rotational velocities of the rigid system. It may be observed that if the vector from the origin to the point x be represented in terms of n unit vectors e_1, e_2, \dots, e_n , satisfying the equations $e_k e_k = e_k e_k e_k^2 = 1$, then the velocity of the rigid body may be represented in terms of the $\frac{1}{2}n(n+1)$ products $e_k e_l$; namely, we may write

$$\rho = \sum e_k \dot{x}_k = 2\rho = \sum p_{\alpha k} \dot{\alpha}_k,$$

and then the equation (1) may be put into the form

$$\rho = Uq_{\alpha}.$$

If dm be the element of mass at the point x , its kinetic energy is

$$\frac{1}{2} \sum \dot{x}_k^2 dm = \frac{1}{2} dm (\sum p_{k\alpha}^2 \dot{\alpha}_k^2 + \sum \dot{\alpha}_k^2) + \sum \sum p_{\alpha k} p_{\beta l} \dot{\alpha}_k \dot{\alpha}_l.$$

Let then $\int \frac{1}{2} \sum \dot{x}_k^2 dm = \frac{1}{2} \rho^2$, $\int \sum p_{k\alpha}^2 dm = \rho_{\alpha\alpha}$,

the integrations extending over the whole rigid system; then, if T be the kinetic energy of the system,

$$2T = \sum \rho_{\alpha\alpha} \dot{\alpha}_k^2 + \sum \dot{\alpha}_k^2 + 2 \sum \sum p_{\alpha k} p_{\beta l} \dot{\alpha}_k \dot{\alpha}_l. \quad (2)$$

Write now $q_{\alpha} = \frac{\partial T}{\partial p_{\alpha k}}, \quad \rho = 2T = \sum q_{\alpha k} \dot{\alpha}_k$,

then q is the momentum of the system; it is a linear function of the velocity, or $q = q_0 + p\dot{\theta}$, and twice the kinetic energy is the scalar part of the product of velocity and momentum, $2T = \sum q_{\alpha k} \dot{\alpha}_k$. The equations of motion are $\dot{q} = f$, where f is the system of applied forces, or in the present case of no forces $\dot{q} = 0$; viz., this is equivalent to $\frac{1}{2}n(n+1)$ equations. From this we get the first integrals, $q = \text{constant}$, and (since $0 = \sum g_j \dot{q} = 2\dot{T}$) $T = \text{constant}$.

But these equations are inconvenient, because the α and β are variable, depending upon the position of the body. We must therefore follow Euler in referring the motion to axes moving with the body, and coinciding with the principal axes at the fixed point. This will make all the β vanish; and we shall have

$$q_{hk} = (\alpha_h + \alpha_k) p_{hk}.$$

From equation (1) we obtain

$$\dot{q}_{hk} = \int dm (\ddot{x}_h x_k - x_h \ddot{x}_k) = \Sigma_i \int dm \{ \dot{x}_i (p_{hi} x_k - p_{ki} x_h) + x_i (\dot{p}_{hi} x_k - \dot{p}_{ki} x_h) \}.$$

Assuming that $\int dm x_h x_k = 0$ when h and k are different, and remembering that $\dot{q}_{hk} = 0$, we may write this equation in the form

$$(\alpha_h + \alpha_k) \dot{p}_{hk} + (\alpha_h - \alpha_k) \Sigma_i p_{hi} p_{ki} = 0 \dots\dots\dots (3).$$

Here \dot{p}_{hk} means the rate of change of that component of velocity which coincides at the moment with one of the principal components; it must be distinguished from the rate of change of the principal component, which we shall call $(\dot{p})_{hk}$. In general, if the system of axes has the rotational velocities r_{hi} , we shall have

$$\dot{p}_{hk} = (\dot{p})_{hk} + \Sigma (p_{hi} r_{ki} - p_{ki} r_{hi});$$

but in the present case the r are equal to the p , because the axes move with the body, and so $\dot{p}_{hk} = (\dot{p})_{hk}$. Thus in the equations (3), which are analogous to Euler's equations for three dimensions, the symbols \dot{p}_{hk} may be understood to mean the rates of change of the principal components of rotation.

If the symbol V_2 is regarded as selecting all the binary products of $\iota_1 \iota_2 \dots \iota_n$ out of any rational integral function of them, the last equations may be written

$$\dot{q} = V_2 p \dot{q};$$

and it may be observed that we have also

$$q = \int dm V_2 p \dot{p}.$$

II.

These equations may be integrated by means of 6 functions of $n+2m+s$ arguments, one of which is a linear function of the time. It is most convenient to use these in the form employed by Cleop for the case $s=0$. Let u_1, u_2, \dots, u_s be the arguments, and let $R_1, R_2, \dots, R_p, \dots, R_q, \dots, R_n$ be n^2 constant quantities, and let

$$E_s = (u_s + 2m_1 R_s + 2m_2 R_{2s} + \dots + 2m_p R_{ps})^2 = (u_s + 2\sum m_k R_{ks})^2;$$

then we shall write

$$G(u_1, u_2, \dots, u_s) = \sum_{k=1}^s 2R_k E_k,$$

where the whole numbers m are to take independently all values from $-\infty$ to $+\infty$. It is clear that the function G is unaltered if we simultaneously increase u_1, u_2, \dots, u_s by equimultiples of the quantities $2R_1, 2R_2, \dots, 2R_n$, for this is only increasing m_k by an integer. Moreover, if we determine s^2 quantities A so that

$$4\sum_k A_k R_k = \pi i, \quad \sum_k A_k R_k = 0,$$

then the function $e^{2\pi i G}$ is unaltered if we simultaneously increase u_1, u_2, \dots, u_s by equimultiples of $4A_1, 4A_2, \dots, 4A_s$. In what follows we shall write for shortness G ($n+X$) instead of $G(u_1+X_1, u_2+X_2, \dots, u_s+X_s)$, omitting always the last suffix, and mentioning only one argument u .

A linear function of the A, R with coefficients 0 or 1 will be called a *quadrant*; there are clearly 2^n quadrants, if none be included among them. Let X and Y be two quadrants, A_X the difference between those parts of them which involve the quantities A , then the function

$$e^{2\pi i A_X} \frac{G(u+X)}{G(u+Y)} = A_{X,Y}(u)$$

is 2 π periodic in the arguments u , the periods being $4A_1, 4A_2, \dots, 4A_s$. It is convenient to speak of the *distance* of two quadrants X and Y , meaning the number of coefficients of the A and R which must be changed from 0 to 1, or vice versa to make one of them into the other. This distance may be any of the numbers

1, 2, ... 2s; and accordingly there are 2s really distinct 2s periodic functions.

It is, however, possible to form a group of $n+1$ 2s equal roots X_1, X_2, \dots, X_n having such a relation to the operator n that if we write $Al_{\alpha}(u)$ for n to X_{α} and $X_{\alpha} + X_{\beta} + \dots + X_n$, then $\frac{1}{2}n(n-1)$ functions $Al_{\alpha}(u)$ satisfy the equation

$$\delta_{\alpha} Al_{\alpha}(u) = \sum_{\beta} c_{\alpha\beta} Al_{\beta}(u) + Al_{\gamma}(u) + \dots + Al_{\delta}(u),$$

where the δ_{α} applies to any one of the arguments; but the values of the c will depend upon which argument is taken. In the case of the hyperelliptic functions, the four operators A may be taken to be the quantities A_1, A_2, F_1, F_2 .

It appears therefore that the equations will always be integrable if we write $p_{\alpha} = \lambda_{\alpha} Al_{\alpha}(u)$, where λ_{α} is a function of u .

III.

From the $\frac{1}{2}n(n-1)$ equations (3) let us pick out $n-1$, namely

$$-(\alpha_1 + \alpha_n) p_{1n} = (\alpha_1 - \alpha_n) \sum p_{12} p_{2n},$$

if we write $(\alpha_1 + \alpha_n) p_{1n} = (\alpha_1 - \alpha_n) l_{1n}$, these equations become

$$-l_{1n} = \sum p_{12} l_{2n}.$$

But these are the equations for the velocity of a fixed point E relative to the moving axes in $n+1$ dimensions. The rest of the equations (3), if we write in them $p_{1n} = 0$, become the equations for the component relations in $n-1$ dimensions. Thus the solution for the rotational velocities and the position of a point fixed in space, for $n+1$ dimensions, are obtained by diminishing the number of periods in the solutions for rotations; it consists accordingly of the Al functions expressing the rotation-velocities, and of Rosenhain's combinations of Al functions and exponentials, expressing the position.

XXVII.

ON THE CANONICAL FORM AND DISSECTION OF A RIEMANN'S SURFACE*.

THE object of this Note is to assist students of the theory of complex functions, by proving the chief propositions about Riemann's surfaces in a concise and elementary manner. To this end I assume only certain results of Poincaré, which are put together at the outset.

I.

Poincaré's Theory of an n -valued Function.

If two variables x and z are connected by an equation of the form $f(x, z) = (z, 1)^n (z, 1)^n = 0$, each is said to be an algebraic function of the other. Regarding z as a complex quantity as in §II, we represent its value by the point whose co-ordinates are x, y , on a certain plane. To every point in this plane belongs one value of z , and consequently, in general, n values of x , which are the roots of the equation $f = 0$. The points of the plane may be divided into those at which the n values of x are distinct, and those at which two or more of them are equal. The latter points are finite in number, and correspond to the roots of the equation which is got by equating to zero the discriminant of f in regard to x . If the roots of this equation are distinct, there are $2(n-1)n$ such points, because the discriminant of the

* (From the *Proceedings of the London Mathematical Society*, Vol. xiii., No. 122, pp. 222-264.)

equation of the n th order in s is of degree $2(n-1)$ in the coefficients, and these coefficients are of the order m in z . But a point at which r values of s become equal corresponds to an $(r-1)$ -fold root of the discriminant-equation.

Let us now consider an arbitrary point O of the plane [fig. 31], corresponding to a value z_0 of z , which is not a root of the discriminant-equation. Then the equation $f(s, z_0) = 0$ will give n different values for s , which we may call $s_1, s_2 \dots s_n$. If we move along any path from the point O to another point P of the plane, the value of z will change continuously, and each of the quantities $s_1, s_2 \dots s_n$ will also change continuously. If therefore the path OP does not go through a point where two values become equal, these n quantities will be distinct all the way, and each of the n values of s at P will belong to a definite one of the values of s at O . But if the path goes through such a point, two or more of the n quantities will become equal and then diverge again, so that it will be impossible after that to distinguish them so as to say which of these belongs to a particular one of the values at the point O . We cannot always avoid this difficulty by going *round* the point, for it is found that the values at P to which the values at O correspond may depend upon the path OP , so that the correspondence is different for a path which goes to the right of the point and for a path which goes to the left of it. When this is the case, the point is called a branch-point. Suppose that, when we go from O to A , the two values p and q of s at O approach one another and become equal at A ; then it is found that the value at P which represents p when we go along the path OBP may represent q when we go along OCP , and *vice versa*. So that, if we travel along $OBPCO$, round the point A and back to O , the values p and q will change continuously into one another. If more than two values are equal at A , the corresponding values at O may be cyclically interchanged by a path going round A . We shall assume, however, that only two values become equal at each branch-point; and, moreover, that no branch-point is at an infinite distance*.

* Roots of the discriminant-equation which are not branch-points correspond to double points on the curve $f(s, z) = 0$. Such points behave, in regard to

A path going along any line from O to very near A , then round A in a very small circle, and then back to O along the same line, will be called a *loop*.

If we start from O and go round any closed curve not including any branch-points, the n values of s at O will be restored in the same order. For the path may be gradually shrunk into a point without crossing any branch-points, so that no two of the n values can become confused at any point of it. The same thing is true if the closed path includes *all* the branch-points. Suppose it a large circle through O ; then it may be gradually increased till it coincide with the tangent at O , then curved over on the other side, and shrunk up into a point; and during the whole process the n values will be distinct at every point of the path.

We shall now go on to show that this n -valued function, which we have spread out upon a single plane, may be represented as a *one*-valued function on a surface consisting of n infinite plane sheets, supposed to lie indefinitely near together, and to cross into one another along certain lines. This surface is called a **RIEMANN'S SURFACE**; we shall demonstrate its existence at the same time that we shew how to construct it in the most convenient form.

II.

Construction of the Riemann's Surface.—Férol's Theorem.

Draw loops from O [fig. 32] to all the branch-points, and let the first, A , interchange the values p and q . If we go round all the loops successively, starting with the value p at O , we must, as we have seen, come back to that value; but this may happen before we have used all the loops. Let B be the first branch-point after going round which the value p is restored. Draw a line from A to B cutting all the loops which alter p , but none of the others. Then, if we go round any of the

the function s , like two coincident branch-points belonging to the same pair of values, and they have no influence on the connection of the different values of s .

branch-points between A and B without crossing the line AB or going round any other branch points, we shall not alter the value p .

Suppose that A interchanges p & q , B interchanges q & r , and that the branch-points between A and B are $1, 2, 3, 4$, interchanging respectively qr, ra, bk, pl . The value q may then be changed into the value p through a loop or other series of values; the loops interchanging lk and pl are good, as an example. Now if we go round 4 by the dotted loop passing round outside A , the effect is the same as going round round $A, 1, 2, 3, 4, 3, 2, 1, A$. By the time we have gone round $A, 1, 2, 3$, we cannot have the value p , for that is first restored by B ; and we cannot have the value l , for then 4 would restore the value p . Hence we have some value which is not altered by the loop to 4 ; and consequently, when we retrace our path, we shall come back to the value p .

Next, let us draw a loop to B which passes within the line AB , but goes round all the included branch-points, as in the figure. The effect of this loop will be to change q into p , for it is the same thing as going round $1, 2, 3, B, 3, 2, 1$. Now the effect of $1, 2, 3, B$ is to change q into p , and this gets not altered in coming back because all the branch-points which alter p are outside the line AB .

Suppose then that all the branch-points of this group which alter p are connected with O by loops going round A , so that they no longer alter p ; and that B is connected with OO by the loop just described, so that no branch-points are contained in the triangle AOB .

Starting now from this new loop OO , with the value p , let us go round all the loops as before from left to right. We know that when all the loops have been gone round, nothing will be left, the value p must be restored. If it is not restored before we have gone round OO , we must draw a line OO cutting all the loops which change the value p but restore it after returning. But if the value p is restored before we have gone round OO , say after going round OO' , then we must draw a new loop OO' going round all the branch-points between A and O' except those which change the value p . This new loop will, by some procedure

reasoning, change p into q . Hence, if the value p is restored before we have gone round OA , we can make a new loop OC which changes p into q ; and this comes next to OB . To those branch-points whose loops have been cut by this new loop we must draw new loops going round to the right of C , so as not to cut OC . The figure comes then into this form [fig. 33], containing

- (1) Loops to the left of OA which do not change the value of p , like the dotted loop OB in the previous figure;
- (2) Three consecutive loops OA , OB , OC which change p into q ;
- (3) Loops to the right of OC which may or may not change p .

If now we start with the loop OC and proceed to the right, the value p must be restored *before* we have gone round OA ; for, starting with OA and going all round, we must restore the value p in the end. Let p then be restored by OB ; and draw a line CD cutting all those loops which change p , but none of the others. Replace the loops which change p by now ones going round between B and C ; and replace OB by a new loop going outside all the branch-points whose loops do not alter p . The figure now consists of these elements:

- (1) Two triangles AOB , COB , containing no branch-points, and such that the loops OA , OB , OC , OB interchange p and q ;
- (2) Loops between OB and OC which do not change p ;
- (3) Unknown loops between OB and OA .

About these unknown loops we may make three suppositions.

First, suppose that none of them change p . Then the value p cannot be altered by any closed curve starting from O and returning to it which does not cut either of the lines AB , CA .

Secondly, suppose that some of these loops change p , but that, when we start with the loop OB and go round to the right, the value p is first restored by OA or OC . All is then that it cannot be first restored by OB , because the two loops OA , OB , taken together, make no change in any value; nor by any loop

between OB and OC , for none of them change p). Then we must join D with A by a line cutting all the loops which change p , but no others; and B with C by a line cutting none of the loops between OB and OC . In that case the value p cannot be altered by any closed curve starting from O and returning to it which does not cut either of the lines BC , DA .

Thirdly, suppose that the value p is restored *before* we come to OA , say at OE . Then we must proceed as before, finding a new line EE' which shall have the properties of AA' in CD . The figure will then consist of three triangles AOB , COE , EOE' , containing no branch-points, and such that the loops OA , OB , OC , OD , OE , OE' interchange p and q ; loops between OB and OC , and between OD and OE , which do not change q ; and unknown loops between OE' and OA .

It is clear that this process must ultimately stop, and then we shall be left with a finite number of lines such that, if we start from O , follow any continuous path, and come back again, without crossing any of these lines, we shall not alter the value p . The lines are either AA' , CC' , EE' , &c., or also they are BB' , DD' , &c.; in either case the loops OA , OB , ... interchange p and q .

It follows that, if we take an infinite plane sheet and cut it through along these lines, we may consider a single value of the function s to be attached to every point of the sheet as well as a way that this value varies continuously when we move about continuously in the sheet; but there will be different values on the two sides of any cut. Namely, we must attach to every point P of the sheet that value of s which changes continuously into p when we go from P to O without crossing any of the cuts. There is only one such value, for if two different paths from O to P gave different values at P , it would be possible to change the value p by means of a closed curve returning to O , and this we have proved not to be the case.

When the lines cut through are AA' , CC' , ... the triangles AOB , COE , ... contain no branch-points; but when the lines are BB' , DD' , ... the triangles BOC , DOE , ... in general contain branch-points. We may, however, draw new loops for C' , E' , ... so as to exclude these branch-points, and the new loops will still change p into q . For no closed curve going round B and C'

to cut BC can change the value p , by what we have proved; but the loop OB changes p into q , therefore change q into p .

shall assume then that the cuture AB, CD, \dots , and that also $A'B', C'D', \dots$ contain no branch-points.

Let us deal with the value q at O in the same way we dealt with the value p . It is first to be observed that going round once or more of the lines AB makes no *any* value at O ; so that, if we agree never to cross AB , we may leave the branch-points A, B, \dots entirely undisturbed.

Forcing us, let us take a loop which changes q into some *new*, say r . There must be such a loop, if the function then two-valued; for otherwise the values p, q would be valued algebraic function of z , and the expression could have a factor of the second degree in z .

Using then with this loop, we may proceed in exactly the same before, and draw lines $A'B', C'D', \dots$ such that a new, starting from O and coming back to it without any of these lines or any of the previously drawn lines, alters the value q . Moreover, we shall have drawn loops γ_1, \dots each of which changes q into r , and such that the $A'B', C'D', \dots$ contain no branch-points. And since our triangles $A'OB', C'OD', \dots$ contained no branch-points, will not have been necessary to cut through them in the new lines $A'B', C'D', \dots$.

shall now speak of the first set of lines AB, CD, \dots as (p, p) , and of the second set as the lines (p, r) .

we take two infinite plane sheets, cut them both through n lines (p, p) , but only the *second* cut along the lines (p, r) ; every point of the *first* sheet we will suppose attached to n of n which is altered at by continuous change of the r (q); and to every point of the *second*, that value which r at by continuous change of the value q at O .

each sheet there will be a finite difference in the values n sides of each of the cuts (p, p) ; but the value on one or upper sheet will be equal to the value on the other or lower sheet. At the cut AB , for example, the value

continuous with p on the side next to O is equal to the value continuous with q on the side remote from O ; because a path taken round A or B from O and back again changes the value p continuously into the value q .

Thus, if we take p, q to denote values at the cut continuous with p, q at O , they will be situated as in the figure [Fig. 34], which represents a section across AB perpendicular to the two sheets. If then we make the two sheets cross one another along the lines p, q , as here represented [fig. 35], then these two values will be continuously distributed on the double sheeted surface so formed.

We may now continue the process with the value r . We must first find a loop which changes r into some other value, say t , and then proceed as before, taking care not to cross the lines qr . (We may cross the lines pq as often as we please, provided that we have not previously crossed the lines qr ; for these lines can have no effect upon r unless it has been previously changed into q .) Then we shall draw lines rt such that the value r cannot be altered by a closed curve not cutting the lines qr or rt , and having their extremities joined to O by loops which change r into t . If we take, then, a third sheet, cut it through along the lines qr and rt , and then join it to the first by any second sheet along the lines qr ; the three values p, q, r may be continuously distributed on this three sheeted surface.

By proceeding in this way it is clear that we shall construct an n -sheeted surface, the sheets of which are connected chain-wise by cross lines, so that the first is connected only with the second, the second with the third, and so on; but there is no direct connection except between consecutive sheets. And the n values of the function may now be attached to the points of this surface, so that one value only belongs to each point, and that in moving this point about on the surface the values belonging to it always changes continuously. Thus, at any point from a given point of the surface (on a given sheet), and as reached by any path so as to come back to the same point (on the same sheet), we shall in all cases return to the former value of the function x .

The theorem that the Riemann's surface belongs to one value

assumed that the sheets are only connected *chainwise*—i.e., so that there are no cross lines except between consecutive sheets—is due to Dr. Liireth.

III.

Uebachs's Theorem.

All the links between successive sheets except the last may be made to consist of one cross line only.

First, we shall prove that, if there are two or more lines (pq) , one of them may be converted into a line (qr) .

The original position of the two lines (pq) and the line (qr) is shown in fig. [36]. If we move the line qr , keeping, of course, its ends fixed, the effect is to interchange the sheets qR in the area over which it moves; so that, by passing it over the line (pq) on the right, we change this into a line (qr) . The position is then as in fig. [37]. If now we pass the remaining line (pq) over this line (qr) , we change it into a line (qr) ; thus we are left with two lines (qr) and one line (pq) . [Fig. 36.]

In this way we may convert all but one of the lines (pq) into lines (qr) . Then we may convert all but one of the lines (qr) into lines (rs) ; and so on. Then the first $n-1$ sheets will be connected chainwise by one crossline each, and the last two by all the remaining cross lines.

The Riemann's surface is now said to be in its canonical form.

The process of transformation may be made clearer by looking at a portion of the three sheets by a plane perpendicular to them cutting the lines pq , qr , pq (figs. 38, 40, 41).

IV.

Transformation of the Riemann's Surface.

The Riemann's surface now consists of n infinite plane sheets, such that the sheet 1 is connected with 2 by a single crossline, 2 with 3 by another crossline, and so on; but $(n-1)$

with (n) by a number of cross-lines which we shall call $p+1$. Thus the whole number of cross-lines is $n-2+p+1=n+p-1$. If w is the number of branch-points, this is twice the number of cross-lines, or $w=2(n+p-1)$. Hence $p=\frac{1}{2}(w-n+1)$.

Let now this n -fold plane be inverted in regard to any point outside it, so that it becomes an n -fold sphere passing through the point. Any two successive sheets of the sphere will be connected by one cross-line, except the two outside sheets, which are connected by $p+1$ cross-lines.

To every point of this n -sheeted spherical surface will correspond one value of the function s , namely, that which belongs to the corresponding point upon the n -fold plane. As for the centre of inversion, it is to be regarded as n distinct points upon the several sheets, corresponding to the n values of s when $z=\infty$.

We shall now prove that this n -fold spherical surface can be transformed without tearing into the surface of a body with p holes in it.

First, suppose we have only two sheets, connected by a single cross-line which joins the branch-points AA' . Let the figure [42] represent a section by the plane which bisects AA' at right angles.

Suppose each hemisphere of the inner sheet to be moved across the plane of the great circle containing AA' (indicated by the dotted line in the figure), so that the points m, n change places. In this process the two hemispheres will have to penetrate and cross each other; but this may be supposed to take place without altering the continuity of either. Each point may be supposed to move on a straight line perpendicular to the dotted plane, till it coincides with what was its reflection in regard to that plane. The effect on the cross-line will be to exchange it from the form drawn in fig. [42] to that shown in fig. [43]; instead of the two sheets crossing along the line, each of them will be doubled under it. The result is that, if we now look down on the double sphere from a point vertically over the line AB , we shall see a spherical shell with a hole in it, in the form of a slit along the line AB [fig. 44]. Construct the spherical shell to be made of india-rubber or some more elastic substance.

then by mere stretching, without tearing, the slit may be opened out until the shell takes the form of a flat plate; that is, of a body with no holes in it.

Next, consider a two-sheeted spherical surface with $p + 1$ cross-lines, and suppose them all arranged along the same great circle; which may obviously be done by stretching, without tearing, the surface. Let this great circle be the one represented by the dotted line in figs. [42] and [43]. Then we may apply to the inner sheet the same process as before; viz., we may interchange the two hemispheres into which the sheet is divided by the dotted plane. The effect is to convert all the cross-lines into slits or holes in a spherical shell; and we have supposed that there are $p + 1$ of these. One of these slits may be stretched out in the same way as AB was before, in order to convert the spherical shell into a flat plate; but in this flat plate there will remain p holes. A double sphere with $p + 1$ cross-lines is thus converted, without tearing, into the surface of a body with p holes in it.

Lastly, suppose that the inner sheet of this two-sheeted sphere is connected by one cross-line with a third inside sheet, the third sheet by one cross-line with a fourth inside it, and so on, until there are n sheets. Let the inner sheet of all be reflected in regard to the plane of the great circle through its crossing line, so that it makes with the sheet next to it a spherical shell with one hole in it. Then, without tearing, the inner sheet may be shrunk up until it merely covers over this hole. The same process may now be applied to shrink up the second sheet into the third, and so on, until we are left with only the two outside sheets connected by $p + 1$ cross-lines. These, however, as we have seen, may be converted, without tearing, into the surface of a body with p holes in it. Hence the proposition follows, that *an n -sheeted Riemann's surface with no branch-points may be transformed, without tearing, into the surface of a body with $p = \frac{1}{2}(n - n + 1)$ holes in it.*

V.

The Number of Irreducible Circuits.

A closed curve drawn on a surface is called a *circuit*. If it is possible to move a circuit continuously on the surface until it shrinks up into a point, the circuit is called *reducible*; otherwise it is *irreducible*. In general there is a finite number of irreducible circuits on a closed surface which are *independent*, that is, no one of which can be made by continuous motion to coincide with a path made out of the others. All other irreducible circuits can then be expressed as compounds of these independent ones. For example, on the surface of a ring (that of a body with one hole through it) there are two independent irreducible circuits; one *round the hole*, as *abc* (fig. 45), and one *through the hole*, as *ade*. If a circuit goes neither round the hole nor through the hole, it can be shrunk up into a point. If it cannot be so shrunk up, it must go a certain number of times round or through the hole or both, that is, it may be made up of circuits like *abc* and *ade*.

In the same way we may see that, on the surface of a body having p holes through it, there are $2p$ independent irreducible circuits; one *round each hole*, and one *through each hole*. For simplicity consider the case $p = 2$. We suppose the body in the form to which we reduced the Riemann's surface, namely, that of a flat plate, represented by figs. (46) and (47), in which A, B, C are the holes. The circuits through each hole are so drawn as to connect the hole directly with the outer rim, like the circuit which is drawn through the hole A . A circuit passing through *two* holes, as B, C (fig. 46), may be moved continuously till it consists of two circuits passing through the two holes separately. Similarly, a circuit passing *three* or *more* holes, as B, C (fig. 47), may be pinched at various points until it is made up of circuits round the separate holes. Thus a circuit as *abcd* (fig. 46) may be moved into the form *abcd* (fig. 47), in which it consists of two circuits passing through the holes A , but in opposite directions. On this account it may be called a *negative circuit*.

VI.

The Canonical Dissection.

Suppose now that it is desired to cut through the Riemann's surface in such a way that it shall still hang together, but that it shall no longer be possible to draw an irreducible circuit upon it. Then we must, of course, prevent the different kinds of irreducible circuits considered in the last section. To prevent the possibility of going round any hole, we must cut the surface along a circuit which goes through the hole. To prevent the passage through a hole, we must cut through a circuit which goes round a hole.

Let us make sections a_1, a_2, a_3 (figs. 13, 14) round the holes, and b_1, b_2, b_3 through the holes. Then we shall have prevented the drawing of any irreducible circuits except perhaps one, like $abcd$ in the previous figures. To prevent these also, we may cut the surface along the line c_1 which goes from p to q , that is, from a point on b_1 to a point on b_2 , and along the line c_2 which goes from q to r , that is, from a point on b_2 to a point on b_3 . We must not cut from r to p also, for then we should divide the surface into two separate parts. We may now open out the upper and under portions of the surface in fig. (13), until it assumes the form of fig. (15). It then becomes obvious that all our cuts form a continuous line, which is now the boundary of the surface, and is made up of the pieces (beginning at p and going round to the right) $a_1, b_1, c_1, b_2, a_2, b_3, c_2, b_3, a_3, b_3, c_2, b_2, a_2, b_1, c_1$. Moreover, it is a matter of intuition that no irreducible circuit can now be drawn on the surface.

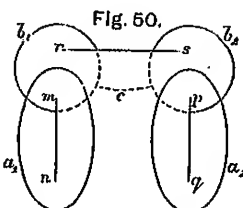
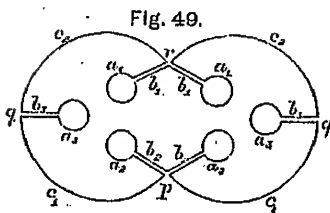
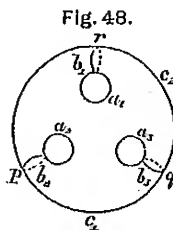
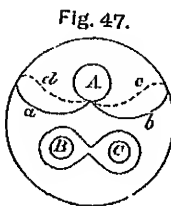
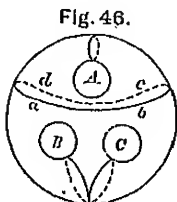
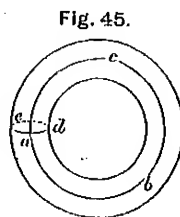
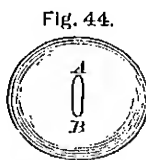
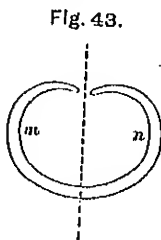
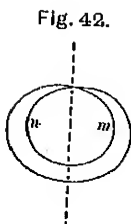
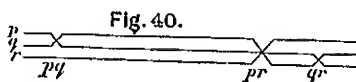
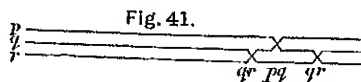
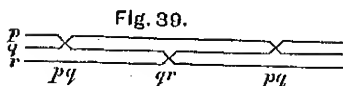
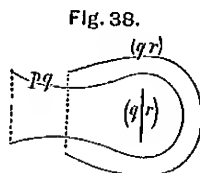
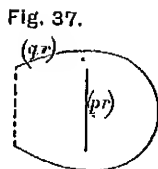
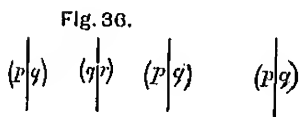
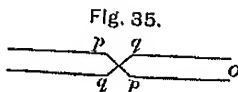
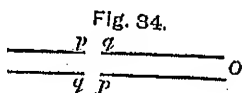
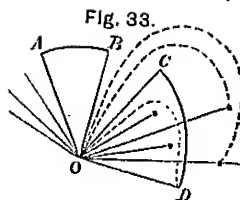
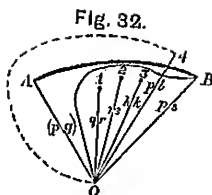
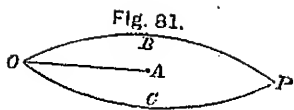
This system of cuts is called a *canonical dissection* of the surface. In the general case it consists of points a going round the holes, points b going through them, and $p-1$ cuts c joining b_1 to b_2, b_2 to b_3, \dots, b_{p-1} to b_p , but not b_p to b_1 . The cuts c may, if we like, join the a cuts together, or generally they may join the system abc together, a system meaning an arc and a b cut belonging to the same hole. In fact, the c cuts are only of importance in completing the single boundary of the surface

and so enabling us to see that no irreducible circuit is any longer possible.

It only remains to translate this result so that it may be applicable to the original form of the Riemann's surface, viz., an n -fold plane. We shall do this in the case $p = 2$, which will sufficiently explain the general case. We have now two sheets connected by three cross-lines mn, pq, rs [fig. 30]. One of these must be chosen to represent the outer rim of our flat plate; the other two will then correspond to the holes in it. Let mn, pq represent the holes, and rs the outer rim, lines in the upper sheet shall be drawn in full, and lines in the lower sheet shall be dotted. Then we must first make cuts a_1, a_2 , which go round the holes mn, pq ; these may be entirely in the upper sheet. Next we must make cuts b_1, b_2 , which connect the holes respectively with the outer rim rs . These cuts lie partly in the upper sheet, where they intersect the cuts a_1 , and partly in the under sheet. Lastly, we must connect the system a_1, b_1 with the system a_2, b_2 by the cut c ; this is shown in the figure by a_1, b_1, c_1 in the under sheet. It is impossible to draw an irreducible circuit on the two-fold plane when it is thus dissected.*

In general, we have proved that in the n -sheeted Riemann's surface which represents the function φ determined by the equation $f(s, z) = 0$, there are $p + 1$ cross-lines such that if one be taken to represent the rim, and the rest holes, of a flat plate, the surface may be dissected into one or more whose irreducible contour is possible by the following process. First the surface along curves a each of which goes round one of the cross-lines taken to represent holes, on one of the sheets of the surface which cross at that line. Connect each of these curves with the one taken to represent the rim by a cut b ending at a fixed curve which crosses each of the two cross-lines again. These connect the systems (ab) exclusively by $p + 1$ cuts c .

* It is to be understood that a circuit is reducible when all points of it can be continuously moved away to infinity without passing any branch-point. The curve in this theory infinitely counts as a single point.



XXVIII.

REMARKS ON THE CHEMICO-ALGEBRAICAL THEORY.

(Extract from a letter to Mr Sylvester*.)

"THE new Journal [see foot-note] I look forward to with the greatest interest; it will be the only English periodical in which one will have room to print formulæ, except the *Philosophical Transactions*. I had designed for you a series of papers on the application of Grassmann's methods, but there is only one of them fit for printing yet†. It is an *explanation* of the laws of quaternions and of my biquaternions by resolving the units into factors having simpler laws of multiplication; a determination of the compounding systems for space of any number of dimensions; and a proof that the resulting algebra is a *compound* (in Peirce's sense) of quaternion algebras. It thus appears that quaternions are the last word of geometry in regard to complex algebras. Another of them was to be about the very thing you speak of, which was communicated to the British Association at Bristol, *not* Bradford. There is no question of reclamation, because the whole thing is really no more than a translation into other language of your own theories published years ago in the *Cambridge Mathematical Journal*. I have a strong impression that you will find there the analogy of covariants and invariants to compound radicals and saturated molecules.

I consider forms which are linear in a certain number of sets of k variables each. To fix the ideas, suppose $k=2$ and that I have altogether 6 sets of 2 variables each, namely

$$x_1x_2, y_1y_2, z_1z_2, u_1u_2, v_1v_2, w_1w_2.$$

* [From the *American Journal of Mathematics, Pure and Applied*, Vol. 1. No. 2, pp. 120—128.]

† [This is xxx. of the present volume.]

Suppose the forms are

$$(xyzv), (yzvw), (xv), (uw);$$

viz. $(xyzv)$ means an expression separately linear and homogeneous in the x , the y , the z , and the v , and so for the rest. I observe that in these four forms each set of variables occurs twice. This being so, there is one invariant of the four forms, which is invariant in regard to *independent* transformations of the six sets of variables. This you knew thirty years ago. All I add is: *to obtain this invariant*, regard the *variables as alternate numbers*, and *simply multiply all the forms together*. By *alternate numbers* I mean those whose multiplication is polar ($xy = -yx$) and whose squares are zero. The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables. The quartic forms may be represented by the symbol ϕ , the quadratics by \circ . Thus the invariant $(xyzv)(yzvw)(xv)(uw)$ will be represented by the figure $\phi\phi\phi\phi$; whereas, $(xyzv)(yzvw)(xu)(vw)$ is this form $\phi\phi\phi\phi$. The former is clearly the product of the two quartic covariants ϕ ϕ got by cutting it across the dotted lines; while the latter is the product of the quadri-covariants $\phi\phi$, $\phi\phi$. A *bond* between two forms means a set of variables common to them. Of course, we may regard two or more of the forms as identical and so form invariants of a single form; thus $\phi\phi$ is the discriminant of a cubic*. Of course, the main thing is to pass from this system of separate variables to that in which the same variables occur to higher orders in the same form, or back again—what you call ‘unravelling’...

The part of the theory which astonished me most is its application to *intergradient* variables when the number in a set is greater than 3,—such as the six co-ordinates of a line in the case of quaternary forms. When the original variables are regarded as alternate numbers, these intergradients are simply their binary products. Thus by simply multiplying the linear forms representing two planes, we get one intergradient form representing their line of intersection. And so generally, whatever be the number of variables in a set, the intergradient variables are merely their products so many together. With

this understanding, the product of a set of forms in which the variables are regarded as alternate numbers is the *only* invariant or covariant of the forms which possess certain definite characters of invariance.

"The ordinary theory of symmetrical forms seems to me to bear the same relation to this one (of forms linear in several sets of variables) that a boulder does to a crystal—all the angles rounded off so that you can't see through it so clearly...."†

† [Dr Sylvester has appended several interesting notes, from which a few extracts are given here. "I think Prof. Clifford overstates the obligations which he alleges to my previous papers. At all events he has more than reconquered his title to the merit of the first conception by the completeness he has imparted to it...In a word he has found the universal pass-key to the *quantification of the graphs*... All that Prof. Clifford adds is the very pith and marrow of the matter which before was wanting." Dr Sylvester further remarks, "I will take the example of this figure [of. * in text] to illustrate Prof. Clifford's rule for finding the *algebraical content* of the graph. Let the bonds be called

x, y, z, t, u, v . Then there will be four forms corresponding to the four apices or atoms, viz.

$$\begin{aligned} & (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) (x_1, x_2) (y_1, y_2) (z_1, z_2), \\ & (b_1, b_2, \dots, b_6) (z_1, z_2) (t_1, t_2) (u_1, u_2), \\ & (c_1, \dots, c_4) (t_1, t_2) (u_1, u_2) (v_1, v_2), \\ & (d_1, \dots, d_6) (v_1, v_2) (y_1, y_2) (x_1, x_2). \end{aligned}$$

where all the x, y, z, t, u, v letters are to be regarded as *polar elements*. [Dr Sylvester objects to the term *alternate numbers* in this connexion.] Take the polar product of these forms; the coefficient of

$$x_1 \cdot x_2 \cdot y_1 \cdot y_2 \cdot z_1 \cdot z_2 \cdot t_1 \cdot t_2 \cdot u_1 \cdot u_2 \cdot v_1 \cdot v_2$$

will be an invariant of three lineo-lineo-linear forms.

If we make the values identical for the same index, whatever the letter which it affects, it becomes an invariant of a single lineo-lineo-linear form; and finally if we make the coefficients of $x_1 y_1 z_1, y_1 z_1 x_1, z_1 x_1 y_1$ all alike, and again the coefficients of $x_2 y_2 z_2, y_2 z_2 x_2, z_2 x_2 y_2$ all alike, and identify the letters x, y, z , the form becomes a binary cubic and the invariant becomes its discriminant. We know *a priori* by my permutation-sum test that the algebraical content above indicated will not vanish because

$$\Sigma (a \dots b)^2 (a \dots c)^2 (a \dots c) (b \dots d)$$

is not zero, whereas the algebraical content of the figure formed by turning round one of each pair of the doubled lines into the position of the two diagonals respectively *will* vanish because the permutation-sum of

$$\Sigma (a \dots b) (b \dots c) (a \dots d) (d \dots a) (a \dots c) (b \dots d)$$

is zero."‡

*XXIX.

NOTES ON QUANTICS OF ALTERNATE NUMBERS,
USED AS A MEANS FOR DETERMINING THE
INVARIANTS AND COVARIANTS OF QUANTICS IN
GENERAL*.

THE term *alternate numbers* means a set (λ_1, λ_2) or sets of numbers which satisfy the following relations :

$$\lambda_1^2 = 0, \lambda_2^2 = 0, \lambda_1\lambda_2 + \lambda_2\lambda_1 = 0 \dots \dots \dots (1) ;$$

to which it is usual to add

$$\lambda_1\lambda_2 = 1.$$

The above set is binary, but there may be ternary, &c. sets, or sets consisting of any number of letters $\lambda_1, \lambda_2, \dots$

By a *Quadratic Form* is here meant an expression—linear in two sets of such numbers regarded as variables, say $\lambda_1, \lambda_2; \mu_1, \mu_2$, such as

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 \dots \dots \dots (2).$$

This may be also denoted, for shortness, by the symbol $a|12$, or even by 12, where the 1, 2 refer to the two sets of variables λ, μ .

* [From the *Proceedings of the London Mathematical Society*, Vol. x. No. 144, pp. 124—129. "This is the substance of some fragments found amongst the papers of the late Professor Clifford. The only published explanation of the method with which I am acquainted, is contained in a letter to Professor Sylvester (see XXVIII.); 'I consider forms which are linear in a certain number of sets of k variables each....The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables.'" Sp.]

In this case there is one invariant only which allows of the variables being separately transformed, namely, the discriminant, which is got by squaring the form. We have, in fact, by the properties of alternate numbers,

$$(a|12)^2 = -2(a_{11}a_{22} - a_{12}a_{21}) = -2D, \text{ suppose } \dots\dots(3).$$

But if the variables are transformed by the same substitution, there is a universal covariant, $\lambda_1\mu_2 - \lambda_2\mu_1$, which may be denoted by (12), or by (21); for $\lambda_1\mu_2 - \lambda_2\mu_1 = \mu_1\lambda_2 - \mu_2\lambda_1$, by the property of alternate numbers.

If we replace the μ 's by x , an ordinary, not an alternate, number, we get a linear function of the λ 's, whose square vanishes in virtue of the relations (1). Thus we have $(a|1x)^2 = 0$, and consequently such a product as $(a|1x)(a|1y)$ must be divisible by $(xy) = x_1y_2 - x_2y_1$, since it vanishes when x and y represent the same point; in fact, if we write the expressions in full, thus,

$$(a|1x) = (a_{11}x_1 + a_{12}x_2)\lambda_1 + (a_{21}x_1 + a_{22}x_2)\lambda_2,$$

$$(a|1y) = (a_{11}y_1 + a_{12}y_2)\lambda_1 + (a_{21}y_1 + a_{22}y_2)\lambda_2,$$

actual multiplication gives

$$(a|1x)(a|1y) = a_{11}x_1 + a_{12}x_2, \quad a_{21}x_1 + a_{22}x_2 = a_{11}, \quad a_{12} \times x_1, \quad x_2 \\ a_{11}y_1 + a_{12}y_2, \quad a_{21}y_1 + a_{22}y_2, \quad a_{21}, \quad a_{22} \quad y_1, \quad y_2.$$

But $a_{11}a_{22} - a_{12}a_{21}$ may be regarded as the discriminant of any quadratic form having a 's for its coefficients, say any form $(a|18)$. And, since it was shown, by equation (3), that the square of such a form is equal to minus twice its discriminant, it follows that the above equation may be written thus,

$$-2(a|1x)(a|1y) = (a|18)^2(xy) \dots\dots\dots(4).$$

Note.—The transformation

$$y_1, y_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} (x_1, x_2)$$

may also be written

$$a_{11}x_1y_1 + a_{21}x_1y_2 + a_{12}x_2y_1 + a_{22}x_2y_2 = 0 = (a|yx),$$

where the y 's are arbitrary functions satisfying identically

$$y_1y_1 + y_2y_2 = 0.$$

In like manner, a second transformation

$$z_1, z_2 = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} (y_1, y_2)$$

may be represented by $(b | zy) = 0$; and the result of the two by

$$(b | zy) (a | yx) = 0.$$

There are one or two other formulæ which it will be convenient to notice before proceeding further. If we multiply the form $(a | 12)$ by (12), we get

$$(12) (a | 12) = a_{21} - a_{12} = s \text{ (suppose)} \dots \dots \dots (5).$$

Also, if we suppose the coefficients to remain unaltered, viz., if $(a | 21)$ represents $a_{11}\mu_1\lambda_1 + a_{12}\mu_1\lambda_2 + \dots$, then

$$(a | 12) + (a | 21) = -s (12) \dots \dots \dots (6),$$

or
$$2 (a | 12) = (a | 12) - (a | 21) - s (12),$$

the quantity s vanishing when the form is symmetrical, i.e. when $a_{12} = a_{21}$. Again, if the coefficients be supposed to remain unaltered, so that

$$(a | 13) = a_{11}\lambda_1\nu_1 + a_{12}\lambda_1\nu_2 + \dots,$$

then it will be found that

$$\begin{aligned} (12) (a | 23) &= - (a | 13) \\ (13) (a | 12) &= - (a | 32) \end{aligned} \dots \dots \dots (7).$$

Such multiplication is in fact tantamount to a substitution of variables, from 2 to 1 (i.e. from μ to λ), or from 1 to 3 (i.e. from λ to ν). This theorem, as will readily be seen, is not restricted to quadric forms; but if a, b be any two different forms, and if 1 be a set of variables in a and not in b , and 2 a set of variables in b and not in a , then, making the same supposition as in b , viz., that the constants a, b remain the same on both sides of the equation, we shall find that

$$(12) (a | 1\dots) (b | 2\dots) = (a | 2\dots) (b | 1\dots) = (a | 1\dots) (b | 1\dots).$$

With reference to the universal covariant $(12) = \lambda_1\mu_2 - \lambda_2\mu_1$, it may be remarked that we may consider such covariants for

more than two sets of variables; and we shall then obtain the following formulæ*:

$$(12)(23) = -(13), \quad (12)(23)(34) = (14), \dots \dots (8),$$

Of which the following is a consequence,

$$(12)(23)(31) = 0 \dots \dots \dots (9).$$

It is clear that quadratic forms can only be combined in a *chain*†, which, when open, gives a quadratic covariant; when closed, an invariant. We can now show that a closed chain of $2n$ sides is equal to $\pm 2D^n$, while a chain of an odd number of sides vanishes.

Take a chain of four sides, (12), (23), (34), (41); then, since

$$(12)(23)(34)(41) = (12)(23) \times (34)(41);$$

and since, by (3) and (4),

$$2(12)(23) = (a|28)^2(31) = -2D(31),$$

$$2(34)(41) = (a|48)^2(13) = -2D(13),$$

$$(31)(13) = 2,$$

it follows that

$$(12)(23)(34)(41) = 2D^2 \dots \dots \dots (10).$$

In like manner any chain of an even number of sides may be resolved into a power of the discriminant multiplied by a chain of determinants of the alternate variables. The product of these latter is ± 2 , according as the number of terms is even or odd. Hence, generally, a chain of $2n$ sides is $2(-D)^n$.

* [This and other parts of the present paper may be compared with Spottiswoode "On Determinants of Alternate Numbers," *Proceedings of the London Mathematical Society*, Vol. vii., p. 100. Sr.]

† [I have, in this paragraph, retained the language of the original MS., although the term *chain* is not here explained. The author appears to have had in his mind a theory which he propounded verbally at the meeting of the British Association at Bristol, and to which, in his latter days, he attached great importance and devoted much time. Some indications of it will be found in the *American Journal of Pure and Applied Mathematics*, Vol. i. p. 127 [cf. XXVIII.]. Several notes relating to the subject have been found amongst his papers, but as they are almost exclusively memoranda without explanation, it is still uncertain whether they can be published. Sr.]

A chain of $2n + 1$ sides may, by analogous processes, be reduced to the product of a determinant of alternate variables by a form containing those variables, and must therefore vanish for symmetrical forms. For unsymmetrical it contains s as a factor. In this case the fundamental formula must be modified as follows :

$$(a | x1) (a | 1y) = (a | xy) s - (xy) D ;$$

or, with alternate numbers in the place of x, y ,

$$(a | 12) (a | 23) = (a | 13) s - (13) D.$$

Multiplying now into $(a | 31)$, we get

$$(a | 12) (a | 23) (a | 31) = (a | 13) (a | 31) s + sD$$

in virtue of equation (5).

$$\begin{aligned} \text{But } (a | 13) (a | 31) &= (a_{11}\lambda_1\mu_1 + \dots) (a_{11}\mu_1\lambda_1 + \dots) \\ &= (2a_{11}a_{22} - a_{12}^2 - a_{21}^2) = 2D - s^2. \end{aligned}$$

$$\text{Hence } (a | 12) (a | 23) (a | 31) = (3D - s^2) s \dots\dots\dots (11).$$

Again,

$$\begin{aligned} (a | 12) (a | 23) (a | 34) &= (a | 13) (a | 14) s - (13) (a | 34) D \\ &= (a | 14) s^2 - (14) sD + (a | 14) D \\ &= (a | 14) (D + s^2) - (14) sD ; \end{aligned}$$

whence also

$$(a | 12) (a | 23) (a | 34) (a | 41) = (2D - s^2) (D + s^2) + s^2 D \dots (12).$$

Suppose, in general, that

$$(a | 12) (a | 23) \dots (a | lm) = A_m (a | 1m) - B_m (1m) ;$$

then

$$\begin{aligned} (a | 12) (a | 23) \dots (a | mn) &= A_m (a | 1m) (amn) + B_m (a | 1n) \\ &= A_m [(a | 1n) s - (1n) D] + B_m (a | 1n) \\ &= (sA_m + B_m) (a | 1n) - A_m D (1n). \end{aligned}$$

$$\text{Hence } A_{m+1} = sA_m + B_m, \quad B_{m+1} = A_m D ;$$

and consequently

$$A_{m+1} = sA_m + DA_{m-1} \dots\dots\dots (13).$$

The form $\lambda_1\mu_1 + 0\lambda_2\mu_2 + s\lambda_3\mu_3 + D\lambda_4\mu_4$ is a form having s and D for invariants; and it may therefore be taken as the

canonical form for a bipartite quantic such as we have been here considering. Again, another useful form is the following,

$$\sqrt{(s^2 - 4D)} (\lambda_1 \mu_2 + \lambda_2 \mu_1) + s (\lambda \mu).$$

We may, however, combine not only two quadratic forms having like coefficients, say, two forms a ; but we may also combine two having different coefficients, say, two forms a and b . Two such forms give rise to an invariant, namely, their product. In fact, we have

$$-(a | 12) (b | 12) = a_{11} b_{22} - a_{12} b_{21} - a_{21} b_{12} + a_{22} b_{11} = D_{ab}, \text{ suppose;}$$

$$\text{and from (5), } (a | 12) (b | 12) + (a | 21) (b | 12) = -s_a s_b,$$

$$\text{or } (a | 21) (b | 12) = D_{ab} - s_a s_b \dots \dots \dots (14).$$

The product of $(a | 12) = a_{11} \lambda_1 \mu_1 + \dots$ and $(b | 13) = b_{11} \lambda_1 \nu_1 + \dots$ gives a covariant; namely,

$$\begin{aligned} (a | 12) (b | 13) &= a_{11} \mu_1 + a_{12} \mu_2, & a_{21} \mu_1 + a_{22} \mu_2 \\ & b_{11} \nu_1 + b_{12} \nu_2, & b_{21} \nu_1 + b_{22} \nu_2 \\ &= \begin{vmatrix} a_{11} b_{21} - a_{21} b_{11}, & a_{11} b_{22} - a_{21} b_{12} \\ a_{12} b_{21} - a_{22} b_{11}, & a_{12} b_{22} - a_{22} b_{12} \end{vmatrix} (\mu_1, \mu_2) (\nu_1, \nu_2) \\ &= D_{ab} | 23, \text{ suppose } \dots \dots \dots (15). \end{aligned}$$

If, in the covariant, we make the forms a and b coincident, it becomes

$$(a | 12) (a | 13) = D_a(23), \quad (a | 12) (a | 23) = s_a (a | 13) - D_a(13);$$

and, multiplying this by, or into, b , we get

$$(a | 12) (a | 13) (b | 43) = (b | 43) (a | 12) (a | 13) = -D_a (b | 42),$$

$$(a | 12) (a | 13) (b | 34) = (b | 43) (a | 12) (a | 13) = -D_a (b | 24),$$

and

$$(a | 12) (a | 13) (b | 23) = D_a s_b,$$

$$(a | 12) (a | 13) (b | 32) = D_a s_b \dots \dots \dots (16).$$

The remaining form to be investigated is

$$(a | 12) (b | 13) (a | 43).$$

The value of this is

$$\begin{aligned}
 & (a_{11}b_{21} - a_{21}b_{11})\mu_1 + (a_{12}b_{21} - a_{22}b_{11})\mu_2, \quad (a_{11}b_{22} - a_{21}b_{12})\mu_1 + (a_{12}b_{22} - a_{22}b_{12})\mu_2 \\
 & \quad \quad \quad a_{11}\rho_1 + \quad \quad \quad a_{21}\rho_2, \quad \quad \quad a_{12}\rho_1 + \quad \quad \quad a_{22}\rho_2 \\
 & = (-a_{21}a_{12}b_{11} + a_{11}a_{21}b_{12} + a_{11}a_{12}b_{21} - a_{11}a_{11}b_{22})\mu_1\rho_1 \\
 & \quad + (-a_{21}a_{22}b_{11} + a_{21}a_{21}b_{12} + a_{11}a_{22}b_{21} - a_{11}a_{21}b_{22})\mu_1\rho_2 \\
 & \quad + (-a_{12}a_{12}b_{21} - a_{22}a_{12}b_{21} - a_{11}a_{12}b_{22} + a_{11}a_{22}b_{12})\mu_2\rho_1 \\
 & \quad + (-a_{22}a_{12}b_{21} - a_{22}a_{22}b_{11} - a_{21}a_{12}b_{22} + a_{21}a_{22}b_{12})\mu_2\rho_2 \\
 & = (-a_{11}D_{ab} + b_{11}D_{aa})\mu_1\rho_1 \\
 & \quad + (-a_{21}D_{ab} + b_{21}D_{aa})\mu_1\rho_2 \\
 & \quad + (-a_{12}D_{ab} + b_{12}D_{aa})\mu_2\rho_1 \\
 & \quad + (-a_{22}D_{ab} + b_{22}D_{aa})\mu_2\rho_2 \\
 & = D_{ab}(a|42) - D_{aa}(b|42) \dots\dots\dots (17).
 \end{aligned}$$

Multiplying this into $(b|45)$, we obtain

$$\begin{aligned}
 & (a|12)(b|13)(a|43)(b|45) \\
 & \quad \quad \quad = D_{ab}(a|42)(b|45) - D_{aa}(b|42)(b|45);
 \end{aligned}$$

that is, referring to equation (15), and dropping the suffix ab ,

$$(\mathfrak{S}|23)(\mathfrak{S}|35) = D_{ab}(\mathfrak{S}|25) - D_{aa} \cdot D_{bb}(25).$$

But $(\mathfrak{S}|23)(\mathfrak{S}|35) = s(\mathfrak{S}_s|25) - D_s(25).$

Hence $s_s = D_{ab}, \quad D_s = D_{aa} \cdot D_{bb} \dots\dots\dots (18),$

as may be easily verified.

If, however, we multiply the equation

$$(a|12) + (a|21) = -s(12) \text{ by } (a|12),$$

we obtain

$$(a|12)(a|21) = 2D_a - s_a^2;$$

hence $(\mathfrak{S}|12)(\mathfrak{S}|21) = 2D_s - s_s^2 = 2D_a D_b - D_{ab}^2 \dots\dots\dots (19),$

and consequently

$$\begin{aligned}
 & \{(\mathfrak{S}|12) - (\mathfrak{S}|21)\}^2 = -4D_a D_b - 2(2D_a D_b - D_{ab}^2) \\
 & \quad \quad \quad = -2(4D_a D_b - D_{ab}^2) \dots\dots\dots (20).
 \end{aligned}$$

The formula $s_s = D_{ab}$ is important as showing that \mathfrak{S} is not made symmetrical by making a and b symmetrical. Hence, in passing from these invariants and covariants to those of sym-

metrical forms, we are obliged to use the symmetrised form of \mathfrak{N} , namely $(\mathfrak{N} | 12) = (\mathfrak{N} | 21)$. Thus, to adapt equation (17) to symmetrical forms, we have

$$\begin{aligned}(\mathfrak{N} | 23) (a | 43) &= D_{aa} (a | 42) = D_{aa} (b | 42), \\(\mathfrak{N} | 23) (a | 43) &= (a | 13) (b | 12) (a | 43) \\&= D_a (14) (b | 12) = D_a (b | 42).\end{aligned}$$

In general, we write a for the mean value of a , i.e., if

$$2 (\bar{a} | 12) = (a | 12) + (a | 21),$$

we shall have $(\bar{a} | 12) = (a | 12) + \frac{1}{2}s(12)$.

Hence, since $(12)^2 = 2$,

$$(a | 12)^2 = (a | 12)^2 + s^2 + \frac{1}{2}s^2 \dots \dots \dots (21),$$

and $D = D + \frac{1}{2}s^2$,

where D is the discriminant of the symmetrical function.

Similarly

$$\begin{aligned}\mathfrak{N}_{ab} &= (\bar{a} | 12) (\bar{b} | 13) = \{(a | 12) + \frac{1}{2}s_a(12)\} \{(b | 13) + \frac{1}{2}s_b(13)\} \\&= \mathfrak{N}_{ab} + \frac{1}{2} [s_a(b | 23) + s_b(a | 23)] + \frac{1}{4}s_a s_b(23),\end{aligned}$$

and therefore

$$\begin{aligned}2\mathfrak{N}_{ab} &= (\mathfrak{N}_{ab} | 23) = (\mathfrak{N}_{ab} | 32) = s_a s_b(23) \} \dots \dots \dots (22), \\&= 2 (\mathfrak{N}_{ab} | 23) = s_b(23) = s_a s_b(23) \} \dots \dots \dots\end{aligned}$$

If we multiply this into $(a | 43)$, we have

$$\begin{aligned}2 (\mathfrak{N}_{ab} | 23) (a | 43) &= 2 (\mathfrak{N} | 23) (a | 43) = D_{ab} (a | 42) = s_a s_b (a | 42) \\&= D_{ab} (a | 42) = 2 D_a D_b (b | 42) = s_a s_b (a | 42).\end{aligned}$$

Putting $s_a = 0$, $s_b = 0$, we get the formula for symmetric functions.

XXX.

APPLICATIONS OF GRASSMANN'S EXTENSIVE ALGEBRA*.

I PROPOSE to communicate in a brief form some applications of Grassmann's theory which it seems unlikely that I shall find time to set forth at proper length, though I have waited long for it. Until recently I was unacquainted with the *Ausdehnungslehre*, and knew only so much of it as is contained in the author's geometrical papers in *Crelle's Journal* and in *Hankel's Lectures on Complex Numbers*. I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science.

The present communication endeavours to determine the place of Quaternions and of what I have elsewhere† called Biquaternions in the more extended system, thereby explaining the laws of those algebras in terms of simpler laws. It contains, next, a generalization of them, applicable to any number of dimensions; and a demonstration that the algebra thus obtained is always a compound of quaternion algebras which do not interfere with one another.

On the Relation of Grassmann's Method to Quaternions and Biquaternions; and on the Generalization of these Systems.

Following a suggestion of Professor Sylvester, I call that kind of multiplication in which the sign of the product is reversed by an interchange of two adjacent factors, *polar multi-*

* [American Journal of Mathematics Pure and Applied, Vol. x, pp. 350—358.]

† Proceedings of the London Mathematical Society [XX, *supra*].

plication*; because the product ab has opposite properties at its two ends, so that $ab = -ba$. The ordinary or commutative multiplication I shall call *Scalar*, being that which holds good of scalar numbers. These words answer to Grassmann's *outer* and *inner* multiplication; which names, however, do not describe the multiplication itself, but rather those geometrical circumstances to which it applies.

Consider now a system of n units $\iota_1, \iota_2, \dots, \iota_n$, such that the multiplication of any two of them is polar; that is, $\iota_r \iota_s = -\iota_s \iota_r$. For geometrical applications we may take these to represent points lying in a flat space of $n-1$ dimensions. A binary product $\iota_r \iota_s$ is then a unit length measured on the line joining the points ι_r, ι_s ; a ternary product $\iota_r \iota_s \iota_t$ is a unit area measured on the plane through the three points, and so on. A linear combination of these units, $\Sigma a_r \iota_r = \alpha$ suppose, represents a point in the given flat space of $n-1$ dimensions, according to the principles of the barycentric calculus, as extended in the *Ausdehnungslehre* of 1844.

In space of three dimensions we may take the four points $\iota_0, \iota_1, \iota_2, \iota_3$ so that $\iota_1, \iota_2, \iota_3$ are at an infinite distance from ι_0 in three directions at right angles to one another.

Now there are two sides to the notion of a product. When we say $2 \times 3 = 6$, we may regard the product 6 as a number derived from the numbers 2 and 3 by a process in which they play similar parts; or we may regard it as derived from the number 3 by the operation of doubling. In the former view 2 and 3 are both numbers; in the latter view 3 is a number, but 2 is an operation, and the two factors play very distinct parts. *The Ausdehnungslehre is founded on the first view; the theory of quaternions on the second.* When a line is regarded as the product of two points, or a parallelogram as the product of its sides, the two factors are things of the same kind and play similar parts. But in such a quaternion equation as $q\rho = \sigma$, where ρ and σ are vectors, the quaternion q is an operation of turning and stretching which converts ρ into σ ; it is a thing totally different in kind from the vector ρ . The only way in

* [American Journal of Mathematics, Vol. I. p. 127 and p. 257, *supra*.]

which the factors q and p can be taken to be of the same kind, is to regard p as itself a special case of a quaternion, viz. a rectangular versor. But in that case the expression does not receive its full meaning until we suppose a *subject* on which the operations p and q can be performed in succession.

The quaternion symbols i, j, k represent, then, *rectangular versors*; that is to say, they are operations which will turn a figure through a right angle in the three co-ordinate planes respectively. It follows that if either of them is applied twice over to the same figure, it will turn it through two right angles, or *reverse* it; we must therefore have $i^2 = j^2 = k^2 = -1$.

To compare these with the symbols for the four points e_0, e_1, e_2, e_3 , let us suppose that i turns the line $e_0 e_2$ into $e_1 e_3$; that j turns $e_0 e_1$ into $e_2 e_3$; and that k turns $e_1 e_2$ into $e_0 e_3$. The turning of $e_0 e_2$ into $e_1 e_3$ is equivalent to a translation along the line at infinity $e_3 e_0$. We may, therefore, write $i = e_2 e_3$, and so $j = e_0 e_1, k = e_1 e_2$. Now i turns $e_1 e_2$ into $e_0 e_3$, that is

$$i \cdot e_1 e_2 = e_0 e_3,$$

or

$$e_3 e_2 = e_0 e_1, \quad e_2 e_1 = -e_3 e_0.$$

We are therefore obliged to write $i^2 = -1$, and in a similar way we may find $i^2 = e_1 e_2 = -1$.

This at once enables us to find the rules of multiplication of the i, j, k . Namely, we have

$$jk = e_2 e_1 \cdot e_1 e_2 = e_2 e_2 = e,$$

$$ki = e_1 e_2 \cdot e_2 e_1 = e_1 e_1 = e,$$

$$ij = e_1 e_2 \cdot e_0 e_1 = e_2 e_0 = -k,$$

and finally

$$ijk = e_2 e_1 \cdot e_1 e_2 \cdot e_0 e_1 = -1$$

In order, therefore, to bring the quaternion algebra within that of the Ausdehnungslehre, we have to make the sequence of units of our units equal to -1 , as pointed out by Grassmann (*Math. Annalen*). But I venture to differ from his authorship in thinking that the quaternion symbols belong to the first place among the "Elementargrößen" of the Ausdehnungslehre, but to

binary products of them; from which supposition, as we have seen, the laws of their multiplication follow at once.

It is quite true that in process of time the conception of a product as derived from the force of the same kind, and so of the product of two vectors, as a thing which might be thought of without regarding them as rectangular vectors, grew upon Hamilton's mind, and led to the gradual replacement of the units i, j, k by the more general selective symbols S and T . To explain the laws of multiplication of i, j, k on this view, we must have recourse to the theory of "Ergänzung," or which comes to the same thing, represent an area by a vector k perpendicular to it. But the explanation in this case is by no means so easy, and it is instructive to observe that the distinction between a quantity and its "Ergänzung," i.e. between an area and its representative vector, which, for a case purposes, it is so convenient to ignore, has to be reintroduced in physics. Thus Maxwell specially distinguishes the two kinds of vectors, which he calls *force* and *flow*, and which in fact are respectively linear functions of the units and of their binary products.

We have regarded the symbols i, j, k as rectangular vectors operating on the quantities x, x_1, x_2, x_3 . These quantities are unit lengths measured anywhere on the axes in the positive directions. They have magnitude, direction, and position, and are thus what I have called *vectors* (not to be confused) to distinguish them from *vectors*, which have magnitude and direction but no position. A vector is of the nature of the translation velocity of a rigid body, or of a couple; it may be represented by a straight line of given length and direction drawn *anywhere*. A vector is of the nature of the rotation velocity of a rigid body, or of a force, it belongs to a definite axis. A vector may be represented as the difference of two points of equal weight (the vector ab may be written $b - a$); this is shown by the principles of the barycentric calculus to represent a point of no weight at infinity. Accordingly the symbols i, j, k may be taken to mean unit vectors along the axes. In fact if we write $x_1 + x_2 = x$, the point x will be situate on the axis unit distance from the origin, and thus $i = x_1 = x_2$ will represent the unit vector from the origin to x_1 .

The versors i, j, k will operate on these vectors in the same way as on the rotors i_0, i_1, i_2, i_3 . We find that

$$i i_0 = i_1 i_3 = i_2 = i_4, \quad j i_0 = i_1 i_2 = i_3 = i_5,$$

These rules of multiplication coincide with those for i, j, k if we write the latter in place of i_1, i_2, i_3 . Thus we may use the same symbols to represent unit vectors along the axes and rectangular versors about them. But it is not in any sense true that the vectors i_1, i_2, i_3 are identical with the versors i, j, k ; it is only sometimes convenient to forget this difference between i_1 and $i_0 i_3$.

In the elliptic or hyperbolic geometry* of three dimensions, the four points a, b, c, d must be taken as the vertices of a tetrahedron self-conjugate in regard to the absolute, so that the distance between every two of them is a quadrant. The product of four points $ab cd$ will then consist of three kinds of terms; (1) terms of the fourth order, being $i_1 i_2 i_3 i_4$, multiplied by the determinant of the co-ordinates of the four points, which is proportional to $\sin(\alpha, \beta) \sin(\gamma, \delta) \cos(\alpha\beta, \gamma\delta)$; (2) terms of the second order, resulting from products of the form $i_0^2 i_1 i_2$, $i_0^2 i_1 i_3$; (3) terms of order zero, resulting from products of the form $i_0^4 i_1^2 i_2^2$. Altogether we may arrange $ab cd$ in eight terms as follows:

$$ab cd = a + \Sigma h_{i,j,k} i_1 i_2 i_3, \quad (i, j, k \text{ distinct}).$$

And it is now easy to see that the product of any even number of linear factors will be of the same form. This form is what I have called a *hyperternion*, and may be easily evaluated as such. Namely, let us write a for $i_0 i_1 i_2 i_3$; then we have

$$i = i_1 i_3, \quad j = i_2 i_4, \quad k = i_3 i_5, \\ ai = ia = i_1 i_0, \quad aj = ja = i_2 i_0, \quad ak = ka = i_3 i_0, \\ a^2 = 1.$$

Therefore, the product of any even number of factors greater than two is a linear function of $1, i, j, k, ai, aj, ak$; that is to say, it is of the form $q + ar$, where q, r are quaternions.

* Dr Klein's name for this theory of geometry of spheres, cylinders, cones, and the evolute. [P. p. 141 seq. et.]

A product of m units, all different, multiplied by any scalar is called a *term* of the order m . The sum of several terms of order m each multiplied by a scalar, is a *form* of order m . The sum of several forms of different orders is a *quantity* and an even quantity when the forms are all of even order, an odd quantity when they are all of odd order. Thus the multiplication of linear functions of the units leads only to even quantities and odd quantities.

The square of a term of the m^{th} order is $+1$ or -1 according as the integer part of $\frac{1}{2}(m+1)$ is even or odd. For the product $t_1 t_2 \dots t_{m-1} t_m$ is transformed into $t_1^2 t_2^2 \dots t_m^2$ by $\frac{1}{2} m(m-1)$ changes of consecutive factors, and therefore equals ± 1 according as $\frac{1}{2} m(m+1)$ is even or odd, which is equivalent to the rule stated.

The multiplication of a term P of order m by a term Q of order n , having k factors common, is scalar or polar according as $mn - k^2$ is even or odd. Let $P = CP'$ and $Q = CQ'$, where C, P', Q' have no common factor; then the steps from $CP' CQ'$ to $CP' Q' C, CQ' P' C, CQ' CP'$ require respectively $k(n-k), (m-k)(n-k), k(m-k)$ changes of consecutive factors; and the sum of these quantities is even or odd as $mn - k^2$ is.

The following cases are worth noticing:

(1) When two terms have no factor common, their multiplication is scalar except when they are both of odd order. (Case $k = 0$.)

(2) The multiplication of two even terms is scalar or polar according as the number of common factors is even or odd.

(3) If one of two terms is a factor in the other, the multiplication is scalar except when the first is odd and the second even.

Theory of Algebras with an odd number of units.

When the number of units is $n = 2m + 1$, there are n terms of the order $n - 1$, and all terms of even order can be expressed by means of these. For the product of any two of these terms is of the second order, since they must have $n - 2$ factors common.

over, the two quaternion sets are commutative with one another. For since the k multiply in the polar manner, we may regard them as elementary units for this purpose; now the terms in the second set are all even, and no term in one set has a factor common with any term in the other set.

In the language of Professor Peirce, then, the five-way algebra is a compound of two quaternion algebras, which do not in any way interfere, because the units of one are commutative in regard to those of the other. A quantity in the five-way algebra is in fact a quaternion $\omega + ix + jy + kz$, whose coefficients ωxyz are themselves quaternions of another set of units ($1, i_1, j_1, k_1$), the i_1, j_1, k_1 being commutative with i, j, k .

I shall now extend this proposition, and shew that *the $(2m+1)$ way algebra is a compound of m quaternion algebras, the units of which are commutative with one another.* To this end let us write $p_6 = k_1 k_2$, and then

$$\begin{aligned} p_1 &= k_1 k_2 k_3 k_4 = p_6 k_5 k_7, & q_1 &= k_3 k_4 k_5, \\ p_2 &= p_1 k_{10} k_{11} & q_2 &= q_1 k_6 k_9, \\ &\dots\dots\dots & &\dots\dots\dots \\ p_r &= p_{r-1} k_{4r+2} k_{4r+3}, & q_r &= q_{r-1} k_{4r} k_{4r+1}. \end{aligned}$$

Consider now the quaternion sets

$$\begin{aligned} &1, k_1, k_2, k_1 k_2 \\ &1, k_4 k_5, k_6 k_7, k_8 k_9 \\ &1, p_1 k_3, p_2 k_7, k_6 k_7 \\ &1, q_1 k_8, q_1 k_9, k_8 k_9 \\ &1, p_1 k_{10}, p_1 k_{11}, k_{10} k_{11} \\ &\dots\dots\dots \\ &1, q_{r-1} k_{4r}, q_{r-1} k_{4r+1}, k_{4r} k_{4r+1} \\ &1, p_{r-1} k_{4r+2}, p_{r-1} k_{4r+3}, k_{4r+2} k_{4r+3}, \\ &\dots\dots\dots \end{aligned}$$

viz.: a p -set and a q -set alternately. I say that if we consider the first m sets of this series, we shall find them to involve $2m+1$ of the k ; that the products of m terms, one from each series, constitute 2^{2m} distinct terms, which are therefore identical

with the terms of the $(2m+1)$ way algebra; and that the terms in any two sets are commutative with each other. The first two remarks are obvious on inspection; the last also is clear for the case of a p -set and a q -set, because the q -set is of even order in the k , and no factors are common to the two sets. It remains only to examine the case of two p -sets and of two q -sets. Compare the two p -sets

$$1, p_{r-1}k_{4r+2}, p_{r-1}k_{4r+3}, k_{4r+2}k_{4r+3},$$

$$1, p_{s-1}k_{4s+2}, p_{s-1}k_{4s+3}, k_{4s+2}k_{4s+3},$$

where $s > r$. All the terms of the first set are contained as factors in each of the terms $p_{s-1}k_{4r+2}$, $p_{s-1}k_{4r+3}$, which are of odd order in the k ; consequently the multiplication is scalar. The term $k_{4r+2}k_{4r+3}$ has no factor common with the first set, and being of even order is commutative in regard to it. Hence the two sets are commutative with one another. Next take the two q -sets

$$1, q_{r-1}k_{4r}, q_{r-1}k_{4r+1}, k_{4r}k_{4r+1},$$

$$1, q_{s-1}k_{4s}, q_{s-1}k_{4s+1}, k_{4s}k_{4s+1}.$$

Here again all the terms of the first sets are factors of $q_{s-1}k_{4r}$ and of $q_{s-1}k_{4r+1}$, and they have no factors in common with $k_{4r}k_{4r+1}$; since then all the terms are of even order in the k , the multiplication is scalar. The proposition is therefore proved.

We may set out a formal proof that the 2^{2m} products of m terms, one from each of the first m sets, are all *distinct*, as follows: suppose this true for the first $m-1$ sets; that is to say, that no two of the products formed from them are identical or such that their product is $\pm k_1 k_2 \dots k_{2m-1}$. Let then a, b be two of these products; and let c, d be two terms of the next set. Then we have to prove that ac can neither be equal to $\pm bd$, nor such that the product $acbd$ is $\pm k_1 k_2 \dots k_{2m-1} k_{2m} k_{2m+1}$. Now if $ac = \pm bd$, multiply both sides by bc ; then $ab = \pm cd$. The product cd is one of the terms of the new set; it is either unity, or contains one or both of the new units k_{2m}, k_{2m+1} , so that it cannot be equal to ab . The product $acbd$ cannot be $\pm k_1 \dots k_{2m+1}$ unless cd is $k_{2m} k_{2m+1}$ and ab is $k_1 k_2 \dots k_{2m-1}$, which is contrary to the supposition. Hence if the products of the first

$m-1$ sets are all distinct for the purposes of the $(2m-1)$ way algebra, the products of the first m sets will be all distinct for the purposes of the $(2m+1)$ way algebra. But it is easy to see that the products of the first two sets are distinct,

Algebras with an even number of units.

Every algebra with $2m$ units is related to the adjacent algebra with $2m-1$ units in precisely the same way as biquaternions are related to quaternions; namely, it is simply that adjacent algebra multiplied by the double algebra $(1, \omega)$ where ω is the product of all the $2m$ units. For clearly all the even terms of the $(2m-1)$ way algebra are also even terms of the $2m$ -way algebra, and so also are their products by ω ; but these are all distinct from one another, and consequently are *all* the even terms of the $2m$ -way algebra.

The multiplication of ω with the k of the $(2m-1)$ way algebra is scalar, because the k are factors in the ω , and they are both even terms.

Hence the $2m$ -way algebra is a product of the $(2m-1)$ way algebra with the double algebra $(1, \omega)$, the two sets of units being commutative with one another.

BINARY FORMS OF ALTERNATE VARIABLES*.

INTRODUCTION.

1. ALTERNATE numbers are such that $\alpha\beta = -\beta\alpha$, $\alpha^2 = 0$, $\beta^2 = 0$. It is easily shown that linear functions of them possess the same properties; i. e., if $\bar{\alpha} = a_1\alpha_1 + a_2\alpha_2 + \dots$, $\bar{\beta} = b_1\beta_1 + b_2\beta_2 + \dots$, where the a , b are scalars, and the α , β alternate numbers, then we shall have $\bar{\alpha}\bar{\beta} = -\bar{\beta}\bar{\alpha}$, and $\bar{\alpha}^2 = 0 = \bar{\beta}^2$. If M , N are homogeneous functions of alternate numbers of degrees m , n respectively, the number of interchanges of consecutive letters necessary to pass from MN to NM is mn ; thus we have

$$MN = (-1)^{mn} NM.$$

Or the product of two functions changes sign when the order of the functions is changed *only when their degrees are both odd*; that is to say, forms of odd degree among themselves behave like alternate numbers, forms of even degree in all cases like scalars. It follows that the square of any form of odd degree is zero.

Determinants of alternate numbers.

2. In expanding a determinant of alternate numbers, the order of the *rows* must be followed in multiplication; that is to say, in every term of the expanded determinant the

* [Communicated to the London Mathematical Society (June 12, 1879) by Dr Spottiswoode, P.R.S., and subsequently printed in the *Proceedings*, Vol. x. pp. 214—221. See note at end of this paper.]

constituent from the first row must be written first, that from the second row second, and so on. Thus, in expanding

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} = (\lambda\mu\nu),$$

the terms are of the form $\pm \lambda_{rst} \mu_s \nu_t$, where rst is a permutation of 123 and the signs follow the ordinary rule. An interchange of two *columns* will then alter the sign of the determinant but an interchange of two rows will leave it unaltered. For the change of sign caused by the interchange is in the latter case counteracted by the change in the order of multiplication. Thus the determinant $(\lambda\mu\nu)$ written above is a symmetrical function of the $\lambda\mu\nu$.

3. Alternate numbers may be considered as given in *sets* of n at a time (like the coordinates of a point in n -fold space), and in that case it is convenient to regard the product of all the numbers of any set as equal to unity. Hence the products of all but one of the numbers make a new set, the *reciprocal* numbers. The n^{th} power of a determinant of *even* order n is $-|n|$; the $(n-1)^{\text{th}}$ power is the determinant of the reciprocal numbers. Considering especially determinants of the second order, we have an important theorem of their multiplication, viz., $(\lambda\mu)(\mu\nu) = -(\lambda\nu)$. For

$$\begin{aligned} (\lambda_1\mu_2 - \lambda_2\mu_1)(\mu_1\nu_2 - \mu_2\nu_1) &= \lambda_1\mu_2\mu_1\nu_2 + \lambda_2\mu_1\mu_2\nu_1 \\ &= (\lambda_2\nu_1 - \lambda_1\nu_2)\mu_1\mu_2 \\ &= -(\lambda_1\nu_2 - \lambda_2\nu_1). \end{aligned}$$

Hence $(\lambda\mu)(\mu\nu)(\nu\rho) \dots (\sigma\tau) \{n \text{ factors}\} = (-)^{n-1}(\lambda\tau)$.

An analogous theorem holds for determinants of the n^{th} order; viz., if we denote a determinant with r rows of λ and s rows of μ by $(\lambda^r\mu^s)$, where $r+s=n$; then

$$(\lambda^r\mu^s)(\mu^s\nu^r) \dots (\sigma^r\tau^s) \{k \text{ factors}\} = (-)^{s(n+r)s}(\lambda^r\tau^s) = (-)^{ks}(\lambda^r\tau^s),$$

since $n+r=2r+s \equiv (\text{mod. } 2)$, and so $s(n+r) \equiv s^2 \equiv s (\text{mod. } 2)$.

Multipartite Forms.

4. A homogeneous function linear in each of n sets of k alternate numbers is called a k -ary multipartite form of the n^{th} order, or shortly, a k -ary form of the n^{th} order. We may consider also forms of any order lower than k in any of the sets; but for the present we restrict ourselves to the case in which the forms are linear. Consider now k forms, each linear in regard to the k alternate numbers $\lambda_1, \lambda_2, \dots, \lambda_k$; viz.,

$$H'_a = a_1\lambda_1 + a_2\lambda_2 + \dots + a_k\lambda_k$$

$$H'_k = k_1\lambda_1 + k_2\lambda_2 + \dots + k_k\lambda_k,$$

where the coefficients a, b, \dots, k are themselves k -ary multipartite forms of alternate numbers.

The product $H'_a H'_b \dots H'_k = \Pi H'$ is an invariant; that is to say, if for the λ we substitute k linear functions of them, say the μ , then the functions H' will be transformed into functions of the μ ; and if we form the same function of the new coefficients that $\Pi H'$ is of the old coefficients, one will be equal to the other multiplied by the determinant of transformation.

For the product is $I \cdot \lambda_1 \lambda_2 \dots \lambda_k$, whether we regard the λ as linear functions of the μ or not; but in the latter case

$$\lambda_1 \lambda_2 \dots \lambda_k = D \cdot \mu_1 \mu_2 \dots \mu_k,$$

where D is the determinant of transformation.

But it is also to a constant factor *près* the only function possessing this property. For let I be such a function, and calculate it for the linear forms $\lambda_1, \lambda_2, \dots, \lambda_k$. As all the coefficients are here either unity or zero, I must be represented by a constant, say I_0 . Now, expressing the λ in terms of the μ , we have, by hypothesis, $I_\mu = D I_0 = \lambda_1 \dots \lambda_k \cdot I_0$.

It is to be remarked that the coefficients of transformation may themselves be forms involving alternate numbers to any *even* order. Otherwise the μ would not be alternate numbers, which is implied.

Moreover, we may regard the λ as no longer linear forms, but forms of any odd order; the new invariant will then be equal to the old one multiplied by a *commutant* of transformation. This leads to a useful theorem in the comparison of invariants; e.g.,

$$c | 456\vartheta . a | \vartheta^2 . b | \vartheta^3 . c | 456\vartheta' = (c | 456\vartheta |)^2 . a | \vartheta^2 . b | \vartheta^3 .$$

The proposition may be further extended by considering forms which involve the λ to an order higher than the first, but less than k ; i. e., linear functions of their products r together. Let $F_a, F_b, \dots F_e$ be forms such that the sum of their orders in the λ is equal to k ; then their product is an invariant (and the only one) in regard to linear transformations of the λ .

If the sum of the orders is less than k , $=h$ suppose, the product is a covariant; viz., it is a linear function of the products of the λ , h together, which, whether derived from the original forms before or after the λ are replaced by linear functions of the μ , has the same value. In this case the r -products of the λ are replaced by linear functions of the r -products of the μ , the coefficients being determinants of the r^{th} order formed with the coefficients of the μ .

Now suppose any number of forms $F_a \dots F_n$ to involve any number of sets of alternate numbers $\lambda_\mu \dots \tau$, yet so that the sum of the orders in any one set is not greater than the number of alternates in that set; then the product of the forms is an invariant or covariant in regard to each of the sets taken separately, in the sense explained above; and it is the only function which possesses this property.

Expression of Unsymmetrical forms in terms of Symmetrical forms and Determinants of the Variables.

5. The binary form in two sets of variables,

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 = a12,$$

will be called *symmetrical* when $a_{12} = a_{21}$. In that case the

interchange of the variables λ, μ only alters the sign of the form; we have $a12 = -a21$. We have, in general,

$$a12 + a21 = (a_{12} - a_{21}) (\lambda_1 \mu_2 - \lambda_2 \mu_1) = s (\lambda \mu).$$

The factor $a_{12} - a_{21}$ is an invariant when both sets of variables are transformed by the same substitutions; in fact, we have

$$(\lambda_1 \mu_2 - \lambda_2 \mu_1) a12 = a_{21} - a_{12} = -s,$$

which exhibits it as a product of the form by the universal covariant $(\lambda \mu)$ or (12). We may make a symmetrical form from $a12$ by adding $-a21$ to it; half this sum shall be called the mean value of $a12$ and denoted by $\overline{a12}$. Thus, we have

$$\overline{a12} = \frac{1}{2} (a12 - a21) = a_{11} \lambda_1 \mu_1 + \frac{1}{2} (a_{12} + a_{21}) (\lambda_1 \mu_2 + \lambda_2 \mu_1) + a_{22} \lambda_2 \mu_2;$$

but also

$$\frac{1}{2} (a12 + a21) = \frac{1}{2} (a_{12} - a_{21}) (\lambda_1 \mu_2 - \lambda_2 \mu_1),$$

therefore

$$a12 = \overline{a12} + \frac{1}{2} s \cdot (12),$$

where

$$-s = (12) a12.$$

It is easy to apply this to forms involving more sets of variables, if we remember that in these results the coefficients may themselves be such forms. We have, for example,

$$a123 = \overline{a123} - \frac{1}{3} \{ (23) a \cdot (23) + (13) a \cdot (13) + (12) a \cdot (12) \},$$

and so, generally,

$$a12 \dots k = \overline{a12 \dots k} - \frac{\sum (12) a \cdot (12)}{n} + \frac{\sum (12) (34) a \cdot (12) (34)}{\frac{1}{2} n (n-1)} - \dots,$$

the coefficients being the reciprocals of the binomial coefficients*.

Theory of Quadratic Forms.

6. In connection with the quadratic form $a12$, we have already considered the invariant

$$a12 + a21 = s_n (12), \text{ where } (12) a12 = -s_n.$$

We have thus the formula

$$a21 = -a12 + s_n \cdot (12) \dots \dots \dots (1).$$

* [In the last two equations of Par. 5, the symbols a are to be considered as abbreviations for $a23, a13, a12; a1234, \&c.$ &c.]

To this if we add

$$(13) a12 = -a32, \quad (23) a12 = -a13 \dots \dots \dots (2),$$

we shall have exhausted all the invariants and covariants of the first order in the coefficients.

The *discriminant* which is the only invariant of the second order, is given by the square of the quadratic form. We may write

$$a12 . a12 = -2D_{aa} \dots \dots \dots (3),$$

where

$$D_{aa} = a_{11}a_{22} - a_{10}a_{21}.$$

We have

$$\begin{aligned} a21 . a21 &= [-a12 + s_a . (12)] [-a12 + s_a . (12)] \\ &= a12 . a12 - 2s_a (12) a12 + s_a^2 . (12)^2 \\ &= a12 . a12 + 2s_a^2 - 2s_a^2 = a12 . a12, \end{aligned}$$

as it obviously should; and this may be regarded as a proof of the formula (2).

Moreover,

$$\begin{aligned} a12 . a21 &= a12 \{-a12 + s_a (12)\} = -a12 . a12 - s_a^2 \\ &= 2D_{aa} - s_a^2 \dots \dots \dots (4); \end{aligned}$$

and again since

$$\begin{aligned} \overline{a12} &= a12 - \frac{1}{2}s_a . (12), \\ \overline{a12} . \overline{a12} &= a12 . a12 + s_a^2 + \frac{1}{4}s_a^2 (12) \\ &= a12 . a12 + \frac{1}{2}s_a^2; \end{aligned}$$

therefore

$$\overline{D_{aa}} = D_{aa} - \frac{1}{4}s_a^2 \dots \dots \dots (5),$$

where $\overline{D_{aa}}$ is the corresponding invariant of the symmetrical function \overline{a} .

Since the product of two *even* forms is independent of their order, we have

$$a12 . a13 = a13 . a12.$$

But

$$\begin{aligned} a12 . a13 + a13 . a12 &= -(23) a12 . a13 . (23) \\ &= a13 . 13 (23) = -2D_{aa} (23), \end{aligned}$$

therefore

$$a12 . a13 = -D_{aa} (23) \dots \dots \dots (6).$$

Hence

$$a12 . a31 = a12 \{-a13 + s_a(13)\} = D_{aa} . (23) - s_a . a32 \dots (7),$$

$$\begin{aligned} a21 . a31 &= \{-a12 + s_a(12)\} \{-a13 + s_a(13)\} \\ &= -D_{aa}(23) + s_a(a32 + a23) - s_a^2(23) \\ &= -D_{aa}(23) = a12 . a13 \dots \dots \dots (8), \end{aligned}$$

$$a21 . a13 = a21 \{-a31 + s_a(13)\} = D_{aa}(23) - s_a . a23 \dots (9).$$

In the case of a symmetrical function $s=0$, and the formulæ reduce themselves to the following:

$$a12 a12 = -2D_{aa}, \quad a21 = -a12, \quad a12 . a13 = -D_{aa} . (23).$$

The following formulæ are also worth noticing:

$$a12 . a34 + a13 . a24 = \{D_{aa} . (14) - s_a . a14\} (23) \dots (10),$$

$$\begin{aligned} a12 . a34 - a21 . a43 &= s_a \{a23 . (14) - a14 . (23)\} \\ &= s_a \{a41 . (23) - a32 . (14)\} \dots \dots (11); \end{aligned}$$

these enable us to make a single or bifid substitution in a product of the form by itself with two different sets of variables.

Passing now to two different forms a, b , we have in the first place the covariant

$$\mathfrak{D}_{ab} 23 = a12 . b13 = \begin{pmatrix} a_{11}b_{21} - a_{21}b_{11}, & a_{11}b_{22} - a_{21}b_{12} \\ a_{12}b_{21} - a_{22}b_{11}, & a_{12}b_{22} - a_{22}b_{12} \end{pmatrix} \chi \mu_1 \mu_2 \chi \nu_1 \nu_2 \dots (12),$$

and next the invariant

$$D_{ab} = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} = -a12 . b12 \dots (13).$$

We may now make the following analysis of the one-place and two-place products of a and b :

$$\left. \begin{aligned} a12 . b13 &= \mathfrak{D}23, \\ a12 . b31 &= a12 \{-b13 + s_b(13)\} = -\mathfrak{D}23 - s_b . a32, \\ a21 . b13 &= \{-a12 + s_a(12)\} b13 = -\mathfrak{D}23 - s_a . b23, \\ a21 . b31 &= \{-a12 + s_a(12)\} \{-b13 + s_b(13)\} \\ &= \mathfrak{D}23 + s_b . a32 + s_a . b23 - s_a s_b . 23 \end{aligned} \right\} (14),$$

$$\left. \begin{aligned} a12 . b12 &= -D_{ab}, \\ a12 . b21 &= a12 \{-b12 + s_b(12)\} = D_{ab} - s_a s_b \end{aligned} \right\} \dots \dots (15).$$

Let us now apply to \mathfrak{S} all those formulæ which we have proved for a single form. Thus from (1), we have

$$\begin{aligned}\mathfrak{S}23 + \mathfrak{S}32 &= -(23)\mathfrak{S}23(23) = -(23)a12.b13.(23) = a13.b13.(23) \\ &= D_{ab}(23) \text{ or } s_2 = D_{ab} \dots\dots\dots (16).\end{aligned}$$

Next, from (3),

$$\begin{aligned}-2D_{\mathfrak{S}\mathfrak{S}} &= \mathfrak{S}23.\mathfrak{S}23 = a12.b13.a42.b43 = D_{aa}D_{bb}(14)^2 \\ &= -2D_{aa}D_{bb} \text{ or } D_{\mathfrak{S}\mathfrak{S}} = D_{aa}D_{bb} \dots\dots\dots (17).\end{aligned}$$

$$\begin{aligned}\text{Hence} \quad & \left. \begin{aligned}\mathfrak{S}23.\mathfrak{S}32 &= 2D_{aa}D_{bb} - D_{ab}^2 \\ \overline{\mathfrak{S}23} &= \mathfrak{S}23 - \frac{1}{2}D_{ab}(23) \\ \overline{D_{\mathfrak{S}\mathfrak{S}}} &= D_{aa}D_{bb} - \frac{1}{4}D_{ab}^2\end{aligned} \right\} \dots\dots\dots (18),\end{aligned}$$

and

which last agrees with the known formula.

We have next the one-place products of \mathfrak{S} by itself; namely,

$$\left. \begin{aligned}\mathfrak{S}23.\mathfrak{S}24 &= -D_{aa}D_{bb}(34) = \mathfrak{S}32.\mathfrak{S}42 = a12.b13.a52.b54 \\ &= a13.b12.a54.b52 \\ \mathfrak{S}23.\mathfrak{S}42 &= +D_{aa}D_{bb}(34) - D_{ab}.\mathfrak{S}43 = a12.b13.a54.b52 \\ \mathfrak{S}32.\mathfrak{S}24 &= +D_{aa}D_{bb}(34) - D_{ab}.\mathfrak{S}34 = a13.b12.a52.b54\end{aligned} \right\} (19);$$

but it is important also to calculate these for the mean value $\overline{\mathfrak{S}}$; namely, we have

$$\begin{aligned}\overline{\mathfrak{S}23}.\overline{\mathfrak{S}24} &= \{\mathfrak{S}23 - \frac{1}{2}D_{ab}(23)\} \{\mathfrak{S}24 - \frac{1}{2}D_{ab}(24)\} \\ &= -D_{aa}D_{bb}(34) + \frac{1}{2}D_{ab}(\mathfrak{S}43 + \mathfrak{S}34) - \frac{1}{4}D_{ab}^2(34) \\ &= -(D_{aa}D_{bb} - \frac{1}{4}D_{ab}^2)(34) = -\overline{D_{\mathfrak{S}\mathfrak{S}}}(34),\end{aligned}$$

as it ought to be, confirming equation (18).

Lastly, from (10) and (11), we get

$$\begin{aligned}\mathfrak{S}12.\mathfrak{S}34 + \mathfrak{S}13.\mathfrak{S}24 &= \{D_{aa}.D_{bb}.(14) - D_{ab}.\mathfrak{S}14\}(23) \\ \mathfrak{S}12.\mathfrak{S}34 - \mathfrak{S}21.\mathfrak{S}43 &= D_{ab}\{23.(14) - \mathfrak{S}14(23)\} \\ &= D_{ab}\{\mathfrak{S}41.(23) - \mathfrak{S}32.(14)\} \dots\dots (20).\end{aligned}$$

There are four one-place products of a, a, b ; namely,

$$\left. \begin{aligned} a12 . a32 . b34 &= D_{aa} . b14, & a12 . a32 . b43 &= D_{aa} . b41, \\ a12 . a23 . b34 &= -D_{aa} . b14 + s_a . \S 14 + s_a^2 . b14 = (s_a^2 - D_{aa}) b14 + s_a . \S 14, \\ a12 . a23 . b43 &= -D_{aa} . b41 - s_a . \S 14 - s_a s_b . a41 - s_a^2 b14 + s_a s_b (14) \\ &= (s_a^2 - D_{aa}) . b41 - s_a . \S 14 - s_a s_b . a41, \end{aligned} \right\} (21)$$

When the forms are symmetrical, all these reduce to

$$\left\{ \begin{aligned} a \\ b \end{aligned} \right\} &= \pm 2D_{ab} & \left\{ \begin{aligned} a \\ b \end{aligned} \right\} \left\{ \begin{aligned} a \\ b \end{aligned} \right\} &= \pm D_{aa} . b14 & \left\{ \begin{aligned} a \\ b \end{aligned} \right\}^{-1} &= \pm \S 12.$$

Three forms a, b, c give rise to the new invariant

$$\begin{aligned} R_{abc} &= a12 . b23 . c31 = + a_{11} (b_{21}c_{32} - b_{22}c_{12}) \\ &\quad - a_{12} (b_{11}c_{22} - b_{12}c_{12}) \\ &\quad - a_{21} (b_{21}c_{31} - b_{22}c_{11}) \\ &\quad + a_{22} (b_{11}c_{21} - b_{12}c_{11}) \dots\dots\dots (22), \end{aligned}$$

and the remaining closed products are of the type

$$a21 . b23 . c31 = \{a12 + s_a (12)\} b23 . c31 = -R_{abc} + s_a D_{bc} - s_a s_b s_c.$$

R calculated for the symmetrical functions a, b, c is

$$\begin{aligned} R_{abc} &= \{a12 - \frac{1}{2}s_a (12)\} \{b23 - \frac{1}{2}s_b (23)\} \{c31 - \frac{1}{2}s_c (31)\} \\ &= R_{abc} + \frac{1}{2} (s_a D_{bc} + s_b D_{ca} + s_c D_{ab}) - \frac{1}{2}s_a s_b s_c \dots\dots\dots (23). \end{aligned}$$

It may be mentioned in this place that the determinant made with the coefficients of four forms a, b, c, d has the value

$$\text{Det. } (a, b, c, d) = s_a R_{bcd} - s_b R_{cda} + s_c R_{adb} - s_d R_{abc} = s_a R_{bcd} - \&c.,$$

and that no additional invariants are introduced by a greater number of forms.

We now consider the open one-place products of 3 forms. There are evidently four such products having b in the middle; and they are in the first place connected by the formulae

$$\begin{aligned} a12 . b23 . c34 + a42 . b23 . c31 &= -R_{abc} (14), \\ a12 . b23 . c34 + a12 . b32 . c34 &= -s_b a13 c34 = s_b \S_{aa} 14 + s_a s_b . a14, \\ a12 . b23 . c34 - a34 . b23 . c12 &= -c32 a34 . b21 + c23 . a21 . b43. \end{aligned}$$

[Here the paper abruptly ends. It has been found very difficult to devise a satisfactory way of dealing with the other notes on *graphs* on account of the fragmentary condition in which they were left. Prof. H. J. S. Smith says 'the papers tied up with the fragment on *Quadric forms* are little more than mere jottings. They serve to show that Clifford had applied the method to the Quintic and Sextic. They should be preserved....If the letters by which Clifford denotes certain concomitants are those used in Clebsch's *Bindre Formen*, or in any other accessible work or memoir, it might be possible to rescue the pages relating to the Quintic and Sextic.' By the advice, and at the expense, of Dr Spottiswoode, these fragments have been lithographed, and fifty copies have been printed off for circulation among the principal libraries.]

XXXII.

ON MR SPOTTISWOODE'S CONTACT PROBLEMS*.

The present communication consists of two parts.

The first part treats of the contact of conics with a given surface at a given point; this class of questions was first treated by Mr Spottiswoode in his paper "On the Contact of Conics with Surfaces," and general formulae applicable to all such questions were given.

The results of that paper are here reproduced with some additions; with the exception of a few collateral theorems, these are all contained in the following Table:—

| | |
|---|-------|
| †Number of five-point conics through fixed point..... | = 6 |
| †Order of surface formed by five-point conics through fixed axis | = 8 |
| Number of six-point conics through fixed axis..... | = 9 ‡ |
| †Number of seven-point conics..... | = 70 |

[* From the *Philosophical Transactions* of the Royal Society of London, Vol. cxiv, Part 2.]

† These results constitute the additions.

‡ [In the Memoir quoted by Professor Clifford, it was stated that the number of conics passing through a given axis and having six-point contact with a surface at a given point is ten. In making this statement I overlooked the fact that, in order to put in evidence that a certain quantity was a factor of the equation which determines the positions of the planes of the conics, the equation was multiplied by a quantity D which is a linear function of the position. In reckoning the degree of the equation this factor must of course be discarded. The degree is consequently less by unity than that stated in the Memoir; viz. it is 9, as proved by Professor Clifford. — See July 3, 1873.]

The second part treats of the contact of a quadric surface with a surface of the order n ; and in particular it determines the number of points at which a quadric (other than the tangent plane reckoned twice) can have four-branch contact with the surface. In his paper "On the Contact of Surfaces," Mr Spottiswoode proves that at an arbitrary point on a surface there is no other solution than the doubled tangent plane, and gives the conditions that must be satisfied by those points at which another solution is possible.

The method here adopted is an extension of that applied by Joachimstal to the contact of lines with curves and surfaces. The co-ordinates of a point on a conic are expressed in terms of a single parameter, those of a point on a quadric by two parameters. To determine the intersection with a given surface we have an equation in the parameter or parameters, and the conditions of contact are expressed in terms of the coefficients of that equation. The special case of the intersection of a quadric with a cubic surface is treated by the method of representation on a plane.

PART I.—THE CONTACT OF CONICS WITH SURFACES OF ORDER 22.

I.

The current plane-coordinates being denoted by

$$X_1, X_2, X_3, X_4,$$

let the equations of the three points A, B, C be respectively

$$0 = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 \equiv \Sigma aX,$$

$$0 = \Sigma bX,$$

$$0 = \Sigma cX.$$

The quantities a_i, b_i, c_i ($i=1, 2, 3, 4$) are the coordinates of the points A, B, C . The symbol A itself I shall use indifferently, as denoting either the form ΣaX or the differential operator

$$a_1\partial_{x_1} + a_2\partial_{x_2} + a_3\partial_{x_3} + a_4\partial_{x_4} \equiv \Sigma a\partial_x,$$

where x_1, x_2, x_3, x_4 are the current point-coordinates. It will be seen in the sequel that this double meaning is useful, while it does not introduce any confusion. Similar interpretations are of course to be given to the symbols B, C , and the like.

Consider now the point

$$P \equiv A + \theta B + \theta^2 C,$$

whose coordinates are $a_i + \theta b_i + \theta^2 c_i$ ($i = 1, 2, 3, 4$). If we suppose θ to take all possible values, the point P will describe a conic section whose tangential equation is

$$0 = 4AC - B^2 \equiv K_3.$$

To the value $\theta = 0$ corresponds the point A , to $\theta = \infty$ the point C ; while the equation shows that B is the intersection of the tangents at A and C . (Fig. 51.)

To find the point at which this conic intersects a given surface u_n of the order n , we must substitute the coordinates of P in the equation of the surface; in this way we shall form an equation in θ of the order $2n$, the solution of which will give the values of θ belonging to the $2n$ points of intersection.

If in this equation the term independent of θ vanishes, then $\theta = 0$ is a root of the equation; consequently the point A is one of the intersections, or the surface u_n passes through the point A . If also the coefficient of θ vanishes, another root of the equation coincides with zero, and *two* points of intersection are at A . And generally if the coefficient of θ^m is the first that does not vanish, m roots of the equation coincide with zero, and m points of intersection are united at A .

The result of substituting the coordinates of any point P in u_n may be conveniently represented by means of the differential operator P . It is known, in fact, that

$$(p_1 \partial_{x_1} + p_2 \partial_{x_2} + p_3 \partial_{x_3} + p_4 \partial_{x_4})^n \cdot (x_1, x_2, x_3, x_4)^n \equiv [n \cdot (p_1, p_2, p_3, p_4)]^n;$$

or, which is the same thing, $P^n u_n$ is $[n$ times the result of substituting the coordinates of P in u_n . Our result may therefore be stated in the following form:—

The necessary and sufficient conditions that the conic $K_3 \equiv 4AC - B^2$ may have m -point contact with the surface u_n

at the point A , are that in the expansion of $(A + \theta B + \theta^2 C)^n \cdot u_n$ in powers of θ , the m th power of θ is the lowest whose coefficient does not vanish.

II.

Equating to zero the coefficients of $1, \theta, \theta^2$ in this expansion, we obtain

$$0 = A^n \cdot u_n,$$

$$0 = nA^{n-1}B \cdot u_n,$$

$$0 = \frac{1}{2}n(n-1)A^{n-2}B^2 \cdot u_n + nA^{n-1}C \cdot u_n.$$

Before proceeding further with these equations, it is convenient to make the following remarks upon their nature, which will serve to simplify the expression of them.

In the first place, then, we have here a series of relations among the coordinates of the points A, B, C and the coefficients of the surface u_n ; and the determination of the coordinates and coefficients so as to satisfy a certain number of the relations presents us with the solution of various geometrical problems. These problems fall naturally into three classes.

1. The surface u_n and the point A are given. In this case the unknowns are the ratios of the eight quantities b, c, a singly infinite number of solutions corresponding to each conic; and we are accordingly able to satisfy seven of the equations*. The problem here is to find the number of conics which have seven-point contact at a given point of a given surface.

We may, however, impose beforehand certain restrictions upon the values of the unknowns, and so consider problems which involve a less number of the equations. While the number of the septactic conics is definite, the sextactic conics form a singly infinite series; and we may ask what is the number of them:

- (a) whose planes pass through a given point,
- (b) which meet a given line, or
- (c) which touch a given plane.

* Viz. six besides the first, which is satisfied identically.

The quintactic conics, again, form a doubly infinite system, and we may inquire about the number of them which satisfy two conditions; *e.g.* which pass through a given point.

2. The surface u_n is given, but not the point A . In this case, as the point A is only restricted to be a point on the surface, we have two more unknowns, making nine in all. The problems here are, to find the order of the curve on the surface at every point of which there is a conic having eight-point contact, and to find the number of points at which there is a conic having nine-point contact.

As before, however, there are certain derived problems coming under this head which involve a less number of equations. We may seek the order of the curve traced out by points of contact of septactic conics satisfying one condition, sextactic satisfying two, &c.; or we may seek the number of septactic conics satisfying two conditions, sextactic satisfying three, &c.

3. The surface is not wholly given. We may here assign a number of relations sufficient to eliminate the quantities a, b, c , leaving one or more relations among the coefficients of u_n . The problems here are such as:—to find the number of surfaces in a pencil $u_n + \lambda u_n$ which admit of ten-point contact with some conic, or one of whose nine-point conics meets a given line.

In the present communication only problems of class 1 will be considered; the formulae in this case may be very considerably simplified. The quantities a and the coefficients of u_n , then, are data of the problem; so that the first of our equations, $A''u_n = 0$, is satisfied by hypothesis. The next equation, $A'''Bu_n = 0$, signifies that the point B lies in the tangent plane at A , as it obviously must if the conic touch the surface at A . We shall suppose this also to be satisfied from the commencement; that is, we shall regard B as a point moving in the tangent plane, and to be determined by construction in that plane. This may be effected analytically if we substitute for B , $\lambda A + \mu B + \nu B'$, where now B, B' are regarded

as fixed points in the tangent plane, and the three unknowns λ, μ, ν take the place of the four quantities b . There is, however, as will be seen, no occasion to make the substitution explicitly.

This being so, any relation involving B only beside the data must be regarded as the equation of a curve in the tangent plane. For example, $A^{n-2} B^2 u_n = 0$, expressing that B lies on the quadric polar of A , is the equation of the two chief tangents at that point. Generally, $B^n u_n = 0$ is the equation of the intersection with u_n of the tangent plane; and the curves $AB^{n-1} u_n = 0$, $A^2 B^{n-2} u_n = 0$, &c. are the successive polars of A in regard to that intersection.

The terms entering into our equations are of the general form

$$\theta^{p+q} \cdot \frac{|n|}{|n-p-q| |p| |q|} A^{n-p-q} B^p C^q \cdot u_n.$$

Any term, therefore, is completely determined by the two numbers p and q , and might, for any thing that has yet appeared, be denoted by (p, q) . In view, however, of subsequent substitutions, we shall keep in evidence the manner in which B and C are involved, and denote the term in question by the symbol $\theta^{p+q} (B^p C^q)$.

As we do not consider any higher than seven-point contact, we have only the five equations:

$$0 = (B^3) + (C) \dots\dots\dots (3),$$

$$0 = (B^3) + (BC) \dots\dots\dots (4),$$

$$0 = (B^4) + (B^2 C) + (C^2) \dots\dots\dots (5),$$

$$0 = (B^5) + (B^3 C) + (BC^2) \dots\dots\dots (6),$$

$$0 = (B^6) + (B^4 C) + (B^2 C^2) + (C^3) \dots\dots\dots (7).$$

III. Conics through a fixed point.

Combining equation (3) successively with (4) and (5), so as to obtain results homogeneous in B , we find

$$0 = (B^3) (C) - (BC) (B^2),$$

$$0 = (B^4) (C)^2 - (B^2 C) (B^3) (C) + (C^2) (B^2)^2.$$

If we regard C as a fixed point, these are equations of a cubic and a quartic curve in the tangent plane, on each of which B must lie if the conic K_2 has five-point contact. But these curves have a common node at A , and common tangents at it; for every term in each has at least one factor of the form (B^m) , which we know to represent a polar of A in regard to the intersection of u_n by the tangent plane—that is, a curve touched by the chief tangents at A . Of their 12 intersections, then, 6 coincide with the point A ; and there remain *six conics having five-point contact at A which pass through an arbitrary point C .*

IV. Conics meeting a fixed axis through A .

If the point C , instead of being altogether given, is movable on a fixed straight line through A , we may represent it by $A + \lambda C$; where now C is really a fixed point, and λ a quantity to be determined. When we make this substitution in our equations, they become*

$$\begin{aligned} 0 &= (B^3) + \lambda (C), \\ 0 &= (B^3) + \lambda (BC), \\ 0 &= (B^4) + (B^2) + \lambda (B^2C) + 2\lambda (C) + \lambda^2 (C^2), \\ 0 &= (B^3) + (B^3) + \lambda (B^3C) + 2\lambda (BC) + \lambda^2 (BC^2), \\ 0 &= (B^6) + (B^4) + \lambda (B^4C) + (B^2) + 2\lambda (B^2C) + \lambda^2 (B^2C^2) + 3\lambda (C) \\ &\quad + 3\lambda^2 (C^2) + \lambda^3 (C^3). \end{aligned}$$

The last three admit of obvious simplifications by aid of the previous ones; and the system may finally be written

$$\begin{aligned} 0 &= (B^2) + \lambda (C) \dots\dots\dots (3'), \\ 0 &= (B^3) + \lambda (BC) \dots\dots\dots (4'), \\ (B^2) &= (B^4) + \lambda (B^2C) + \lambda^2 (C^2) \dots\dots\dots (5'), \\ (B^3) &= (B^5) + \lambda (B^3C) + \lambda^2 (BC^2) \dots\dots\dots (6'), \\ (B^4) - \lambda^2 (C^2) &= (B^6) + \lambda (B^4C) + \lambda^2 (B^2C^2) + \lambda^3 (C^3) \dots\dots (7'). \end{aligned}$$

* It must be remembered that (B) and terms involving A only vanish by hypothesis. The formulæ have also been simplified by the omission of certain coefficients depending on n which do not affect the final results.

Locus of poles of axis in regard to four-point conics.

If we select any plane through the line AC , there will be a singly infinite number of conics in the plane having four-point contact with the surface at A . The line AC will, as is well known, have the same pole in regard to all these conics—that is to say, the point B will be the same for the whole system. If we now allow the plane to turn round the axis, the point B will trace out a curve in the tangent plane. The equation to this curve is got by eliminating λ between equations (3') and (4'); namely, it is

$$0 = (B^3) (BC) - (B^3) (C) \dots\dots\dots (4'').$$

We see, therefore, that *the locus of the poles of an axis in regard to all the four-point conics whose planes pass through it is a cubic curve in the tangent plane touching the chief tangents at A , which point is therefore a node on the curve.*

We might have inferred this from the fact that on any line through A there is only one point B , while this point coincides with A in the case of the two chief tangents; since at a point of inflexion all the four-point conics contain the inflexional tangent.

Number of six-point conics through an axis.

We have now to determine B so that the equations (3'), (4'), (5'), (6') may be simultaneously satisfied. We know already that B must lie on the cubic (4''); it is necessary therefore to find some other locus on which it has to lie. First of all, then, we must eliminate λ between (3') and (5') and between (3') and (6'); the results are,

$$(B^3) (C)^2 = (B^4) (C)^2 - (B^2 C) (B^2) (C) + (B^2)^2 (C^2) \equiv p_4, \text{ say;}$$

$$(B^3) (C)^2 = (B^3) (C)^3 - (B^3 C) (B^2) (C) + (B^2)^2 (BC^2) \equiv q_5, \text{ say.}$$

Here $p_4 = 0$ and $q_5 = 0$ are curves touching the chief tangents at A , and of the degrees four and five respectively. But the equations are not homogeneous; in fact only the ratios and not the absolute values of the quantities a were determined by the

fixing of the point A , and they may be regarded as involving an arbitrary factor whose square affects the left-hand side of the equations. It is, however, at once eliminated, and we obtain the homogeneous result,

$$(B^2) \cdot q_5 = (B^2) \cdot p_4.$$

This is a curve of order 7 having two branches in each of the chief directions at A . Of its 21 intersections with the cubic (\mathcal{A}''), then, 12 coincide with A , and there remain *nine positions of B which give sextactic conics through the fixed axis*; or we may say, *of the sextactic conics at the point A , there are nine whose planes pass through an arbitrary point C .*

System of five-point conics through an axis.

Since there is one five-point conic in every plane, if we consider all the planes through a fixed axis we shall obtain a singly infinite number of five-point conics. Of this system there is only one conic whose plane passes through an arbitrary point, viz. the conic determined by the plane through that point and the axis.

There are eight conics of the system which meet an arbitrary line.

The number of conics which meet an arbitrary line is clearly the same as the order of the surface which they trace out. Now, since through every point on the axis can be drawn six conics of the system (as proved in the last section), the axis is a six-fold line on the surface. The section of the surface, then, by a plane through the axis is made up of the axis taken six times over and the conic in that plane; or it is of the order *eight*. Q. E. D.

V. *Conics not subject to any condition.*

In order to get rid of the restriction of meeting a fixed axis, we must again modify our fundamental equations. We have to put them into a form in which they will represent *any* conic touching the surface u_n at the point A . For this

purpose it is necessary and sufficient that C should be movable over a plane passing through A ; since every cone through A must cut the plane in one other point, but this may be any point of the plane. We attain this analytically by substituting for λC in the second set of equations, $\lambda C + \mu D$, where C and D are now fixed points not in the tangent plane and not in any straight line through A . This is equivalent to still considering the conics which meet a given axis, but allowing that axis to move over a fixed plane.

The transformed equations are:—

$$0 = (B^2) + \lambda(C) + \mu(D) \dots\dots\dots (3''),$$

$$0 = (B^2) + \lambda(RC) + \mu(RD) \dots\dots\dots (4'').$$

$$(B^2) = (B^2) + \lambda(R^2C) + \mu(R^2D) + \lambda^2(C^2) + 2\lambda\mu(CD) + \mu^2(D^2) \dots\dots (5''),$$

$$(B^2) = (B^2) + \lambda(R^2C) + \mu(R^2D) + \lambda^2(RC^2) + 2\lambda\mu(RC'D) + \mu^2(RD^2) \dots\dots\dots (6'').$$

$$(B^2) = \lambda^2(C^2) + 2\lambda\mu(C'D) + \mu^2(D^2) + \lambda^2(R^2C^2) + \lambda\mu(R^2C'D) + \mu(R^2D) + \lambda^2(R^2C^2) + 2\lambda\mu(R^2C'D) + \mu^2(R^2D^2) + \lambda^2(C^2) + 3\lambda\mu(C'D) + 3\mu^2(D^2) + \mu^2(D^2) \dots\dots\dots (7'').$$

Locus of poles of axis in given plane in regard to conics.

From the first two of these equations we obtain

$$1 : \lambda : \mu = (BD)\{C\} : (BC)\{D\} : (B^2)\{D\} : (B^2)\{D\} : (B^2)\{C\} \\ \dots\dots (B^2)\{B^2\} = n_1 : l_2 : m_2 \text{ say,}$$

here $n_1 = 0$ in the equation to a straight line passing through A , while $l_2 = 0$, $m_2 = 0$ are values touching the cone tangents at A .

Substituting these values in (5'') and (6''), we obtain

$$n_1^2(B^2) = n_1^2(B^2) + n_1 l_2(R^2C) + n_1 m_2(R^2D) + l_2^2(C^2) \\ + 2l_2 m_2(C'D) + m_2^2(D^2) = n_1^2 \text{ say,}$$

$$n_1^2(B^2) = n_1^2(B^2) + n_1 l_2(R^2C) + n_1 m_2(R^2D) + l_2^2(RC^2) \\ + 2l_2 m_2(RC'D) + m_2^2(RD^2) = n_1^2 \text{ say.}$$

Here the curves $n_1 = 0$, $n_1 = 0$ are of the sixth and seventh orders respectively, each of them having one branch at A in

each of the chief directions, and one other branch different for the two curves.

The equations

$$n_1^2(B^2) = u_a,$$

$$n_1^2(B^3) = v_7$$

must hold for six-point contact, but they are not homogeneous. Eliminating n_1^2 , however, there results

$$(B^2) \cdot v_7 = (B^3) \cdot u_a,$$

a curve of the ninth order, locus of the poles in regard to the sextactic conics of an axis moving in a fixed plane. This curve has two branches at A in each of the chief directions, and one other branch; and consequently is met by the plane ACD in five points coinciding with A , and in four other points. Now the plane ACD does not in general contain a sextactic conic; the pole of the axis can therefore only be in this plane when the axis itself is in the tangent plane. In this case there is a certain number of proper sextactic conics in planes through the axis, and it is clear that the pole of the axis in regard to any such conic is the point A . These conics, therefore, correspond to the five intersections of the plane ACD with the locus of poles which coincide with A ; or *through an axis in the tangent plane can be drawn five proper sextactic conics.* In the tangent plane itself there are four improper conics having six-point contact; viz. the pair of chief tangents, which (as a sharp conic or line-pair reckoned among conics given tangentially) counts for two, and each chief tangent doubled.

Number of septactic conics.

There is a finite number of septactic conics at the point A ; each of these meets the plane ACD in a determinate point $A + \lambda C + \mu D$, and fixes thereby a position of the point B . These positions of the point B must necessarily lie in the 9th locus just investigated; it remains only to find a homogeneous relation which shall determine another locus for B , and to count the number of their intersections.

To this end we must first substitute for $l_1 : l_2 : \mu$ their values $n_1 : l_2 : m_2$ in equation (i''). The result is

$$\begin{aligned} n_1^3 (B) - n_1 l_2^3 (C^2) - 2n_1 l_2 m_2 (C'D) - n_1 m_2^2 (D^2) \\ + n_1^2 (B^2) + n_1^2 l_2 (B^2 C) + n_1 m_2 (B^2 D) \\ + n_1 l_2^2 (B^2 C^2) + 2n_1 l_2 m_2 (B^2 C'D) + n_1 m_2^2 (B^2 D^2) \\ + l_2^3 (C^3) + 3l_2^2 m_2 (C^2 D) + 3l_2 m_2^2 (C^2 D^2) + m_2^3 (D^3), \end{aligned}$$

or $n_1 l_2 : m_2$ may.

The curve m_2 has one branch at A in each of the chief directions and one other branch. The curve l_2 has one branch in each of the chief directions and two other branches.

The equations (5), (6), (7) have now become

$$n_1^2 (B^2) = v_2,$$

$$n_1^2 (B^2) = v_2,$$

$$n_1 l_2 = v_2.$$

The first two of these give us the curve already considered,

$$(B^2), v_2 = (B^2), v_2,$$

which has at A two branches in the chief directions and one other. The first and third give

$$n_1 (B^2), v_2 l_2 n_1$$

a curve of order 12, having two branches in each of the chief directions and two other branches. Of the 124 intersections of these curves, then, $24 + 5 + 4 + 2 = 35$ coincide with A , leaving 89 for the number of separate curves.

PART II. THE CONTACT OF QUADRIC SURFACES WITH SURFACES OF ORDER n .

1. Conditions of contact

Let A, B, C, D be four points forming a tetrahedron, whose tangential equations are

$$0 = \Sigma aX, \quad \Sigma bX, \quad \Sigma cX, \quad \Sigma dX$$

respectively, their coordinates being a_i, b_i, c_i, d_i ($i = 1, 2, 3, 4$). Then the point

$$P = A + \theta B + \phi C + \psi D,$$

whose coordinates are

$$p_i = a_i + \theta b_i + \phi c_i + \psi d_i \quad (i = 1, 2, 3, 4),$$

will, if we suppose θ and ϕ to take all possible values, trace out a quadric surface whose tangential equation is

$$0 = AD + BC - Q_0.$$

To the pair of values

$$\theta = 0, \quad \phi = 0, \quad \text{corresponds the point } A,$$

$$\theta = \alpha, \quad \phi = 0, \quad \text{ " " " } B,$$

$$\theta = 0, \quad \phi = \epsilon, \quad \text{ " " " } C,$$

$$\theta = \epsilon, \quad \phi = \epsilon, \quad \text{ " " " } D.$$

The equation shows that AB, AC, BD, CD are generating lines.

If, now, we wish to find the nature of the curve in which this quadric intersects a given surface u_n of the order n , we must substitute the coordinates of P in the equation of the surface; in this way we shall form an equation which is of the order n in θ and in ϕ separately. If we regard θ and ϕ as coordinates of a point on the quadric surface, the equation just found is that of the curve of intersection.

Suppose that in this equation the term independent of θ and ϕ vanishes, then the equation is satisfied by the pair of values $\theta = 0, \phi = 0$, or the curve of intersection passes through the point A . Now the various directions in which we may start from the point A are determined by the initial value of the ratio $\theta : \phi$ when we move along them. The direction in which the intersection curve starts from A is therefore that obtained by neglecting in the equation terms of higher order than the first; and we see that there is only one such direction.

If, however, not only the constant term but the coefficients of θ and ϕ in the equation vanish, the initial directions are

The further conditions for three-branch contact are

$$0 = A^{n-2} B^2, \quad 0 = A^{n-2} C^2, \quad 0 = A^{n-1} D + (n-1) A^{n-2} BC.$$

The first two of these show that B and C are points on the chief tangents at A . If we regard the absolute values of the coordinates of A as given, then it appears from the third equation that B and C may be chosen arbitrarily on the chief tangents and D anywhere in space, the equation giving the relation between the absolute values of their coordinates which determines the particular surface $AD - BC = 0$.

For four-branch contact we have the additional equations,

$$\begin{aligned} 0 &= A^{n-2} B^2, & 0 &= A^{n-2} C^2, \\ 0 &= 2A^{n-2} BD + (n-2) A^{n-2} B^2 C, \\ 0 &= 2A^{n-2} CD + (n-2) A^{n-2} BC^2. \end{aligned}$$

The first two of these indicate that B and C lie on the polar cubic of A in regard to the section u_n by the tangent plane. Now this polar cubic has a node at A , whose tangents are the chief tangents. Each of these lines therefore meets the cubic in three points at A , and cannot have any other point on the curve unless it be itself a part of the cubic. But the points B and C have to lie one on each of the chief tangents. In order, therefore, that all the equations may be satisfied, the polar cubic in question must break up into the two chief tangents and some other line.

This condition may be put into another form. For if we seek the points in which the line AB meets the surface, by substituting the coordinates of $A + \lambda B$ in $u_n = 0$, the conditions $A^n u_n = 0$, $A^{n-1} B u_n = 0$, $A^{n-2} B^2 u_n = 0$, $A^{n-3} B^3 u_n = 0$ indicate that *four* roots of the equation are equal to zero, or that the line meets the surface in four consecutive points at A . We find, therefore, that

Those points of a surface at which a quadric may have four-branched contact are the points at which each chief tangent meets the surface in four consecutive points, or, which is the same thing, the points whose polar cubic contains the chief tangent.

The number of these points has been counted by Clebsch, *Crelle*, LXIII, 14*, and turns out to be

$$n(44n^2 + 162n + 102).$$

A point of this nature being given, one quadric surface having four-branch contact at it may be drawn through another arbitrary point.

The coordinates of the point A being given as to their absolute values, let us substitute for B and C , $A + \lambda R$, $A + \mu U$; where now B and C are fixed points on the chief tangent, whose coordinates are given absolutely. This being so, the following equations are satisfied *ex hypothesi*:

$$\begin{aligned} A^{\mu}u_{\alpha} &= 0, & A^{\mu-1}Ru_{\alpha} &= 0, & A^{\mu-2}Uu_{\alpha} &= 0, \\ A^{\mu-1}Ru_{\alpha} &= 0, & A^{\mu-2}Uu_{\alpha} &= 0, & A^{\mu-3}R^2u_{\alpha} &= 0, & A^{\mu-4}U^2u_{\alpha} &= 0, \end{aligned}$$

from which it follows at once, for example, that

$$A^{\mu-2}(A + \lambda R)u_{\alpha} = 0.$$

For D also let us substitute νD , where the coordinates of D are now given absolutely. Our three remaining equations are (omitting for shortness the mention of A)

$$0 \equiv \nu D + (\mu - 1)\lambda\mu, RR^{\mu}, \dots \dots \dots (1),$$

$$0 \equiv \nu D + (\mu - 2)\lambda\mu, RR^{\mu} + \lambda A, RD + \frac{1}{2}(\mu - 2)\lambda\mu^2, R^2U^{\mu}, \dots (2),$$

$$0 \equiv \nu D + (\mu - 2)\lambda\mu, RR^{\mu} + \nu\mu, U^2D + \frac{1}{2}(\mu - 2)\lambda\mu^2, R^2U^{\mu}, \dots (3);$$

and it remains only to show that these equations determine uniquely λ, μ, ν .

If we subtract (1) from (2) and (3) successively, we obtain

$$0 \equiv -\mu, RR^{\mu} + \nu, RD + \frac{1}{2}(\mu - 2)\lambda\mu, R^2U^{\mu}, \dots \dots \dots (4),$$

$$0 \equiv -\lambda, RR^{\mu} + \nu, U^2D + \frac{1}{2}(\mu - 2)\lambda\mu, R^2U^{\mu}, \dots \dots \dots (5),$$

the values $\lambda = 0, \mu = 0$ not being admissible. But if we substitute in (4) and (5) the value of $\lambda\mu$ derived from (1), we obtain two simple equations which determine the ratios $\lambda : \mu : \nu$, after which the absolute values are uniquely determined from (1).

* The investigation is given by Salmon, *Lessons, &c.*, p. 444.

[11] is otherwise obvious that if $0 \rightarrow q_0$ be the point equation four-branch pencil, and $0 \rightarrow t_0$ to the tangent plane, $0 \rightarrow q_0 + t_0^2$ will be the equation to a pencil of quadrics having four-branch contact, one of which may be made to pass through an arbitrary point.

Special investigation for $n = 3$.

[12] In the case in which u_0 is a cubic surface, the points in question are clearly the 120 points of contact of the 45 triple tangent planes, namely, the intersection of the 27 lines. This case may be conveniently studied by means of the representation of that surface on a plane. The plane sections of the surface here correspond to a system of cubics having six intersection points; the quadric sections therefore to sextics having nodes at these points. The problem is then to draw a sextic curve having six given nodes and a quadruple point as a limit.

[13] The twenty-seven lines of the cubic are represented by

- the 15 fixed points,
- the 15 lines joining them, and
- the 15 conics each through five of them.

[14] It shall now be proved that the sextic must include two

elements; and that for each pair that meet there is a singly fixed number of sextics.

[15] The sextic cannot be a proper curve; for six nodes and a quadruple point are equivalent to twelve nodes, and a proper sextic can have only ten. Nevertheless we may apply a quadratic transformation to it, where principal points are the quadruple point and two of the nodes. The sextic is thus reduced to a sextic, passing 2, 0, 0 times through the principal points respectively and having four other nodes. But a sextic having six nodes must be made up of a cubic and two straight lines, since, if the node at a principal point is the intersection of the two lines, the original sextic was made up of two lines, passing through two of the six points, and a quartic having nodes

at their intersection and the remaining two points and passing through those four. Let a, b, c, d, e, f be the six points, p the intersection of ab, cd ; then the sextic in this case made up of

lines ab, cd ,

quartic p^2ef^2abcd .

If, however, in the transformed figure the node at a principal point was the intersection of a line with the conic, the original sextic was made up of a line, a conic, and a nodal cubic, viz. if p be the intersection of the line ab and the conic bcd^2ef , the nodal cubic is p^2acdef .

Now we know that we can draw a singly infinite number of quartics with three fixed nodes and four fixed points, or of cubics with a fixed node and two fixed points. Hence in both these cases the sextic includes two representatives of straight lines in the cubic, together with another curve which may be arbitrarily chosen from a pencil.

XXXIII.

ON THE CLASSIFICATION OF LOCI*.

PART I. CURVES.

By a *curve* we mean a continuous one-dimensional aggregate of any sort of elements, and therefore not merely a curve in the ordinary geometrical sense, but also a singly infinite system of curves, surfaces, complexes, &c., such that one condition is sufficient to determine a finite number of them. The elements may be regarded as determined by k coordinates; and then, if these be connected by $k-1$ equations of any order, the curve is either the whole aggregate of common solutions of these equations, or, when this breaks up into algebraically distinct parts, the curve is one of these parts. It is thus convenient to employ still further the language of geometry, and to speak of such a curve as the complete or partial intersection of $k-1$ loci in flat space of k dimensions, or, as we shall sometimes say, in a k -flat. If a certain number, say h , of the equations are linear, it is evidently possible by a linear transformation to make these equations equate h of the coordinates to zero; it is then convenient to leave these coordinates out of consideration altogether, and only to regard the remaining $k-h+1$ equations between $k-h$ coordinates. In this case the curve will, therefore, be regarded as a curve in flat space of $k-h$ dimensions. And, in general, when we speak of a curve as in flat space of k dimensions, we mean that it cannot exist in flat space of $k-1$ dimensions.

The whole aggregate of linear complexes may be regarded

[* From the *Philosophical Transactions of the Royal Society*.—Part II, 1876, pp. 663–681.]

na constituting a space of five dimensions, in which the *special* complexes, or straight lines, constitute a quadric locus. A ruled surface, or scroll, will be thus regarded as a curve lying in a quadric locus in a flat space of five dimensions. If, however, the generators of the scroll all belong to the same linear complex, the scroll must be regarded as a curve lying in a quadric locus in a flat space of four dimensions. And if, further, the scroll has two linear directrices, so that the generators belong to a linear congruence, then the scroll may be regarded as a curve lying on an ordinary quadric surface in three dimensions. Thus, for example, quartic scrolls having two linear directrices correspond either to pinch-quadric curves of deficiency 1 (that is, they are elliptic curves whose coordinates may be expressed as elliptic functions of one variable), or to the curves of deficiency 0 which are the partial intersections of a quadric and a cubic surface (that is, they are unicursal curves).

This view of ruled surfaces is made excellent use of by Veit, "Zur Theorie der windschiefen Flächen," *Math. Annalen*, Vol. VIII, p. 54.

Similar considerations apply to surfaces. By a *surface* we shall mean, in general, a continuous two-dimensional aggregate (which may also be called a *two-space* or a *two-way locus*) of any elements whatever, curves, surfaces, complexes, &c., defined by the whole or a portion of the system of solutions of $k-2$ equations among k coordinates. We shall assume that none of these equations are linear, and then shall speak of the surface as in a flat space of k dimensions. We shall in certain cases go further, and speak of an k -space or k -way locus, *viz.* a locus determined by the whole or an algebraically separate portion of the system of solutions of $k-k$ equations among k coordinates; if none of these equations are linear, the k -way locus will be said to be in k dimensions. The general point of view is that of Professor Cayley, "On the Curves which satisfy given Conditions," *Phil. Trans.*, Vol. LXX (1869), pp. 75-144; the methods of construction are those of Dr Salmon, "Solid Geometry," p. 261.

THEOREM A. *Every proper curve of the n^{th} order is in a flat space of n dimensions or less. For through $n+1$ points of it we can draw a flat space of n dimensions, which must therefore contain the curve, since it meets it in a number of points greater than its order.*

Thus, for example, there is no curve of the second order, in space of any number of dimensions, except a plane conic. If, therefore a system of curves, in a plane or on any surface, is such that two curves of the system can be drawn through an arbitrary point, then the coordinates of a varying curve of the system may be represented by $x_i + 2\theta y_i + \theta^2 z_i$, ($i = 1, 2, 3 \dots k$), and the envelope of the system is, in the case of plane curves, a curve having the equation $\sqrt{U} + \sqrt{V} + \sqrt{W} = 0$, where U, V, W are three curves of the system; in the case of curves on a surface, it is the intersection of the surface with another having an equation of that form*.

* Professor Henrici has kindly written for me the following notes in elucidation of this argument:—

“In the first sentence of the paper it is stated that by a *curve* is meant any one-dimensional aggregate of any sort of elements. The definitions given are algebraical, but the reasoning later on becomes more and more geometrical.

“In this note the connexion between the algebraical definition and the geometrical reasoning will be shown in the case where the elements are plane curves of order n .

“If we suppose a curve given by its equation in point coordinates we may take the coefficients as homogeneous coordinates of the curve.

“As there are $\frac{n(n+3)}{2}$ ratios of these coefficients, it follows that all curves of order n in a plane constitute a $\frac{n(n+3)}{2}$ spread, and this will be a *flat* spread as no relation has been supposed between the coordinates.

“To determine in this spread a k -flat, $k < \frac{n(n+3)}{2}$, we have to assume a sufficient number of equations between the coordinates, or denoting by n_1, n_2, \dots curves of order n we may write down the equation of one element in the k -flat in the form $a_1 n_1 + a_2 n_2 + \dots + a_{k+1} n_{k+1} = 0$, and take the k ratios of the a as the coordinates of a variable curve.

“For $k=2$ we get a *net* as the flat space of two dimensions or as the *plane* in this space, and for $k=1$ a pencil corresponding to the *line*.

“If, on the other hand, we assume in the k -flat $k-1$ equations between the coordinates a , there remains a singly infinite number of curves, that is according

To particularise still further, a system of conics having the characteristic $\mu = 2$ must always have quadruple contact with a quartic curve; and the different species may be enumerated by studying the successive degeneration of the curve, ending with the fundamental system $\mu = 1$, when it breaks up into four straight lines.

So again, there is no quadric scroll, in any number of dimensions, except the ordinary quadric surface which is in flat space of three dimensions.

A curve of the third order must be either the known skew cubic in three dimensions, or a plane cubic. Hence, if a system

to Professor Clifford a *curve* (with curves as elements), according to the usual nomenclature a *series* of curves.

"To determine the order of this curve we have to find the number of elements on it which satisfy a linear relation between the coordinates. In any case the condition that a curve shall pass through a given point gives such a relation, and the number of curves through a point is the order of the series.

"Hence, if we wish to extend a theorem relating to a series in the ordinary sense with points as elements, but in any number of dimensions to a property relating to a series of curves, or if we wish to illustrate in a plane a theorem relating to a curve in more dimensions than three, we have instead of a point on the curve to take a curve in the series, and to replace the order of the curve by the index of the series.

"The theorem that every curve of order two in a line space has upon it three curves in a series of index 3 belongs to a *net*.

"Further, the coordinates of a point on a conic can be represented as expressions of the second degree in a variable parameter, and x, y, z, w, v, \dots where $t=1, 2, 3$. If the coordinates are taken in the plane of the conic, t, t^2 if they are taken in space we have to take t, t^2, t^3, t^4 , and so on for more dimensions. The locus of these points, that is, the conic, is then expressed as an equation of the form

$$\sqrt{U} + \sqrt{V} + \sqrt{W} = 0,$$

where U, V, W are three of the points.

"If we apply this to our series we obtain the results stated in the text, viz., the coordinates of any curve of a series such that two curves pass through a given point are of the form quoted, and the equation of the envelope is of the form

$$\sqrt{U} + \sqrt{V} + \sqrt{W} = 0,$$

U, V, W being three of the curves.

"Similarly, if the series is such that three pass through any point, then the series may be considered as a 'curve' of order three, and the same results as in the text follow at once from the known properties about cubic curves, which are either unicursal (twisted, or plane twist, cubics) or they are plane curves of deficiency one."—January, 1870.

of curves be such that three of them can be drawn through an arbitrary point, the equation of any curve of the system is of one of the two forms—

$$U + 3Vt + 3Wt^2 + Xt^3 = 0,$$

$$U + V \operatorname{sn}^2 u + 2W \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u = 0,$$

where t, u are parameters. Hence it is easy to write down the equations to the envelopes in the two cases, and to enumerate the distinct species.

A cubic scroll must be of the nature of the skew cubic, because it is a curve (with complexes for elements) which is obliged to lie on a quadric locus (that of the special complexes, or straight lines).

THEOREM B. *A curve of order n in flat space of k dimensions (and no less) may be represented, point for point, on a curve of order $n - k + 2$ in a plane.*

The proposition is obvious when $k = 3$. The cone standing on a curve of order n (in ordinary space of three dimensions), and having its vertex at a point of the curve, is of order $n - 1$; if then we cut this cone by a plane, we have the tortuous curve represented, point for point, on a plane curve of order $n - 1$.

Now this process is applicable in general. Starting with an arbitrary point, P , of a curve in any number of dimensions, let us join this point to all the other points of the curve; we shall thus get a cone of order $n - 1$. For any flat locus of $k - 1$ dimensions drawn through the point P must meet the curve in n points, of which P is one; and therefore it must meet the cone in $n - 1$ lines. Hence, if we cut this cone by such a flat ($k - 1$) way locus not passing through P , we shall get a curve of order $n - 1$ in flat space of $k - 1$ dimensions, which is a point-for-point representation of the original curve. By continuing this process we may go on diminishing the order of the curve and the number of dimensions by equal quantities, until we have subtracted $k - 2$ from each; when we are left with a curve of order $n - k + 2$ in a plane.

The reduction may, however, be effected in one step. A

flat ($k=2$) way locus may be drawn through $k-1$ arbitrary points. Suppose it to contain $k-2$ consecutive points of the curve at P , and another variable point, Q , of the curve. Such a locus will meet an arbitrary plane in one point, R . As Q then moves about on the curve, R will trace out on the plane a curve which corresponds to it, point for point. But this curve is of order $n-k+2$, for a flat ($k=1$) way locus, passing through $k-2$ consecutive points of the original curve at P , will meet that curve in $n-k+2$ other points, and therefore will meet also the locus of R in $n-k+2$ points. This locus is, therefore, of order $n-k+2$, as was to be proved.

The fixed points through which the variable ($k=2$) way locus passes need not all be united at P , but they may be any $k-2$ arbitrary points on the curve.

We will now consider some examples of this general.

1. *Universal curve of order n in n -dimensional space.*

A curve of order n in flat space of n dimensions (and no less) is always universal. We may prove this independently by considering a variable in n -flat which passes through $n-1$ fixed points on the curve. Its equation will be of the form $A + tC = 0$, where t is a variable parameter, and it will meet the curve in one other point, which is thus associated with a value of t .

The equations to such a curve may always be written in the form—

$$\begin{aligned} 0 &= A, B, C, \dots, K, L, \dots, \\ &= B, C, D, \dots, L, \dots, \end{aligned} \quad (1),$$

where the $A, B, C, \dots, K, L, \dots$ are linear functions of the coordinates, and the number of columns is $=n$. For the $n \geq 4$ loci are generic coordinates are proportional to rational integral functions of t of the n^{th} order. Solving these $n+1$ equations for $1, t, t^2, \dots, t^n$ we find

$$1, t, t^2, \dots, t^n = A, B, C, \dots, L,$$

which is equivalent to the system written above, where

The more general system of equations—

$$0 = \begin{vmatrix} A, & B \dots K \\ A', & B' \dots K' \end{vmatrix} \dots\dots\dots (2),$$

where the $A \dots K, A' \dots K'$ are linear functions as before, may always and easily be reduced to the former, for they are got by eliminating t from the n equations.

$$\begin{aligned} A + tA' &= 0, \dots\dots\dots (3). \\ B + tB' &= 0, \\ &\vdots \\ K + tK' &= 0. \end{aligned}$$

We may, however, solve these equations for the ratios of the coordinates, which will thus be expressed as rational functions of t of the n^{th} order. Solving these for $1, t, t^2 \dots t^n$ we come back to the previous system.

The equations (3) exhibit the curve as the locus of the intersection of corresponding elements in n projective pencils.

The equation to the $(n-1)$ flat which passes through the n points whose parameters are $t_1, t_2 \dots t_n$, is easily seen to be—

$$0 = \begin{vmatrix} A, & B, & C, & \dots & L \\ 1, & t_1, & t_1^2, & \dots & t_1^n \\ 1, & t_2, & t_2^2, & \dots & t_2^n \\ \vdots & & & & \\ 1, & t_n, & t_n^2, & \dots & t_n^n \end{vmatrix}.$$

But this equation is manifestly divisible by the coefficient of L , which is the product of the differences of all the t . If we write—

$$\begin{aligned} \Sigma_1 &= t_1 + t_2 + t_3 + \dots + t_n, \\ \Sigma_2 &= t_1 t_2 + t_1 t_3 + t_2 t_3 + \dots + t_{n-1} t_n, \\ &\text{etc.} = \text{etc.} \\ \Sigma_n &= t_1 t_2 \dots t_n, \end{aligned}$$

then the equation is

$$0 = L - K\Sigma_1 + \dots \pm B\Sigma_{n-1} \mp A\Sigma_n \dots\dots\dots (4).$$

If we omit the suffixes of the t in this formula we obtain

the equation to the osculant is ± 1 that at the point $t = 0$. Namely (beginning at the other end, at $t = 1$)

$$0 = At^2 + nBt^{n-1} + \frac{1}{2}Ct^{n-2} + \dots + (-1)^n n! Bt + Z, \quad (5)$$

and we see at once that *the class of such a curve is always equal to its order*.

We thus obtain a very useful representation (the *building*) of the points of the n -dimensional space by means of groups of n points on such a minimal curve; namely, each point in the space is represented by the points of contact of the n -osculant $(n+1)$ flats which pass through it. The use of such a representation of ordinary three-dimensional space by means of a skew cubic was pointed out by Dr. Hurw., and the corresponding representation of a plane by means of a curve has been used by M. Darboux (*Sur une classe remarquable de courbes et de surfaces algébriques*, Paris, 1873, Note II., p. 187), and by me ("On the Transformation of Elliptic Functions," *Proc. Lond. Math. Soc.*, Vol. VII. (1875) (ser. 2), pp. 367-379), *supra*. It may be worth while to mention that our extension to all spaces of the theory of the *in* and *osculant* cubic polygon may be obtained by this means.

A curve of this kind determines when a dyadic correspondence exists in the space of n dimensions. Through every point may be drawn a osculant $(n+1)$ flat, and through this a point of contact another $(n+1)$ flat, which shall be called the *polar* of the point. If the point moves along a straight line its polar will pass through a fixed $(n-2)$ flat, the *polar* of the line. And generally if the point lies in any k flat the polar will pass through a fixed $(n-k-1)$ flat.

When $n=2$ we have the ordinary system of polar reciprocals in regard to a plane cubic. When $n=3$ we have that system in regard to a skew cubic which is described by Schläfli, (*Grelle*, Vol. LXV., p. 39). These two systems are typical respectively of the cases in which n is even and odd. When n is even, the relation between the coordinates of two points, which expressed that each lies in the polar of the other, is a symmetric one; consequently those points which lie in their own polars are points on a certain quadratic locus, and the system is merely

that of the poles and polars in regard to this quadric locus upon which the curve lies. The equation to this locus is at once obtained by equating to zero the quadrinvariant of the form $(I, t)^n$ which occurs in the equation (5) of the osculant $(n-1)$ flat, namely, it is

$$0 = AL + nBK + \frac{1}{2}n(n-1)CH + \text{etc.}, \dots \dots \dots (6).$$

To prove this, observe that if in the equation (5) we substitute the coordinates of any point p , the values of t which satisfy the equation are the parameters of the points of contact of the osculant $(n-1)$ flat which passes through the point. If t_1, t_2, \dots, t_n be these values, the equation (4) represents the $(n-1)$ flat which passes through the points of contact, that is to say, the polar of the point. Now if we denote by A', B', \dots the results of substituting the coordinates of the point p in A, B, \dots then we shall have

$$\begin{aligned} A'\Sigma_1 &= nB' \dots \dots \dots (7), \\ A'\Sigma_2 &+ \frac{1}{2}n(n-1)C', \\ &\vdots \\ A'\Sigma_n &= B' \end{aligned}$$

so that, when n is even the equation of the polar is—

$$0 = AL + A'L + n(BK' + B'K) + \frac{1}{2}n(n-1)(CH' + C'H) + \text{etc.}, \dots (8),$$

that is, it is simply the polar of the point in regard to the quadric (6).

It is to be observed that the quadric is completely determined when the curve is given. I reserve the question of the conditions to which the curve is subject when the quadric locus is given, or, say, the discussion of the problem to represent the relation of poles and polars in regard to a quadric locus (in spaces of an even number of dimensions) by means of a unicursal curve.

But when n is odd, the last term of equation (4) is negative, and the equation of the polar is—

$$0 = AL + A'L + n(BK' - B'K) + \frac{1}{2}n(n-1)(CH' - C'H) + \text{etc.}, \dots (9),$$

that is, it is skew symmetrical, and *every point lies upon its polar*. It is convenient to use the term *coflat* for $n+1$ points,

We shall treat these two cases in succession. They are exemplified by the two species of quartics in ordinary tri-dimensional space.

The coordinates of a point on the unicursal curve are proportional to rational integral functions of a parameter t . This representation may be simplified in a manner due to Rosanes, *Crelle*, Vol. LXXV, p. 106. We have n binary quantities of order n ; now these may be linearly combined in n different ways so as to produce a perfect n^{th} power. Hence the original quantities may be expressed each as a linear function of the n^{th} powers of the same n linear quantities. Thus, for example, three binary cubics may be simultaneously reduced to the forms

$$\begin{aligned}aw^3 + b\ v^3 + c\ w^3, \\a'w^3 + b'\ v^3 + c'\ w^3, \\a''w^3 + b''v^3 + c''w^3,\end{aligned}$$

where $u + v + w = 0$ identically. If the x, y, z of a point in a plane are respectively proportional to these cubics, we may, by solving the equations for w^3, v^3, w^3 , obtain three linear functions X, Y, Z of the coordinates, which are respectively proportional to w^3, v^3, w^3 . Transforming them to the new triangle whose sides are $X=0, Y=0, Z=0$ we must have the equation of a unicursal cubic expressed in the form

$$X^3 + Y^3 + Z^3 = 0.$$

It is clear that the lines $X=0, Y=0, Z=0$ are tangents at the three points of inflexion.

In general, let the n quantities be

$$\begin{aligned}a_0 + na_1t + \dots + a_nt^n, \\b_0 + nb_1t + \dots + b_nt^n, \\ \vdots \\h_0 + nh_1t + \dots + h_nt^n,\end{aligned}$$

then the linear quantities $u, v, w \dots$ are the factors of

$$\begin{vmatrix} t^n, & -nt^{n-1}, & \frac{1}{2}n(n-1)t^{n-2}, & \dots & t \\ a_0, & a_1, & a_2, & \dots & a_n \\ b_0, & b_1, & b_2, & \dots & b_n \\ \vdots & \vdots & \vdots & & \vdots \\ h_0, & h_1, & h_2, & \dots & h_n \end{vmatrix}.$$

Since there are $n-2$ identical relations between n linear quantities, the $n-2$ equations of the unicursal curve may be written in the form

$$\begin{vmatrix} X_1^{\frac{1}{n-2}}, X_2^{\frac{1}{n-2}}, \dots, X_n^{\frac{1}{n-2}} \\ a_1, a_2, \dots, a_n \\ \beta_1, \beta_2, \dots, \beta_n \end{vmatrix} = 0;$$

it is evident that the equations $X_1, X_2, \dots, X_n = 0$ represent stationary osculant ($n-2$) flats, that is to say, $(n-2)$ flats which pass through n consecutive points of the curve.

The properties of this curve may be very conveniently studied by regarding it as a projection of the curve considered in the last section. If all the points of that curve be joined to a point O , not situated upon it, the joining lines will form a cone of order n ; and on cutting this cone by an $(n-1)$ flat, we shall obtain the curve now under discussion.

The n points of supersensation, whose existence has just been proved, are then clearly the projections of the points of contact of osculant $(n-1)$ flats to the full skew curve drawn through the point O . It follows that when n is odd, *there n points of supersensation are on the same $(n-2)$ flat*; but when n is even this is not the case, unless the point O lies on the quadric cone associated with the full skew curve, in which case we have a special variety of the projection. Thus the three points of inflexion of a nodal cubic are in one straight line, but a unicursal skew quartic in ordinary space has not in general the property that the points of contact of its four stationary osculating planes are in one plane. The property established above for the full skew curve shows that this will be the case if the four points form an equianharmonic system, or if the quadratic invariant of the quartic which determines them is equal to zero. And generally when n is even, the n points of supersensation will be co-flat if, and only if, the quartic in t which determines them has its quadratic invariant zero.

By using the values of the coordinates of a variable point of the curve expressed in terms of a parameter t , we may obtain an expression of this quadratic invariant and also of its product by

the discriminant in terms of the roots of the quantic. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the values of t which belong to the points of supersculation, and x_1, x_2, \dots, x_n the coordinates of a variable point on the curve. Then we may write

$$x_i = (t - \alpha_i)^n, \quad i = 1, 2, \dots, n,$$

and the coordinates of the point α_i are $(\alpha_i - \alpha_1)^n, (\alpha_i - \alpha_2)^n, \dots, (\alpha_i - \alpha_n)^n$. If for shortness we write (hk) instead of $\alpha_h - \alpha_k$, then the condition that the n points shall be co-flat is

$$0 = \begin{vmatrix} 0 & , & (12)^n, & (13)^n, & \dots & (1n)^n \\ (21)^n, & 0 & , & (23)^n, & \dots & (2n)^n \\ (31)^n, & (32)^n, & 0 & , & \dots & (3n)^n \\ \vdots & \vdots & & \vdots & & \vdots \\ (n1)^n, & (n2)^n, & (n3)^n, & \dots & 0 \end{vmatrix}.$$

This is obviously always satisfied if n is odd, for then the determinant is skew symmetrical, and being of odd order it necessarily vanishes. If, however, n is even, the determinant is a symmetrical function of the roots which vanishes when any two of them are equal; and consequently it must contain as a factor the product of the squares of their differences. Now the determinant is of the order $2n$ in each root, and the discriminant is of order $2(n-1)$; therefore the remaining factor is of order 2 in each root, and being a symmetrical invariant must be a function of the squares of their differences. It can therefore be no other than $\Sigma (\alpha_1 - \alpha_2)^2 (\alpha_1 - \alpha_3)^2 \dots (\alpha_{n-1} - \alpha_n)^2$; this is, to a factor près, equal to the quadrinvariant of the form

$$(t - \alpha_1)(t - \alpha_2) \dots (t - \alpha_n).$$

The equation to the $(n-2)$ flat passing through two consecutive points of the curve at t , and through $n-3$ other points p, q, \dots, n , is clearly

$$0 = \begin{vmatrix} x & dx & p & q & \dots & y \\ 1 & 2 & 3 & 4 & \dots & n \end{vmatrix},$$

where the y are current coordinates, and the determinant is expressed in umbral notation. Writing in this for $x_i, (t - \alpha_i)^n$, and for $dx_i, n(t - \alpha_i)^{n-1} dt$, we may observe that the determinant

$$\begin{vmatrix} (t - \alpha_1)^n & , & (t - \alpha_2)^n \\ (t - \alpha_1)^{n-1} & , & (t - \alpha_2)^{n-1} \end{vmatrix} = (\alpha_2 - \alpha_1) (t - \alpha_1)^{n-1} (t - \alpha_2)^{n-1},$$

so that the equation is of order $2(n-1)$ in t . It then follows that $2(n-1)$ different $(n-2)$ flats may be drawn through $n-2$ arbitrary points to touch the curve; or that the developable traced out by the tangent lines is of the order $2(n-1)$.

Similarly, from the value of the determinant

$$\begin{vmatrix} (t-a_1)^n & (t-a_2)^n & \dots & (t-a_{n-1})^n \\ (t-a_1)^{n-1} & (t-a_2)^{n-1} & \dots & (t-a_{n-1})^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ (t-a_1)^{n-k+1} & (t-a_2)^{n-k+1} & \dots & (t-a_{n-1})^{n-k+1} \end{vmatrix}$$

which is equal to the product of the differences of a_1, a_2, \dots, a_{n-1} multiplied by

$$\{(t-a_1)(t-a_2)\dots(t-a_{n-1})\}^{n-k},$$

we may conclude that the number of $(n-2)$ flats which can be drawn through k consecutive points of the curve and through $n-k$ other arbitrary points is $(k+1)(n-k)$; so that the k -th locus which is traced out by $(k+1)$ flats passing through k consecutive points is of the order $(k+1)(n-k)$. For the equation of an $(n-2)$ flat passing through k consecutive points is easily

$$0 = \begin{vmatrix} x & dx & d^2x & \dots & d^{n-1}x & y & dy & \dots & d^{n-1}y \\ 1 & 2 & 3 & \dots & k & & & & \end{vmatrix}$$

where we must substitute for the a_1, a_2, \dots, a_{n-1} the corresponding powers of $t-a_i$ beginning at the a_i^k . Making k equal to $n-1$ we obtain the equation of the osculant $(n-2)$ flat at any point of the curve; it is

$$0 = \frac{P_1 y_1}{(t-a_1)^n} + \frac{P_2 y_2}{(t-a_2)^n} + \dots + \frac{P_{n-1} y_{n-1}}{(t-a_{n-1})^n}$$

where P_i = product of the differences of all the a 's except a_i . Thus the class of the curve is $2(n-1)$.

3. *Universal curve of order n in $n-k$ dimensions*

The characteristic numbers belonging to this series may at once be obtained by regarding it as a projection of the full skew curve. The number of ranks is $n-k+2$, and their corresponding values of them are respectively $2(n-1)$, $3(n-2)$, \dots , $(k+1)$; the class is $(k+1)(n-k)$; and the number of points of repeated osculation is $(k+2)(n-k-1)$. For example, the universal

curvilinear in three dimensions is of rank $2 \cdot 4 = 8$, and of class $3 \cdot 2 = 6$, and it has $4 \cdot 2 = 8$ superosculant planes.

Convenient forms of the equations may be got by eliminating some of the variables from the equations of the full-skew curve; but care must be taken to select these variables so that the resulting system is sufficiently general.

b. *Elliptic (or bicursal) curve of order n in $n - 1$ dimensions.*

We have proved already that a curve of order n in $n - 1$ dimensions can be represented, point for point, on a plane. If, therefore, it is not unicursal, its coordinates can be expressed in terms of elliptic functions of a single parameter. Now, it follows from the investigations of Clebsch, *Crelle*, Vol. IV. (1864), pp. 210—270, that if n points of the curve are flat, the sum of their parameters will differ from a certain constant by a sum of integer multiples of the two periods of the elliptic function. Let the periods be ω and ω' , then if t_1, t_2, \dots, t_n the parameters of the points,

$$t_1 + t_2 + \dots + t_n = c + a\omega + b\omega',$$

where c is a constant, and a, b are integers. To find the points of superosculation, we must suppose the n points to become coincident, or the t , still satisfying this equation, to become equal. Thus we obtain

$$nt = c + a\omega + b\omega'$$

$$t = \frac{c}{n} + \frac{a}{n}\omega + \frac{b}{n}\omega',$$

values of t , representing distinct points, will be got by giving to the numbers a, b the values $0, 1, \dots, n - 1$ independently. Hence there are n^2 points of superosculation.

Thus a plane cubic has nine inflexional tangents, and a tri-quadric curve has sixteen superosculant planes.

Propositions hold good in general in regard to the grouping of these points, which are analogous to those which relate to inflexions of a cubic. Thus, an $(n - 2)$ flat drawn through 1 of them will always pass either through another besides, or through the tangent line at one of the $n - 1$. This is obvious

from the values already given for the parameters of points of superosculation.

Through any point of the curve can be drawn $(n-1)^2$ osculant $(n-2)$ flats. This is proved in the same way as the preceding proposition, which is, in fact, the projection of it; for if through the given point we draw a cone containing the curve, and cut it by an $(n-2)$ flat, the section will be an elliptic curve of order $n-1$ in $n-2$ dimensions, and the projections of the points whose osculant $(n-2)$ flats pass through the given point will be points of superosculation on the projected curve. Hence, also, the lines joining the given point to the points of contact are grouped in respect of co-flatness in the same way as the points of superosculation in the curve of next lower order.

More generally, through k given points of the curve there can be drawn $(n-k)^2 (n-2)$ flats which have $(n-k)$ pointic contact with the curve. If u_1, u_2, \dots, u_k are the parameters of the k given points, those of the required points are given by

$$u = \frac{1}{n-k} (u_1 + u_2 + \dots + u_k + a\omega + b\omega'),$$

where the integers a, b may take independently the values

$$0, 1, \dots, n-k-1.$$

From these results we may now determine the various ranks and the class of the curve. Suppose that we know the number of $(n-2)$ flats which can be drawn through $n-2$ arbitrary points in space—or, which is the same thing, through an arbitrary $(n-3)$ flat P —to touch a certain curve. Then, if the arbitrary $(n-3)$ flat meets the curve in any point, two of these will coincide at that point. For taking an $(n-4)$ flat in the $(n-3)$ flat, and joining it to all the points of the curve by $(n-3)$ flats, we may cut this figure by a plane or 2 flat. Every $(n-3)$ flat will cut this plane in a single point. The problem is then reduced to drawing tangents from a point (viz., the intersection of P by the plane) to a plane curve; and we know that when this point lies on the curve, two of the tangents coincide at it.

In general, a certain number of $(n-2)$ flats can be drawn

through an arbitrary $(n - k - 1)$ flat to have k -pointic contact with a given curve; this number is, in fact, the $(k - 1)^{\text{th}}$ rank of the curve. If the arbitrary $(n - k - 1)$ flat meets the curve at any point, then k of these $(n - 2)$ flats coincide at that point. For we may project the whole figure from an $(n - k - 2)$ flat lying in the $(n - k - 1)$ flat on to a k flat. The problem is then reduced to drawing $(k - 1)$ flats through a given point to have k -pointic contact with a curve in k dimensions. Now we know, from the example of the full-skew curve, that, when the point lies on the curve, k of these coincide at the point.

If the arbitrary $(n - k - 1)$ flat meet the curve in more points than one, k of the osculants will coincide at each of them; and this result is not affected by the union of the points into one. In particular, if it meet the curve in $n - k$ coincident points, the number of osculants which there coincide is $k(n - k)$.

Applying now these general considerations to the elliptic curve, we find at once that the $(k - 1)^{\text{th}}$ rank of it is nk . For we must add to the number $k(n - k)$, just obtained, the number, k^2 , given by the theory of elliptic functions for the $(n - 2)$ flats drawn through $n - k$ consecutive points of the curve to have k -pointic contact elsewhere. In particular, the class of the curve is $n(n - 1)$; we have observed already that the number of superosculants is n^2 .

Thus, a plane cubic is of order 3, class 6, and has 9 inflexions; a quadri-quadric is of order 4, rank 8, class 12, and has 16 superosculant planes. We learn, moreover, that a quintic curve in four dimensions, when not unicursal, is of first rank 10, second rank 15, class 20, with 25 points of superosculation. Hence a quintic in three dimensions, with five apparent dps., is of rank 10, class 15, and has 20 superosculant planes; this follows by projection from the former case.

A curve of this kind, viz., an elliptic curve of order n in an $(n - 1)$ flat has its coordinates $x_1, x_2 \dots x_n$ determined by the equations

$$x_1, x_2, \dots, x_n = 1, t, t', t^2, tt' \dots$$

(the last term on the right being $t^{(n-1)}t'$ or else t^{2n} , according as n is odd or even), where $t = \operatorname{sn}^2(u + iK')$ and

$$t' = \frac{dt}{du} = 2 \operatorname{sn}(u + iK') \operatorname{cn}(u + iK') \operatorname{dn}(u + iK') \\ = \sqrt{2t(1-t)} (1-t).$$

{If n is even, we may write $t = \operatorname{sn}^2 u$ instead of $\operatorname{sn}^2(u + iK')$. The condition for n points u_1, u_2, \dots, u_n to be co-flat is then

$$u_1 + u_2 + \dots + u_n = 0.$$

See Lindemann; Clebsch's *Lectures on Geometry*, vol. II.

{Theory of Derived Points on an Elliptic (or Bicursal) Curve

Sylvester's theory of derived points on a plane cubic follows:—Starting from any given point on the curve, we construct its *tangential*, or point where the tangent to the original point meets the curve again; similarly we may construct the tangential of the tangential, or second tangential, and so on. By joining any two non-consecutive points of the series, we can find their *residual*, the point where the line joining them meets the curve again. In this way we obtain an infinite group of points derived from (and including) the original point such that the line joining any two of them is either tangent to the curve or passes through a third point of the group. It is to be observed that all points on the curve uniquely determined from the given point by any geometrical process (e.g., the point where the conic of five-pointic contact meets the curve again, the point where cubics of eight-pointic contact meet the curve again, &c.) are included in the group.

The coordinates of any derived point may be expressed rationally in terms of the coordinates of the original point, the order of the functions to which they are proportional is always a square number. Thus the three coordinates of the tangential are proportional to quartic functions of the coordinates of the original. If the square root of the order of the functions be called the order of the derived point, then we have the theorem that when three derived points are in a straight line, the sum of their orders is a multiple of 3.

line, the order of one of them is equal to the sum of the orders of the other two. It is observed that there is no derived point whose order is divisible by 3. By help of this observation it is easy to make out a scheme of the orders; for when we join two points, the order of their residual must be the sum or the difference of the orders of the points, and one or the other of these is always divisible by 3.

This theory is really a geometric representation of the multiplication of elliptic functions. The coordinates x_1, x_2, x_3 of any point on the cubic curve may by proper choice of axes be made proportional to elliptic functions of a parameter u , so that $x_1 : x_2 : x_3 = 1 : \operatorname{sn}^2(u + iK') : \operatorname{sn}(u + iK') \operatorname{cn}(u + iK') \operatorname{dn}(u + iK')$. This being so, if u, v, w are parameters of three points in a straight line, we shall have $u + v + w = 0$. If v be the tangential of u , the three points u, u, v are in a straight line, and $2u + v = 0$, or $v = -2u$. Hence the series of tangentials has for parameters

$$u, -2u, +4u, -8u, \&c.;$$

and in general the parameter of any derived point is of the form nu , where n is a positive or negative integer. The number n , taken positively, coincides with what was called the order of the derived point. For the elliptic functions of nu are of the order n^2 in the elliptic functions of u .

In this way all points of the theory are explained, excepting the fact that no derived point has its order divisible by 3.

Moreover, we see at once that the theory can be extended to other curves of deficiency 1; as, for example, the quadri-quadric curve. Starting with any point on this curve, we may find the point where the osculating plane at that point meets the curve again; then repeat the process with the point so found, and so on. The plane joining any three of these points will meet the curve in another derived point, or else touch it at one of the three points. The plane drawn through one derived point to touch the curve at another derived point will meet it again in a derived point, or touch at the first point, or osculate at the second. The coordinates of any derived point are of the order n^2 in those of the original point, where $\pm n$ may be called the

order of the derived point. In this case the order of no derived point is divisible by 2.

I was desirous of finding a similar representation of the multiplication of hyper-elliptic and Abelian functions; and therefore sought for cases in which derived elements might be found on *curves* (in the sense explained in the beginning of this paper) of deficiency greater than 1. For this purpose I considered scrolls. Taking an arbitrary generator on a quartic scroll having two linear directrices, we may draw a one-sheeted hyperboloid through three consecutive generators at that place; this will meet the quartic scroll in one other generator, which is thus uniquely derived from the given one. Similarly on a quintic scroll contained in a linear complex, the two tractors of four consecutive generators meet the scroll in two other points lying on a generator. And on a sextic scroll not contained in a linear complex, the linear complex having five-line contact at a given generator (containing five consecutive generators) will contain one other generator of the scroll. In these three cases, then, from any three, four, or five generators we may uniquely derive a fourth, fifth, or sixth generator respectively; and the whole theory of derived elements may be applied to the generators of these scrolls.

Unfortunately, however, each of the scrolls considered is at most of deficiency 1, so that we merely get more illustrations of the multiplication of elliptic functions. And it may be shown, in general, that a *curve* on which such a theory of derived points is possible, is at most of deficiency 1.

Suppose that it has no singular points, and that $k - 1$ points on it being given, there is uniquely determined one other point. If this is effected (as in the above examples) by drawing a flat space through the $k - 1$ points, which meets the curve in one other point, then it must be of the order k . Moreover, it must be in a flat space of so many dimensions that the flat of one dimension less is determined by $k - 1$ points. Now a $(k - 2)$ flat is determined by $k - 1$ points; therefore, the curve is in a $(k - 1)$ flat.

Thus the impossibility of extending the theory of derivation

to curves of deficiency greater than unity is equivalent to the proposition that a curve of order k in $k+1$ dimensions is at most of deficiency 1. This failure was the starting point of the present paper.

It remains to explain why, in the group of numbers expressing the orders of the derived points, only certain forms present themselves. Let that number which, with $k+1$ other numbers, makes up zero, be called the *residual* of those numbers; it is, in fact, their sum taken negatively. Then the process of forming the group is to start from unity, and add the residual of every $k+1$ numbers of the group, repetitions being allowed. I say that by this process we shall only get numbers of the form $mk+1$. For let m_1k+1, m_2k+1 , &c. be $k+1$ such numbers, then their residual is $-(m_1+m_2+\dots)k-k+1$, which is a number of the same form. Now as unity, with which we start, is of this form, it follows that all the numbers of the group must be of the form $mk+1$. {January, 1879.}

CURVES OF DEFICIENCY p .

Theorems relating to Abelian Functions.

It will be convenient to put together shortly those propositions relating to the application of Abelian functions to curves which will be wanted in the sequel.

The aggregate of the real and imaginary points on a curve constitutes a two-way spread, or surface, which may be transformed, by stretching without tearing, into the surface of a body with p holes in it. On this surface there are $2p$ distinct closed curves which cannot without breaking be shrunk into a point, namely, one round each hole, and one through each hole. Any other irreducible circuit must be made up of combinations of these.

If any rational function of the coordinates be integrated from one point to another along the curve-spread, the value of the integral will depend upon the path of the integration. If the integral becomes infinite at any points, it may be altered in

value by making the path go round one or more of these; but in any case it may be altered by incorporating into the path any of the $2p$ closed circuits just mentioned. It is found that there are p distinct rational functions of the coordinates whose integrals do not become infinite at any point of the curvspread. Any other integral which is everywhere finite must be a linear combination of these. Of such linear combinations it is convenient to take a certain set as the *normal* set; they are so chosen that each of them, when integrated along a closed path which goes *round* a hole, gives zero for all the holes but one, and πi for that one; thus, the p integrals, which we may call $u_1, u_2, u_3, \dots, u_p$, are associated one by one with the p holes $1, 2, \dots, p$. If they are integrated along a closed curve passing *through* the hole h , let the values be called a_{h1}, a_{h2}, a_{hp} ; then it is found that $a_{hk} = a_{kh}$, or the integral of u_k through the hole h is equal to the integral of u_h through the hole k .

If we now take all the integrals from a point x to a point y along the same path, and if u_1, u_2, \dots, u_p are the set of values for one such path, and U_1, U_2, \dots, U_p for another path, then we must have

$$\begin{aligned} U_1 &= u_1 + m_1 \pi i + q_1 a_{11} + q_2 a_{12} + \dots + q_p a_{1p}, \\ U_2 &= u_2 + m_2 \pi i + q_2 a_{21} + q_3 a_{22} + \dots + q_p a_{2p}, \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ U_p &= u_p + m_p \pi i + q_1 a_{p1} + q_2 a_{p2} + \dots + q_p a_{pp}, \end{aligned}$$

where the numbers m, q are integers; namely, m_h is the additional number of times the new path has gone round the hole h , and q_h is the additional number of times it has gone through that hole. We shall write these equations thus

$$U_1, U_2, \dots, U_p \equiv u_1, u_2, \dots, u_p \pmod{\pi i, a},$$

and shall say that the quantities U are *congruent* to the quantities u in respect of the periods $\pi i, a$.

Suppose now that the curve has no actual nodes, and that a locus of any order intersects it in the points $\omega_1, \omega_2, \dots, \omega_m$. Then, if another locus of the same order intersects it in the points y_1, y_2, \dots, y_m , and we take any one of the integrals, say u ,

from x_1 to y_1 , from x_2 to y_2 , ..., from x_m to y_m , the sum of these results will be congruent to zero. That is to say

$$\sum \int_x^y du_h \equiv 0.$$

Here the \sum refers to the suffixes of the x and y , not to h . There are p such equations. This is Abel's Theorem.

When the curve is in a k -flat and of the order n , we shall use this theorem chiefly for its n intersections with a $(k-1)$ flat. If we regard the lower limits of the integrals (the point x) as fixed, the integrals for any point y may be regarded as parameters belonging to that point, and then Abel's Theorem gives us p equations between the parameters of n points which lie on a $(k-1)$ flat. The truth of these equations is *necessary* to the points lying on a $(k-1)$ flat, but it may not be sufficient. Thus in a bicircular quartic curve, $p=1$, we have one equation to express that four points are in a straight line, and if the points are collinear the equation is true. But it does not follow from the equation that the points are collinear; in fact, the equation holds equally good if the points are in a circle.

If the sums of the parameters of p points are given, that is, if we have the p equations

$$\left(\int^{x_1} + \int^{x_2} + \dots + \int^{x_p} \right) du_h \equiv v_h \quad [h=1, 2, \dots, p],$$

the v_h being arbitrary constants, and the lower limits of the integrals being supposed constant; then the upper limits x_1, x_2, \dots, x_p may be expressed in terms of the quantities v_h —namely, they are the roots of an equation of degree p whose coefficients are products of \mathfrak{A} -functions of the v . If

$$\phi(m_1, m_2, \dots, m_p) \equiv \sum m_h u_{hk} + 2 \sum m_h v_h,$$

then

$$\mathfrak{A}(v_1, v_2, \dots, v_p) \equiv \sum v_h \phi(m),$$

the \sum indicating that each of the p integers m_1, m_2, \dots, m_p is to take all integral values positive and negative. When the lower limits are so chosen that the sum of the parameters is zero for the complete intersection by any locus, this \mathfrak{A} -function has remarkable properties. If we sum each of the parameters

for any $p+1$ points on the curve, the W -function whose arguments are these sums will vanish. That is it

$$v_k = \left(\int^{x_1} + \int^{x_2} + \dots + \int^{x_{p+1}} \right) dx_1,$$

then $W(p_1, \dots, p_{p+1}) = 0$. If these sums are taken for any $p+2$ points, not only will the W vanish, but also its differential coefficient in regard to any one of the points. And generally, if we take for the x the sums of the parameters for $p+r$ points, the W and its first $r-1$ differential coefficients in regard to any of the points will vanish.

Conversely, if the p quantities x are such that $W(p)$ and its first $r-1$ differential coefficients vanish, then it is possible to find $p+r$ points x such that

$$\left(\int^{x_1} + \int^{x_2} + \dots + \int^{x_{p+r}} \right) dx_1 = 0.$$

Although here the number of equations is greater by r than the number of unknown quantities, yet it is possible to satisfy them all in virtue of the relations existing between them.

Relation between the Order and Deficiency of a Curve.

We shall now apply these theorems to the study of curves existing in k dimensional, of the order n and deficiency p . A $(k+1)$ that cuts such a curve in n points, such that the sum of each of the p parameters for the n points, is zero. But a $(k+1)$ that is determined by k points, so that, k arbitrary points being chosen on the curve, it is always possible to find $n-k$ other points, so that the sum of each parameter for the whole n points shall be zero. Let then $x_1, \dots, x_k, \dots, x_n$ be the sums of the parameters for the given k points; then to find the remaining $n-k$ points we have the p equations

$$\left(\int^{x_1} + \int^{x_2} + \dots + \int^{x_n} \right) dx_1 = 0.$$

If p is not greater than $n-k$, we know that these equations can be solved, although the solution may be indeterminate. But if $p > n-k$, the equations cannot be solved unless certain

conditions are satisfied by the v . Let $p = n + k = r$, then r conditions must be satisfied; namely the v must be sums of the parameters of not more than $p - r$ (or $n - k$) points. But they are sums of the parameters of k points; therefore k is not greater than $n - k$, or $2k$ is not greater than n . We have proved then that

If $p = n - k$, then $2k$ is not greater than n .

Conversely, if $k > \frac{1}{2}n$, p is at most equal to $n - k$.

We may also state the proposition in this way. *A curve of order n and deficiency p , not greater than $\frac{1}{2}n$, can at most exist in $n - p$ dimensions.*

[It appears, therefore, that the theorems at the beginning of the paper may be extended, and that in n dimensions we have the curve of order n which is unicursal, the curve of order $n+1$, and deficiency at most 1, of order $n+2$, and deficiency at most 2, and so on till we come to the order $2n$, which is the first case of exception, and may have deficiency $n+1$. This curve is the natural geometric representation of the general Abelian functions, its multiple tangent flats playing the same part as the double tangents of the quartic curve in Riemann's beautiful paper on the case $p = 3$. H. Weber has noticed that in four dimensions this curve is the complete intersection of three quadric loci. January, 1879.]

(ABSTRACT OF XXXIII)*.

"A CURVE," is to be understood to mean a continuous one-dimensional aggregate of any sort of elements, and therefore not merely a curve in the ordinary geometrical sense, but also a singly infinite system of curves, surfaces, complexes, &c., such that one condition is sufficient to determine a finite number of them. The elements may be regarded as determined by k coordinates; and if these be connected by $k - 1$ equations

* [From the *Proceedings of the Royal Society*, No. 167, 1878. This abstract, I believe, contains the communication on the same subject made to the London Mathematical Society, Feb. 6th, 1877.]

If we consider a curve of the order n and deficiency p , existing in k dimensions, a $(k-1)$ flat cuts such a curve in n points, such that the sum of each of the p parameters (Abel's theorem gives p equations between the parameters of n points which lie on a $(k-1)$ flat), for these n points is zero. And we obtain the theorem.

VII. A curve of order n and deficiency p , not greater than $\frac{1}{2}n$, can at most exist in $n-p$ dimensions.

Under the name *sphere* must be included as particular cases points and planes. The power of a point in regard to a sphere is the squared tangent to it; the power of two points is the squared distance between them. The powers of spheres in regard to planes are infinite quantities which are proportional to the distances of their centres from the planes. The powers of planes in regard to one another are infinities of the second order which are proportional to the cosines of the angles of intersection.

The symbol (PQ) is linear in regard to the coefficients of the two spheres involved, and therefore distributive. That is to say, we have

$$(P, \lambda Q + \mu R) = \lambda (PQ) + \mu (PR).$$

It follows directly from this that if we form a determinant whose constituents are such symbols, of the form

$$(ABCDE) = \begin{vmatrix} (AP), & (AQ), & (AR), & (AS), & (AT) \\ (BP), & \text{\&c.} & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ (CP), & & & & \\ (DP), & & & & \\ (EP), & & & & (ET) \end{vmatrix}$$

(the letters denoting any ten spheres) it must be equal to the product of the determinants $(ABCDE)$, $(PQRST)$, multiplied by a linear factor. The ordinary theorem for multiplication of determinants shows that we have in fact

$$\frac{(ABCDE)}{(PQRST)} = s (ABCDE) (PQRST) \dots\dots\dots (3).$$

If we divide the rows of the determinant by (AO) , (BO) , etc. and the columns by (PO) , (QO) , etc., its constituents become actually the powers of the spheres involved. Now, for the right-hand side of the equation (3), we have

$$\frac{(ABCDE)}{(AO)(BO)(CO)(DO)(EO)} = \frac{(ABCD, E)}{(ABCD, O)(EO)} \frac{(ABCD, O)}{(ABCD, O)(EO)} (AO)(BO)(CO)(DO)$$

The quantity

$$\frac{(ABCD O)}{(AO)(BO)(CO)(DO)},$$

is easily seen to be 48 times the volume of the tetrahedron whose vertices are at the centres of the four spheres. The other factor

$$\frac{(ABCDE)}{(ABCD O)(EO)}$$

is clearly the power of the sphere E in regard to a sphere cutting A, B, C, D orthogonally. We see then that *given any five spheres, the product of the tetrahedron whose vertices are at the centres of any four into the power in regard to the fifth of a sphere cutting them orthogonally is a symmetrical function of the five spheres.* I shall call this the *apospheric function* of the five spheres; it vanishes when they are all orthotomic of the same sphere.

This being so, the theorem (3) informs us that *the determinant formed with the powers of two sets of five spheres is equal to 6144 (= $8 \times 48 \times 48$) times the product of their apospheric functions.*

The corresponding determinant formed with two sets of six spheres vanishes identically; or we have always

$$\begin{pmatrix} ABCDEF \\ PQRSTU \end{pmatrix} \equiv 0 \dots\dots\dots (4).$$

Power-Coordinates.

DEF. Coordinates of a sphere are quantities proportional to certain multiples of its powers in respect of five fixed spheres.

It is convenient for many purposes to take as these multiples the reciprocals of the radii of the fixed spheres. Thus if the coordinates of a sphere X are $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$, its powers in respect of the fundamental spheres are $\alpha_1 r_1, \alpha_2 r_2$, etc., where

*XXXV

A FRAGMENT ON MATRICES.

[In explanation of the following paper observe that the matrix $\phi_i = (a_i, b_i, c_i)$ substituted for the point having the coordinates (a', b', c')

coordinates (a, β, γ) , the point having the coordinates

$$(\phi\phi a, \beta, \gamma) = (a + b\beta + c\gamma, a' + b'\beta + c'\gamma, a'' + b''\beta + c''\gamma);$$

in particular for the points i, j, k coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively, it substitutes the points whose coordinates are (a, a', a'') , (b, b', b'') , (c, c', c'') respectively. An indeterminate matrix or more definitely, a matrix indeterminate in the first degree is a matrix the determinant of which vanishes, but for which the first minors do not all of them vanish; such a matrix substituted for a given point a point in a determinate line called in this paper, the axis—but for one position of the given point, called the null-point, the position of the substituted point is altogether arbitrary. A matrix indeterminate in the second degree is a matrix for which all the first minors vanish, in what is the same thing, one for which the second and third rows are mere multiples of the first row. (1.)

AN indeterminate matrix ϕ_i substituted for the fundamental points i, j, k three collinear points a_{ij}, a_{jk}, a_{ki} . As these must satisfy an identical equation $\lambda_{ij}x_i + \lambda_{jk}x_j + \lambda_{ki}x_k = 0$, there is a point $I_j = (I_{ij} + I_{jk}) + I_{ki}k$ which is reduced by ϕ_i to zero. But forasmuch as upon the line ij there is a (1, 1) correspondence (viz. to

every point $x_1(i+x_2j+x_3k)$ there corresponds one point Σx_i , to every point Σx_i on the line corresponds the straight line

$$(x_1 + x_2)(i + (x_1 + x_2)j + x_3 + x_2k)$$

which has one point on the line o_1 , this must be two united points, say μ, v . Thus if λ represents the null-point and λ, μ, v are now taken as fundamental points, the matrix ϕ is reduced to the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}.$$

This clearly corresponds to that mode of projection in which the centre of projection is taken on one of the two planes but not on the other.

Now consider two indeterminate matrices ϕ_1, ϕ_2 , and let λ, μ be their null-points, μ, v and σ, τ the respective united points. The product $\phi_1\phi_2$ will have μ for a united point, and σ for axis; while $\phi_2\phi_1$ will have λ for null-point, and σ for axis. In general therefore the product will still be indeterminate in the first degree only. But if λ is on the line o_1 , i.e. if the null-point of ϕ_1 is on the axis of ϕ_2 , then there is a constant line through μ any point on which ϕ_2 will convert into λ , any point on which, therefore, $\phi_2\phi_1$ will destroy. Hence in this case $\phi_2\phi_1$ is indeterminate in the second degree.

Let

$$\phi_1 = \begin{pmatrix} \phi_{111} & \phi_{112} & \phi_{113} \\ \phi_{121} & \phi_{122} & \phi_{123} \\ \phi_{131} & \phi_{132} & \phi_{133} \end{pmatrix}$$

$$\text{i.e.} \quad \phi(x_1i + x_2j + x_3k) = (\phi_{111}x_1 + \phi_{121}x_2 + \phi_{131}x_3)i + \dots$$

may

$$\phi_2(i) = \Sigma \phi_{21k} \phi_{1k1} i_{21} x_k$$

Then if ϕ_1 is indeterminate, $\phi_{111} = 0$, and the products $\phi_{112}, \phi_{113}, \phi_{211}$ are all on the axis. The axis is the relation

$$\begin{pmatrix} \phi_{112} & \phi_{113} & \phi_{211} \\ \phi_{122} & \phi_{123} & \phi_{212} \end{pmatrix}.$$

and the null-point is

$$\begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix}.$$

If then the null-point of ϕ is on the axis of ψ , we must have

$$0 = \begin{vmatrix} \phi_{11}\psi_{11} + \phi_{12}\psi_{21} + \phi_{13}\psi_{31}, & \phi_{11}\psi_{12} + \phi_{12}\psi_{22} + \phi_{13}\psi_{32} \\ \phi_{21}\psi_{11} + \phi_{22}\psi_{21} + \phi_{23}\psi_{31}, & \phi_{21}\psi_{12} + \phi_{22}\psi_{22} + \phi_{23}\psi_{32} \end{vmatrix},$$

i.e. the first minors of $|\phi| \times |\psi'|$ must vanish, the rows of $|\phi|$ being multiplied into the columns of $|\psi'|$. If the null-point of ψ is on the axis of ϕ , the first minors of $|\phi'| \times |\psi|$ must vanish, the columns of $|\phi'|$ being multiplied into the rows of $|\psi|$. Hence it is possible for $\phi\psi$ to be indeterminate in the second degree, while $\psi\phi$ is so only in the first. But $\phi\psi$ and $\psi\phi$ may both be doubly indeterminate without being equivalent. If however the null-point of each matrix is a united point on the axis of the other, and the remaining united points coincide at the intersection of the axes, then ϕ , ψ are commutative and their product is indeterminate in the second order.

In general two matrices are commutative when and only when they have the same united points. For let λ be a united point of ψ and not of ϕ , and suppose ϕ alters it to λ' . Then $\phi\psi$ will change λ to λ' , but $\psi\phi$ will not do this unless λ' be another united point of ψ . Now there can be no cycle of changes, because the powers of ϕ have the same united points as ϕ . Hence, &c. Q.E.D.†

From this Cayley's theorem follows, because any set of n quantities $g_1 \dots g_n$ can be expressed linearly in terms of n other sets, $1, \phi, \phi^2 \dots \phi^{n-1}$. Consequently any matrix commutative with ϕ is of the form

$$a + b\phi + \dots + k\phi^{n-1}.$$

To form a matrix which shall have three given points for united points. Let the points be l, m, n ; viz. $l = l_1 i_1 + l_2 i_2 + l_3 i_3$.

* [Observe that the rows of the unaccented symbol are multiplied into the columns of the accented symbol, the accents being used only for the purpose of marking this distinction. O.]

† [The argument seems to be, if ϕ and ψ are commutative, then $\phi\lambda = \lambda'\psi$ a united point of ψ , and in like manner $\phi^2\lambda, \phi^3\lambda$, &c. are all of them united points of ψ , and being thus finite in number, there is some power $\phi^m\lambda$ which is $=\lambda$: viz. λ is a united point of ϕ^m , and therefore a united point of ϕ . O.]

$m \in \mathbb{N}$, $n \in \mathbb{N}$. Then $(lmm)l_1 = (m_2n_1)l + m_2l_1m + (l_2n_1)n$, etc. Thus $(lmm)\phi_1 = \lambda(m_2n_1)l + \mu(m_2l_1)m + \nu(l_2n_1)n$ and

$$\begin{aligned}(lmm)\phi(x) &= (lmm)\lambda l + (lmm)\mu m + (lmm)\nu n \\ &= \{ (lmm)\lambda l + (lmm)\mu m_1 + (lmm)m_2l_1 \} x_1 + \dots\end{aligned}$$

$$\therefore (lmm)\phi_1 = \lambda l_1 \frac{(lmm)}{l_1} + \mu m_1 \frac{(lmm)}{m_1} + \nu n_1 \frac{(lmm)}{n_1}.$$

$$\phi = \begin{pmatrix} \lambda_1 & M_1 & N_1 \\ \lambda_2 & M_2 & N_2 \\ \lambda_3 & M_3 & N_3 \end{pmatrix} \begin{pmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$

[Some lines, inserted apparently by way of verification, are omitted. The values obtained for the matrix ϕ is incorrect; we ought to have

$$\begin{aligned}(\phi\bar{0} \mid l_1, l_2, l_3) &= \lambda(l_1, l_2, l_3), \\ (\phi\bar{0} \mid m_1, m_2, m_3) &= \mu(m_1, m_2, m_3), \\ (\phi\bar{0} \mid n_1, n_2, n_3) &= \nu(n_1, n_2, n_3)\end{aligned}$$

where ϕ is the required matrix

$$\begin{pmatrix} \lambda & \mu & \nu \\ \mu & \lambda & \nu \\ \nu & \lambda & \mu \end{pmatrix}$$

and (λ, μ, ν) are arbitrary constants.

There are thus nine linear equations for the determination of the nine coefficients of ϕ , and the solution is obtained in the formula

$$\phi = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix} \cdot \begin{pmatrix} \lambda & \mu & \nu \\ \mu & \lambda & \nu \\ \nu & \lambda & \mu \end{pmatrix} = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix}$$

In fact starting from this formula, and substituting for ϕ its signification we have

$$\begin{pmatrix} a & h & e \\ a' & h' & e' \\ a'' & h'' & e'' \end{pmatrix} \cdot \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix} = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix} \cdot \begin{pmatrix} \lambda & \mu & \nu \\ \mu & \lambda & \nu \\ \nu & \lambda & \mu \end{pmatrix}$$

ON TRICURVEAR SURFACES.

The focal properties of aspherical curves of order $2n$ are most easily studied by inverting their plane in regard to an external point; the plane then becomes a sphere, and the inverse curve is the general intersection of this sphere by a surface of the n^{th} order, so that the problem is reduced to the study of the intersection of a spheroid with a surface of order n in its relations with a certain plane section of the quadric. The method is equivalent to either of the two following:

(1) The method of representation of a uniserial surface upon a plane used by Clebsch and Cremona. Given a quadric surface (sphere), its plane sections are uniserial curves involving three variables, any two of which intersect in two points; they may therefore be fitly represented by plane conics passing through two fixed points (circles). The two fixed points represent two generators on the surface, and the line joining them represents the point of intersection of the generators; these constitute the exceptional system in the surface and in the plane. A section of the quadric surface by a surface of order n is represented by a plane curve order $2n$ passing n times through each of the fixed points; because the curve on the surface is not n times by each of the special generators; and conversely a curve of order $2n$ in the plane is in general represented by a curve of order $4n$ on the surface, but when the plane curve passes n times through each of the fixed points, the two special generators are each thrown off n times from the curve on the surface; this makes a reduction $2n$ in the order, and there remains only a curve of order $2n$ meeting every generator in n points, that is

to say, the intersection of the quadric by an n^{th} surface. A tricircular sextic is in this way regarded as the representation on a plane of a quadicubic curve.

(2) The method of circular co-ordinates. If we write

$$X = x^2 + y^2 + 2a_1x + 2b_1y + c_1,$$

$$Y = x^2 + y^2 + 2a_2x + 2b_2y + c_2,$$

$$Z = x^2 + y^2 + 2a_3x + 2b_3y + c_3,$$

$$W = x^2 + y^2 + 2a_4x + 2b_4y + c_4,$$

then the equation of every circle can be put into the form $lX + mY + nZ + sW = 0$. The quantities $XYZW$ satisfy identically a homogeneous equation of the second order, which may be written as follows, (XY) meaning the cosine of the angle of intersection of the circles X and Y : viz.,

$$\begin{vmatrix} 1, & (XY), & (XZ), & (XW), & Xr_1^{-1} \\ (YX), & 1, & (YZ), & (YW), & Yr_2^{-1} \\ (ZX), & (ZY), & 1, & (ZW), & Zr_3^{-1} \\ (WX), & (WY), & (WZ), & 1, & Wr_4^{-1} \\ Xr_1^{-1}, & Yr_2^{-1}, & Zr_3^{-1}, & Wr_4^{-1}, & 0 \end{vmatrix} = 0$$

where r_1, r_2, r_3, r_4 are the radii of the four circles. If they cut orthogonally this reduces to

$$X^2r_1^{-2} + Y^2r_2^{-2} + Z^2r_3^{-2} + W^2r_4^{-2} = 0.$$

In any case we may write this relation $\Omega_2 = 0$. An n circular curve of order $2n$ may always be represented by an equation of order n in $XYZW$, say $U_n = 0$, and this in an infinity of ways. For if $U_n = 0$ represent any curve, $U_n + K_{n-2}\Omega_2 = 0$ will represent the same curve, whatever quantity of order $n-2$ in X, Y, Z, W is represented by K_{n-2} . But regarding $U_n = 0$, $\Omega_2 = 0$, since they contain four variables, as equations to surfaces, this amounts to saying that we have only to pay attention to their curve of intersection. So that this method is merely a translation into analytical language of the former one. It will however exhibit the results in a convenient form; for if we succeed in drawing through the curve of intersection some surface of the n^{th} order whose equation is of a simple form, this amounts to finding an equation for the plane curve of order

deficiency 4. The same thing appears by considering the cone standing on the curve and converging to a point of it. This is of the fifth order and has two nodal lines, namely the generators of the quadric through the point. The tangent cone, then, drawn through an arbitrary point of space, being of the sixth order with six nodal lines and no cusps, is of class $18 = 30 - 12$; and this is also the class of the tricircular sextic. In the latter case the number of tangents which can be drawn to the curve from one of the triple points is $12 = 18 - 6$, 2 tangents being swallowed up by each of the 3 branches through the triple point. Consequently the number of foci is 144, of which only 42 are real.

Returning to the quadricubic curve, we see that the tangent cone from an arbitrary point must have $36 = 6 \times 12 = 6 \times 6$ inflexions, and $102 = \frac{1}{2} (18 \times 17 - 6 \times 36 - 6)$ double tangents. The former number is the class of the torse traced out by tangents to the curve, and the latter is the class of the torse enveloped by planes doubly tangent. In the reciprocal problem, this latter number is the order of the nodal curve of the torse generated by common tangent planes to a quadric and a surface of the third class; that is to say, *the focal curve of a surface of the third class is of order 102*.

*XXXVII.

ON BESSEL'S FUNCTIONS.

1. I CONSIDER the function

$$f(x) = 1 + x + \frac{x^2}{(1!)^2} + \dots + \frac{x^k}{(k!)^2} + \dots \text{ ad inf.};$$

the series is evidently one-valued and convergent for all values real or complex of the variable x .

2. The n^{th} derived function of $f(x)$ is

$$f_n(x) = \frac{1}{\Pi n} + \frac{x}{\Pi(n+1)} + \frac{x^2}{\Pi(n+2) \cdot \Pi 2} + \dots + \frac{x^k}{\Pi(n+k) \cdot \Pi k} + \dots \text{ ad inf.};$$

and if we integrate n times from 0 to x we obtain

$$f_n(x) = \frac{x^n}{\Pi n} + \frac{x^{n+1}}{\Pi(n+1)^2} + \frac{x^{n+2}}{\Pi(n+2) \cdot \Pi 2} + \dots + \frac{x^{n+k}}{\Pi(n+k) \cdot \Pi k} + \dots \text{ ad inf.} f'_n(x).$$

3. Hence we derive the differential equations

$$\begin{aligned} \partial_x^{n+1} \{x^n f'_n(x)\} &= x^n f'_n(x), \\ \partial_x \{x^{n+1} \partial_x f_n(x)\} &= x^n f'_n(x). \end{aligned}$$

4. We may generalize the definitions of (2) so as to obtain a value of $f_n(x)$ for all values of n , if we take Πn to mean $\Gamma(n+1)$, according to Cauchy's notation. The differential equations of (3) will hold good in the general case.

5. By multiplying together the series for e^{xy} and $e^{\frac{x}{y}}$, as follows :

$$\begin{aligned}
 & 1 + xy + \frac{x^2 y^2}{1!2} + \frac{x^3 y^3}{1!3} + \frac{x^4 y^4}{1!4} + \dots \\
 & + \\
 & xy^{-1} + x^2 + \frac{x^3 y}{1!2} + \frac{x^4 y^2}{1!3} + \frac{x^5 y^3}{1!4} + \dots \\
 & + \\
 & \frac{x^2 y^{-2}}{1!2} + \frac{x^3 y^{-1}}{1!2} + \frac{x^4}{(1!2)^2} + \frac{x^5 y}{1!3 \cdot 1!2} + \frac{x^6 y^2}{1!4 \cdot 1!2} + \dots \\
 & + \\
 & \frac{x^3 y^{-3}}{1!3} + \frac{x^4 y^{-2}}{1!3} + \frac{x^5 y^{-1}}{1!3 \cdot 1!2} + \frac{x^6}{(1!3)^2} + \frac{x^7 y}{1!4 \cdot 1!3} + \dots \\
 & \frac{x^4 y^{-4}}{1!4} + \frac{x^5 y^{-3}}{1!4} + \frac{x^6 y^{-2}}{1!4 \cdot 1!2} + \frac{x^7 y^{-1}}{1!4 \cdot 1!3} + \frac{x^8}{(1!4)^2} + \dots
 \end{aligned}$$

we find

$$\begin{aligned}
 e^{x(y + \frac{1}{y})} &= f(x^2) + \left(y + \frac{1}{y}\right) \cdot x f_1(x^2) + \left(y^2 + \frac{1}{y^2}\right) \cdot x^2 f_2(x^2) + \dots \\
 &+ \left(y^k + \frac{1}{y^k}\right) \cdot x^k f_k(x^2) + \dots
 \end{aligned}$$

6. In this formula write $-ix$ for x , and iy for y ; then

$$\begin{aligned}
 e^{x(y - \frac{1}{y})} &= f(-x^2) + \left(y - \frac{1}{y}\right) \cdot x f_1(-x^2) + \left(y^2 - \frac{1}{y^2}\right) \cdot x^2 f_2(-x^2) - \dots \\
 &+ \left\{y^k + \left(-\frac{1}{y}\right)^k\right\} x^k \cdot f_k(-x^2) + \dots
 \end{aligned}$$

a result which may also be obtained by direct multiplication.

7. In the equation of (5) and (6), put $y = e^{i\phi}$; thus we obtain

$$\begin{aligned}
 e^{2ix \cos \phi} &= f(x^2) + 2 \cos \phi \cdot x f_1(x^2) + 2 \cos 2\phi \cdot x^2 f_2(x^2) + \dots \\
 &+ 2 \cos k\phi \cdot x^k f_k(x^2) + \dots \\
 e^{2ix \sin \phi} &= f(-x^2) + 2i \sin \phi \cdot x f_1(-x^2) + 2 \cos 2\phi x^2 f_2(-x^2) + \dots \\
 &+ 2i \sin (2k-1)\phi \cdot x^{2k-1} f_{2k-1}(-x^2) + 2 \cos 2k\phi \cdot x^{2k} f_{2k}(-x^2) + \dots
 \end{aligned}$$

and separating real and imaginary parts of the field equation,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$\begin{aligned} \sin(2\pi \sin \theta_0^0) &= 2 \sin \theta_0^0 \cos \theta_0^0 = 2 \sin \theta_0^0 \sqrt{1 - \sin^2 \theta_0^0} \\ &= 2 \sin \theta_0^0 \sqrt{1 - \frac{1}{4} \sin^2 \theta_0^0} = 2 \sin \theta_0^0 \sqrt{\frac{3}{4} + \frac{1}{4} \cos^2 \theta_0^0} \\ &= 2 \sin \theta_0^0 \sqrt{\frac{3}{4} + \frac{1}{4} \cos^2 \theta_0^0} = 2 \sin \theta_0^0 \sqrt{\frac{3}{4} + \frac{1}{4} \cos^2 \theta_0^0} \end{aligned}$$

[illegible]

[illegible]

and by addition of the last two of these

$$\int_{\partial M} \psi^*(\omega) = (-1)^{n-1} (\det D\psi) \int_M \omega$$

[illegible]

* XXXVIII.

ON GROUPS OF PERIODIC FUNCTIONS.

I. INTRODUCTION.

THE simplest periodic functions are the circular and elliptic functions; these may be regarded as built up out of exponential and θ -functions, and the latter again as built up of circular functions. Let us write *

$$\kappa u = a^u + a^{-u},$$

then $\mathfrak{N} u = \sum_0^\infty q^{nu} \kappa(mu) = \sum_{n=1}^\infty q^n (n^2 Q + nu),$

where $q = a^2.$

In the same way if we write

$$\kappa(u, v) = a^{uv} (v + u) + a^{-uv} (v - u),$$

we may form the function

$$\mathfrak{N}(u, v) = \sum_0^\infty q^{mu} \kappa(mu, v) = \sum_{n=1}^\infty \sum_{m=1}^\infty q^n (n^2 Q + n^2 Q + 2nmv + nu + mv,$$

where it is to be understood that $\kappa(mu, v)$ means $a^{muv} (v + mu) + a^{-muv} (v - mu).$

By proceeding in this manner, we may form an ascending series of functions κ, \mathfrak{N} , of an increasing number of variables. Let m, m_1, \dots, m_p be p whole numbers, and let

$$\phi m = \sum Q_1 m_1^2 + 2 \sum Q_2 m_1 m_2;$$

then we may form a function \mathfrak{N} of the p arguments u, u_1, \dots, u_p by adding together all the exponentials of the form

$$e^{\phi(m) + 2 \sum mu_1}$$

* Rosenhain's Method, *Mem. Die. Sav.* t. xi.

venient form considered by Göpel, in which I now replace the letters v by the u ;

$$G(u_1 \dots u_p) = \Sigma^p e^{|\Sigma_i (u_i + 2\Sigma_j m_j B_{ij})^2|}.$$

It is clear that this is unaltered if we simultaneously increase the arguments $u_1 \dots u_p$ by the quantities $2B_{1j} \dots 2B_{pj}$, where j may be any of the numbers 1, 2, p . There is also a set of quantities A in regard to which the function is quasi-periodic, or reproduces itself with an exponential factor. In regard to those quantities the quotient of G by $e^{|\Sigma u^2|}$ is actually periodic; viz., if we write

$$G(u_1 \dots u_p) = e^{|\Sigma u^2|} \Theta(u_1 \dots u_p),$$

then we shall have

$$\Theta(u_1 + 2A_{1j}, \dots, u_p + 2A_{pj}) = \Theta(u_1 \dots u_p),$$

provided that

$$4\Sigma_j A_{ij} B_{ij} = \pi i, \quad 4\Sigma_i A_{ij} B_{ij} = 0,$$

and these equations suffice to determine the quantities A . We have then

$$\begin{aligned} G(u_1 + 2A_{1j}, \dots, u_p + 2A_{pj}) &= e^{|\Sigma_i (u_i + 2A_{ij})^2|} \Theta(u_1 \dots u_p) \\ &= e^{|\Sigma_i 4A_{ij} u_j^2 + 4\Sigma_i u_i A_{ij}|} G(u_1 \dots u_p). \end{aligned}$$

We shall now leave out in all cases the first suffix, writing for instance $G(u + A_j + B_k)$ instead of

$$G(u_1 + A_{1j} + B_{1k}, \dots, u_p + A_{pj} + B_{pk}),$$

and so in similar cases.

II.

A linear function of the quantities A, B appertaining to a particular argument u , the coefficients being either 0 or 1, is called a *quadrant*, e.g. for u_1 a certain quadrant is

$$A_{11} + A_{12} + B_{13} + B_{19} + B_{1p}$$

Since for every argument there are p A 's and p B 's, the whole number of quadrants is 2^{2p} , if 0 be included among them. We have now to consider the 2^{2p} functions $G(u + X)$, where X

quadrant. The quantities A, B themselves may be called *prime* or *elementary* quadrants.

Since quadrants of u are disposed in two rows, thus

$$\begin{array}{c} A_1, A_2, A_3, \dots, A_p, \\ B_1, B_2, B_3, \dots, B_p, \end{array}$$

be considered as forming the p pairs A_1B_1, A_2B_2, \dots , and every quadrant will be composed of a certain number of pairs, a certain number of single A 's and a certain number of single B 's.

If the number of pairs in a quadrant X is odd, then $G(u+X)$ is an odd function of the a_i and $G(X) = 0$.

$$\begin{aligned} X &= a_1A_1 + \dots + a_pA_p + b_1B_1 + \dots + b_pB_p, \\ u \text{ and the } b \text{ are each of them either 0 or 1. We have} \\ G(u+X) &= \sum a \left[\sum (u+X + 2\sum mB)^p \right] \\ &= \sum a \left[\sum \{u + \sum aA + \sum (2m+b)B\}^p \right], \\ G(X) &= G(u+X) = \sum a \left[\sum \{u + \sum aA + \sum (2m+b)B\}^p \right] \\ &\quad (m+1 \text{ written for } m), \end{aligned}$$

the exponent on the right is the same in these two expressions except for the term $2\sum a_i b_i A_i B_i$ in the first, and $A_i B_i$ in the second. The differences $a_i b_i A_i B_i = a_i b_i \pi i$ have

$$e^{2\pi i a_i b_i} G(u+X) = e^{2\pi i \sum a_i b_i} G(u+X),$$

so $e^{-2\pi i \sum a_i b_i} G(u+X)$ is an odd or even function of a as $\sum a_i b_i$ is odd or even.

The number of odd functions is $2^{p-1}(2^p-1)$. We have to

$$a_1 b_1 + a_2 b_2 + \dots + a_p b_p = 2n+1,$$

the a, b are either 0 or 1. Let k of the a be zero, the $p-k$ unity; then the b belonging to the first k may be 0 or 1, which gives 2^k combinations.

* Clebsch and Gordan, p. 260.

The sum of the remaining $p - k$ b 's must be odd, so that the last is determined when all the others are chosen, which may be done in 2^{p-k-1} ways; thus we have 2^{p-1} systems of the b . But the k a 's which are zero may be chosen in $\frac{*[p]^k}{k}$ ways; and k may have all values from 0 to $p - 1$. Hence the whole number of odd functions is

$$2^{p-1} \left(1 + 2 + \frac{*[p]^2}{2} + \frac{1}{3} \frac{*[p]^3}{3} + \dots + p \right) = 2^{p-1} (2^p - 1), \quad \text{Q.E.D.}$$

It follows that the number of even functions is $2^{p-1} (2^p + 1)$.

Let X , Y , Z be any three quadrants, and consider the functions

$$G(u + Y + Z), G(u + Z + X), G(u + X + Y).$$

If they are all even, or if one is even and two odd, then each is of the same character as the product of the other two. But if they are all odd, or if one is odd and two even, then the fluxion of each is of the same character as the product of the other two. We shall now show that

$$G'(u + X + Y) Gu = G(u + X + Y) G'u$$

may be expressed as the sum of a number of terms such as

$$G(u + X + Z), G(u + Y + Z).$$

Since Gu is an even function, it follows from what has just been said that of the three functions

$$G(u + Y + Z), G(u + Z + X), G(u + X + Y),$$

either all must be odd, or one odd and two even.

To this end it is important to consider the function which is derived from Gu by adding together the squares of all the terms; namely, we write

$$Gu = \sum e | 2 \sum (u + 2 \sum m, H_e)^2.$$

If now in $G(u + X + Z)$ we select the term involving the numbers m , and multiply it by the term in $G(u + Y + Z)$ involving the numbers n , the product is

$$e | \sum (u + X + Z + 2 \sum m H)^2 + \sum (u + Y + Z + 2 \sum n H)^2.$$

* [Auctoris.]

Now let $\mu_i = m_i + n_i$, $\nu_i = m_i - n_i$;
then we have

$$\mu^2 + \nu^2 = 2(m^2 + n^2), \quad \mu\mu' + \nu\nu' = 2(mm' + nn'),$$

and consequently

$$(u + X + Z + 2\Sigma mB)^2 + (u + Y + Z + 2\Sigma nB)^2$$

= twice square of half sum + twice square of half difference

$$= 2(u + \frac{1}{2}X + \frac{1}{2}Y + Z + \Sigma\mu B)^2 + 2(\frac{1}{2}X - \frac{1}{2}Y + \Sigma\nu B)^2.$$

In this expression each corresponding pair of μ, ν must be either both even or both odd. We may put separately the B 's belonging to the odd ones; let \mathfrak{A} be written for the sum of them, viz. \mathfrak{A} is the sum of any selection of the B 's; then $\Sigma\mu B$ may be written $\mathfrak{A} + 2\Sigma m'B$ and $\Sigma\nu B = \mathfrak{A} + 2\Sigma n'B$, where the m' and n' are now independent. Then we shall have

$$G(u + X + Z) = G(u + Y + Z) \\ = \Sigma P(u + \frac{1}{2}X + \frac{1}{2}Y + Z + \mathfrak{A}) P(\frac{1}{2}X - \frac{1}{2}Y + \mathfrak{A}),$$

where the possible number of terms on the right is the number of different selections \mathfrak{A} , viz. 2^n .

We must however investigate the effect of adding half-quadrants to the arguments of the functions P . We have, namely,

$$\begin{aligned} P(u + \frac{1}{2}X) &= \Sigma e^{i[2\Sigma\{u + \frac{1}{2}\Sigma a_i A + \Sigma(2m + \frac{1}{2}b_i) \cdot B\}^2]} \\ &= P(u + \frac{1}{2}X) = P(u + \frac{1}{2}X) \\ &= \Sigma e^{i[2\Sigma\{u + \frac{1}{2}\Sigma a_i A + \Sigma(2m + \frac{1}{2}b_i) \cdot B\}^2]} \\ &= \Sigma e^{i[2\Sigma\{u + \frac{1}{2}\Sigma a_i A + B_x + \Sigma(2m + \frac{1}{2}b_i) \cdot B\}^2]} \\ &\qquad\qquad\qquad (B_x = \Sigma b_i B.) \end{aligned}$$

But

$$P(u + B_x + \frac{1}{2}X) = \Sigma e^{i[2\Sigma\{u + \frac{1}{2}\Sigma a_i A + B_x + \Sigma(2m + \frac{1}{2}b_i) \cdot B\}^2]},$$

the difference of these exponents consists of a term $4\Sigma a_i A u$, and a term $-4\Sigma a_i(4m + b_i)AB = \Sigma a_i(4m + b_i)\pi i$. This therefore gives rise to a factor $e^{i[\Sigma a_i(4m + b_i)\pi i]}$ which is ± 1 according as Σab is even or odd, i.e., according as $e^{-2\Sigma ab} G(u + X)$ is even or odd. Thus

$$P(u + \frac{1}{2}X) = \pm e^{-2\Sigma ab} P(u + B_x + \frac{1}{2}X),$$

according to the character of X .

* XXXIX.

THEORY OF MARKS OF MULTIPLE THETA
FUNCTIONS.

[A mark (p) is a symbol $\left(\begin{smallmatrix} a, \beta, \gamma, \dots \\ a', \beta', \gamma', \dots \end{smallmatrix}\right)$, composed of the $2p$ integers $a, \beta, \gamma, \dots, a', \beta', \gamma', \dots$, distinguished only as even or odd integers; and the mark is even or odd according as $ax' + \beta\beta' + \gamma\gamma' \dots$ is even or odd $(\S 1)$.

NUMBER of marks (p) is 2^{2p} .

Number of odd marks $(p+1)$ may be calculated from number [of odd marks] p . Let N_p, E_p be numbers of odd and even marks for p ; $N_p + E_p = 2^{2p}$. We may prefix to each of these odd marks $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}, \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$ or $\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$, and to each of the even marks $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$, to get an odd mark for $p+1$. Therefore

$$N_{p+1} = 3N_p + E_p = 2^{2p} + 2N_p \\ = 2^{2p} + 2 \cdot 2^{2(p-1)} + 2^2 \cdot 2^{2(p-2)} + \dots + 2^2 = 2^2 (2^{2p-2} + 1);$$

and consequently $N_p = 2^{p-1} (2^p - 1)$. These numbers are 1, 6, 28, 120, ...

Any mark (p) except 0 may be divided into two odd marks in N_{p-1} ways. For let it begin with $\begin{smallmatrix} a \\ b \end{smallmatrix}$. Take any odd mark $(p-1)$ for the places after the first, and add it to those places of the given mark. If the result is even, we must divide $\begin{smallmatrix} a \\ b \end{smallmatrix}$ into an even and an odd part; if odd, we must divide it into two even parts. The former can be done in one way and one

only, unless $\frac{a}{b} = \frac{0}{0}$; viz. the parts are $\frac{a+1}{b+1}$ and $\frac{1}{1}$. The latter can be done in one way only, viz. $\frac{a}{b} = \frac{0}{0} + \frac{a}{b}$ or $\frac{1}{1} = \frac{1}{0} + \frac{0}{1}$; but here we may attach either of the even parts to either of the odd succeeding parts, and we have consequently *two* divisions of the given mark corresponding to the *two* odd parts. Hence if $\frac{a}{b}$ is not $\frac{0}{0}$, the number of divisions is exactly N_{r-1} . If it is, we may select any other place of the given mark, which is not $\frac{0}{0}$.

Thus every mark g , except 0, is the sum of each of N_{r-1} pairs of odd marks. The $2N_{r-1}$ odd marks form the group g . We may include the group 0, consisting of all the odd marks. Then every odd mark occurs in N_r groups.

Every two odd marks l, m are common to $2N_{r-1}$ groups. A group to which l belongs is of the form $l + n$, where n is odd; if m belongs to this, $l + n = m + p$, and therefore $l + m = n + p$, or n, p form a pair of the group $l + m$. There are two groups for each such pair, except l, m ; adding to these the group 0, we have $2N_{r-1}$.

To find the odd marks common to two groups, g, h , let $g + h = l + m$, where l, m are odd. Then

$$g + l = h + m, \quad g + m = h + l.$$

Now $(g + l) \cdot (g' + l') + (h + l) \cdot (h' + l') = gh' + g'h$

$$+ (gg' + hh') + (g + h)(g' + h').$$

If this = 1, then $g + l, g + m$ are one even and one odd; therefore of every pair in the group $g + h$, one belongs to the group g and one to the group h ; and the odd marks which make up these pairs in g, h are identical. In this case, therefore, every two of the groups $g, h, g + h$ have N_{r-1} marks in common, no two of which form a pair. For this, *one or all* of the marks $g, h, g + h$ must be odd. If however $gh' + g'h = 0$, so that *none* or *two* of $g, h, g + h$ are odd, then $g + l, g + m$ are both even or

both odd. Now in this case the three groups $g, h, g + h$ contain every odd mark between them. For

$$\begin{aligned}(g + l)(g' + l) &= (h + l)(h' + l) \\ &= gg' + hh' + (g + h)l + (g' + h')l \\ &= (g + h + l)(g' + h' + l) + gh' + g'h + ll.\end{aligned}$$

Therefore of $g + l, h + l, g + h + l$, either one is odd or all are odd. Let then each group have x marks in common with the other two, and g to itself. Then

$$x + 3(g - N_g) + x + g - 2N_g =$$

so that $g = \frac{1}{3}N_g + N_{g+g} = 2^{p-2} \{2^{p-2} - 2^{p-3} + \dots + 2^{p-p}\},$

$$g = 2^{p-1} \{2^{p-1} - 1\} = 2^{p-2} + \dots + 2^{p-3} + \dots + 2^{p-p} \{2^{p-2} - 1\} = 1N_{g+g}.$$

These common marks are distributed in sets of 4, l, m, n, p , so that

$$g = l + n + m + p, \quad h = l + p + m + n, \quad g + h = l + m + n + p.$$

Relations of two groups.

Two groups g, h stand in one of two possible relations to each other. If they have N_{g+h} odd marks in common, no two of which are paired in either group, they shall be called *near* groups; if they have in common N_{g+h} sets of two marks, each set forming two pairs in each group, they shall be called *far* groups. The sum of two groups is near to or far from both of them, according as the groups are near or far. If they are near, no marks are common to the two groups and their sum; if they are far, the N_{g+h} sets occur in the sum divided into pairs in the only remaining way.

Hence, l, m, n being odd marks, the groups $l + m, l + n$ are near if l , which is common to them, does not belong to their sum, or if $l + m + n$ is even; far when $l + m + n$ is odd.

Relations of three groups.

Let f and g be two near groups, and h another group near to f but not equal to $f + g$; then if h be near to $g, f + h$ will be

far from g , and *vice versa*. The group h cannot contain all the marks common to f and g , for then $f + g$ would have absolutely no marks in common with h , which is impossible. Let then a be a mark common to f and g but not to h , and let $f \rightarrow a + \alpha_1$, $g \rightarrow a + \alpha_2$. Then if h is near to g , α_1 and α_2 both occur in h , but not paired, because $\alpha_1 + \alpha_2 \rightarrow f + g$, which is not h by hypothesis. If $h \rightarrow \alpha_1 + \beta_1 + \alpha_2 + \beta_2$, then $a + h \rightarrow f + \beta_1 + g + \beta_2$, and therefore $f + \beta_1 + g + \beta_2 \rightarrow h$, suppose. Now h is odd, because either α_2 or β_1 must occur in the group f , and α_2 does not. Hence we have these equations,

$$\begin{array}{llll} f \rightarrow a + \alpha_1 & g \rightarrow a + \alpha_2 & h \rightarrow \alpha_1 + \beta_1 & f + h \rightarrow a + \beta_1 \\ \rightarrow h + \beta_1 & \rightarrow h + \beta_2 & \rightarrow \alpha_2 + \beta_2 & \rightarrow h + \alpha_2, \end{array}$$

from which it appears at once that $f + h$ is far from g , as was to be proved.

It follows that one half of the groups which are near to f are also near to g , the complementary half (counting $f + h$ complementary to h) being far from g . We shall now consider more closely those groups which are near both to f and to g .

We observe that two pairs of the group h are obtained by crossing the constituents from two pairs of the groups f, g having in common the marks a, b . These marks shall be called a *cross-pair* of h ; neither of them is contained in h . Since every cross-pair gives rise to a set of four marks common to the far groups g and $f + h$, and *vice versa*, there are $N_{f, g}$ cross-pairs. Every mark common to f and g either belongs to a cross-pair or occurs in h . Hence there are $N_{f, g} + 2N_{f, g} = 2^{2k-1}$ marks common to f, g, h .

Relations of four groups.

If j be another group near to f, g and h , it contains one mark and one only out of every cross-pair of h . If it did not contain a or b , it must contain $\alpha_1, \beta_1, \alpha_2, \beta_2$, and therefore be far from h ; if it contained both a and b , it must contain none of these four, with the same result. For the same reason h must contain one and one only out of every cross-pair of j . Hence of the marks

common to f, g , there are N_{p-2} occurring neither in h nor j , and N_{p-2} occurring in each but not in the other. Thus $2^{2p-4} - N_{p-2}$ are common to all four; these are also common to $f + g + h + k$.

The system $fghj$ is formed as follows. Given f and g near to one another, we find h near to both; then it has N_{p-2} cross-pairs and 2^{2p-4} marks common with f and g . Let a belong to one of the cross-pairs, and c be one of the common marks; then a group formed to have ac for a cross-pair will be near to h . For it will contain α_1 and α_2 separated from each other, and associated with marks which do not occur in h . Let j be this group; we have seen that it has $2^{2p-4} - N_{p-2}$ marks common with fgh .

Now let a be one of these common marks; then a group k formed to have ad for a cross-pair will be near to h and j . All those N_{p-2} marks which are common to the cross-pairs of i and j will belong separately to the N_{p-2} cross-pairs of k . For if pq be a cross-pair of h , and pr the corresponding one of j , then k cannot have the cross-pair qr , or it would be $= f + g + h + j$, which we do not suppose; hence it must have a cross-pair ps . Then s must occur in the groups h and j . Consequently the number of marks common to $fghjk$ is $2^{2p-4} - 2N_{p-2}$. These are also common to $f + g + h + j$, $f + g + j + k$, $f + g + h + k$.

If j be near to f and g and far from h , then of the two marks in every cross-pair of h it must contain neither or both. If it contains a , for example, it cannot contain α_1 or α_2 and therefore not β_1 or β_2 , so that it must contain b . Being near to f and g it is far from $f + g$, so that the three groups $f + g$, h , j are far from one another.

We may now find the groups to which three odd marks l, m, n are common. If g be one of these, $g + l$ is common to the groups g , $l + m$, $l + n$. We have therefore to find the marks which are common to the groups $l + m$, $l + n$. If

$$1 \equiv (l + m)(l' + n') + (l' + m')(l + n),$$

$$\equiv lm' + l'n + mn' + m'n + lm' + l'm,$$

$$\equiv (l + m + n)(l' + m' + n') + ll' + mm' + nn',$$

i.e. if $l+m+n$ is even, then there are N_{p-1} marks common to the groups $l+m$, $l+n$; and by adding l to each of these, we obtain the N_{p-1} groups to which l, m, n are common. If however $l+m+n$ is odd, there are $4N_{p-2}$ marks common to the groups $l+m$, $l+n$, and consequently $4N_{p-2}$ groups to which the three marks l, m, n are common. In the former case, no two of the N_{p-1} marks common to $l+m$, $l+n$ are paired in either of them; let $l+g$, $l+h$ be any two of these, then $g+h$ is not equal to $l+m$, or $l+n$, or $m+n$; hence $l+g$, $m+g$, $n+g$ are different from $l+h$, $m+h$, $n+h$. Changing now the notation, let l_1, m_1, n_1 be three odd marks whose sum is even, and let $0, g_2, g_3 \dots g_n$, ($n=N_{p-1}$) be the groups to which they are common. Also let $g_r = l_1 + l_r = m_1 + m_r = n_1 + n_r$; then all the marks l, m, n are different. Moreover the three odd marks l_r, m_r, n_r are common to the N_{p-1} groups $l_r + l_s = m_r + m_s = n_r + n_s$. These three marks are identical, because

$$l_1 + l_r = m_1 + m_r = n_1 + n_r,$$

and $l_1 + l_s = m_1 + m_s = n_1 + n_s$. Hence $l_r + m_r + n_r$ is even.

All the groups to which three odd marks are common shall be called a *set*. Then we have N_{p-1} sets 1, 2, 3 ..., all containing the group 0, and any two of them r, s having also in common the group $(rs) = l_r + l_s = \text{etc.}$

Consider now the case of three odd marks l, m, n , whose sum is odd, and which are therefore common to $4N_{p-2}$ groups. The marks common to 0, $l+m$, $l+n$ arrange themselves in sets of fours, $pqr's$, so that

$$l+m=p+q=r+s, \quad l+n=p+r=q+s.$$

Hence the groups containing l, m, n arrange themselves in sets of fours,

$$\begin{array}{l|l|l|l} f=l+p & g=l+q & h=l+r & k=l+s \\ =m+q & =m+p & =m+s & =m+r \\ =n+r & =n+s & =n+p & =n+q \\ =0+s & =0+r & =0+q & =0+p \end{array}$$

so that

$$l+m=f+g=h+k, \quad l+n=f+h=g+k, \quad m+n=g+h=f+k,$$

Let $I_1 \equiv m_1 \equiv a, I_2 \equiv p_1 \equiv a$, so that we have to deal with four odd marks whose sum is zero. These will be common to N_{p_1} tetads of groups f, g, h, k ; and if

$$f \equiv I_1 \equiv I_2 \equiv a, \quad g \equiv m_1 \equiv m_2 \equiv p_1 \equiv p_2,$$

we shall have N_{p_1} tetads f, m_1, m_2, p_2 . The order of any tetad must only be altered by a total substitution, such a tetad has therefore four forms, by adding these tetads of odd marks in various forms we obtain always the same total of groups f, g, h, k . Any such tetrad of odd marks as is common to the N_{p_1} tetads of groups f, g, h, k .

We pass now to the consideration of three groups g, h, k which are not such that $g + h + k = 0$, but which are such that every two of them are contiguous, i.e. they have N_{p_1} marks in common; and we propose to determine the number of marks common to all three groups. Let a be a mark common to g and h but not to k , so that $g = a + p_1, h = a + p_2$; then a and a , both seen in k , but not paired, because $a_1 + a_2 = g + h$, which is not k by hypothesis. Hence $g + h$ and k are non-contiguous groups, since two marks which are paired in $g + h$ occur in k . If $k = a_1 + p'_1 + a_2 + p'_2$, then $a + h = g + p'_2 + k + p'_1$, and therefore $g + p'_1 = h + p'_2 + k$, impossible. Thus $g = k + p'_1, h = k + p'_2$, so that the six marks $a, k, a_1, p'_1, a_2, p'_2$ are distributed as follows:

$$\begin{array}{cccc} g = a + p_1 & h = a + p_2 & k = a_1 + p'_1 & g + h = a_2 + p_2 \\ = k + p'_1 & = k + p'_2 & = a_1 + p'_1 & = p'_2 + p'_2 \end{array}$$

Since $g + h$ and k are non-contiguous, there are N_{p_1} sets of six marks. If a be a mark common to g and h , not belonging to any of these sets, it must also belong to k ; and consequently we must have $g = a + p_1, h = a + p_2, k = a + p_3$. On such marks there are $N_{p_1} \cdot 3N_{p_3} = N_{p_1} \cdot 3N_{p_3} = 2^{12}$. Thus in this the number of marks common to three such groups.

Hence four odd marks f, m_1, m_2, p_2 such that $f + m_1 + m_2 + p_2$ are contiguous two and two but their sum is zero, are common to 2^{12} groups; and we have a theory of 2^{12} such sets of four odd marks, any two sets being common to one group besides the group 0.

Let λ be another mark common to the groups $l+m$, $l+n$, $l+r$, and let $l+m \equiv \lambda+\mu$, $l+n \equiv \lambda+v$, $l+r \equiv \lambda+p$; then λ, μ, v, p form another such set. If g is a group containing l, m, n, r , then $g+l$ belongs to the groups $l+m$, $l+n$, $l+r$, that is, to the groups $\lambda+\mu$, $\lambda+v$, $\lambda+p$. To find the groups g , then, we must take the marks p belonging to the groups $l+m$, $l+n$, $l+r$ and add them to l . Similarly, to find the groups containing λ, μ, v, p , we must add the same marks p to λ . Suppose two of them, p, q are such that $p+l \equiv q+\lambda$, then this group $p+l$ contains all the eight marks $l, m, n, r, \lambda, \mu, v, p$. So also will the group $p+\lambda \equiv q+l$. For this, we must have $p+q \equiv l+\lambda$; we have therefore to find the number of marks common to the four groups $l+m$, $l+n$, $l+r$, $l+\lambda$. The last is non-vanishing; to the first three, we know that

$$l+l \equiv m+\mu \equiv n+v \equiv r+p.$$

$$\text{Let } g+h+k=0, \text{ and } g \equiv a+\alpha \equiv b+\beta \equiv c+\gamma,$$

$$h \equiv a+\beta \equiv b+\alpha, \quad k \equiv a+\beta \equiv \alpha+\beta;$$

then if $a+b+c$ is odd, c occurs in the group k ; it therefore occurs among the $4N_{\nu-2}$ marks common to g and k . Thus the marks in g are such that the sum of two from the same tetrad, not paired, and one belonging to no tetrad is always even.

Hence if we take two unpaired marks a, b , and a third c not belonging to the group $a+b$, then a will not belong to the group $b+c$, etc. How many are common to the groups $g, a+b, a+c$? If d is common to them, we have

$$g \equiv d+\delta \equiv a+\alpha \equiv b+\beta \equiv c+\gamma,$$

$$d+\alpha \equiv a+b \equiv \alpha+\beta, \quad d+\theta \equiv a+c \equiv \alpha+\gamma.$$

Common to $g, a+b$ are $4N_{\nu-2}$ including $ab\theta$,

$$a \quad g, \quad a+c \quad a \quad 4N_{\nu-2} \quad a \quad \omega\alpha\gamma.$$

Suppose that

$$g \equiv a+\alpha \equiv b+\beta, \quad c+\gamma \equiv d+\delta, \quad c+\epsilon \equiv f+\phi,$$

$$h \equiv a+b \equiv \alpha+\beta, \quad c+d \equiv \gamma+\delta, \quad c+f \equiv e+\phi,$$

$$k \equiv a+\beta \equiv \alpha+b, \quad c+\delta \equiv \gamma+d, \quad c+\phi \equiv e+f.$$

a mark may be divided in 2^{p-1} ways into two marks, namely $N_{p,1}$ ways into two odd marks

$$2^{p-2}(2^{p-1}-1) = 2^{p-2} \cdot 2^{p-2};$$

$N_{p,1} = 2^{p-2} + 2^{p-2}$ ways into two even marks; and consequently 2^{p-2} ways into one odd and one even mark. Thus every mark g can be divided in 2^{p-2} ways into h and k so that their groups shall have $N_{p,2}$ marks common.

This is the same number as if we were permitted to select the marks to be made common in any way we liked from $2p-3$ of the pairs in g .

Thus, $p=3$, we may divide every mark in 32 ways, and therefore in 16 ways, so that three groups shall have six marks common.

Every [triple] θ -function has 23 zero-values; every value annuls 23 θ -functions.

Every two θ -functions have 12 zero-values in common; every two values annul 12 θ -functions.

The three functions g, h, k have 6 zero-values in common, if

$$gg' + hh' + (g + g')(h + h') = 1,$$

or if $gh' + g'h = 1 \pmod{2}$, in this case also the values g, h, k annul 6 θ -functions.

This is true of the functions g, h, k , if

$$(g + h)(g' + k') + (g' + k')(g + h)$$

$$\text{or} \quad hh' + hk' + kg' + kh' + gh' + g'h = 1,$$

$$\text{or} \quad (g + h + k)(g' + h' + k') = gg' + hh' + kk' = 1.$$

If however this quantity is even, the three functions have only 4 zero-values in common.

Six θ -functions having 3 common zeros shall be called a *set*. If the set includes $\theta_{g,1}$ the 3 zeros are all odd; whence it appears that their sum must be even. Hence they belong to one or more of Weber's "groups," so that no two form a "pair." Every even mark may be expressed in 241 ways as the sum of three different odd marks; thus the number of sets including $\theta_{g,1}$

is 30×36 ; and generally this is the number of sets including any given mark.

The number of sets including θ_a and θ_b is $160 \div 8 = 20$. For the six pairs in the group (a) give us 20 triads of pairs, and each of these gives eight triads of odd marks constituting a set.

If $0, g, h, 0, h, k; 0, k, g$ have each 0 zeros in common, there are 4 zeros common to $0, g, h, k$, unless $g+h+k=0$, when there are none common (Weber).

Suppose then lmm, lnr, lmn to be zeros belonging to three sets which include θ_a . Then

$$0, l \pm m, l \pm n; 0, l \pm n, l \pm r; 0, l \pm r, l \pm m$$

are zeros belonging to three sets which include θ_r . Now

$$l \pm m \pm l \pm n \pm l \pm r = l \pm m \pm n \pm r;$$

consequently this must not be zero, if the three sets are to have but θ_a common. Of these three sets, then, any two can only have one common such θ as belong also to the third.

Let $g = l \pm l' = m \pm m' = n \pm n'$; then the sets $lmm, l'm'n'$ have no θ in common except θ_m and $\theta_{n'}$. For two groups g, h have 4 or 0 zeros in common, the latter only when no two of them form a pair in either group. Let now h be a mark of the set lmm , so that $h = l \pm u = m \pm v = n \pm \gamma$. Then either u/v must be different from $l'm'n'$; or, suppose $u = m'$, then $v = l'$, and $\gamma = l \pm m' \pm n = l \pm m \pm n$. But $l \pm m' \pm n$ is necessarily even, so that $m'l'\gamma$ is not a set. Starting then from the set $l'l_1l_2$, let the marks of it be $0, u, h, v, l_1, g$; then

$$u = l \pm m, \quad h = l \pm n, \quad v = l \pm p, \quad f = l \pm q, \quad g = l \pm r;$$

$$\text{or,} \quad u = l_1 \pm m_1 = l_2 \pm m_2 = l_3 \pm m_3,$$

Any two of the six sets l, m, n, p, q, r have one mark in common besides 0. For (e.g.)

$$\begin{aligned} p_1 \pm q_1 &= v \pm f = p_2 \pm q_2 = p_3 \pm q_3, \\ g &= u \pm a_1, & h &= u \pm a_2, & k &= a_1 \pm \beta_1, \\ &= h \pm \beta_2, & &= h \pm \beta_3, & &= a_2 \pm \beta_3. \end{aligned}$$

If g, h, k are contiguous but sum not zero, the same is true of $g, g+h, g+k$, etc.

$$\begin{aligned} h+k &= a + \beta_2 & k+g &= a + \beta_1 & g+h &= a_1 + a_2 \\ &= b + a_1 & &= b + a_2 & &= \beta_1 + \beta_2. \end{aligned}$$

$$\left. \begin{aligned} g+h+k &= a_2 + \beta_1 \\ &= a + b \\ &= a_1 + \beta_2 \end{aligned} \right\} \text{ is non-contiguous to all six.}$$

If $l+m, l+n, l+r$ are contiguous two and two but their sum not zero, then $m+n, m+r$ are also contiguous. Hence the sum of every three of the four marks l, m, n, r is even.

Now suppose a fourth group f , contiguous to g . If it is non-contiguous to h , then $f+g$ will be contiguous to both of them. Hence one-half of the groups contiguous to g are also contiguous to h . Suppose then that f is contiguous to g and h , then it is non-contiguous to $g+h$.

If the mark a is common to g and h but not to f ,

$$g = a + a_1, \quad h = a + a_2,$$

then $f = a_1 + \gamma_1 = a_2 + \gamma_2$, so that $g + \gamma_2 = h + \gamma_1$. Now suppose e is odd, because either a_2 or γ_2 must occur in the group and a_2 does not. Hence we have

$$\begin{aligned} f &= \beta_1 + \gamma_2 & g &= a + a_1 & h &= a + a_2 & k &= a_1 + \beta_1 & f &= a_1 + \gamma_1 \\ &= \beta_2 + \gamma_1 & &= b + \beta_1 & &= b + \beta_1 & &= a_2 + \beta_2 & &= a_1 + \gamma_2 \\ &= a + a_2 & &= c + \gamma_1 & &= c + \gamma_1 & &= c + \gamma_2 & &= b + \beta_2 \\ & & &= d + \delta_1 & &= d + \delta_1 & &= d + \delta_2 \end{aligned}$$

If therefore f is contiguous to k, γ_1 and γ_2 do not occur in k , and therefore e then; also b in f . But e cannot occur in f , or b in k .

To make a system of four groups, contiguous two and two, and having common marks; take two contiguous groups

$$g, \quad = a + a_1 = b + \beta_1 = c + \gamma_1, \text{ etc.},$$

$$h, \quad = a + a_2 = b + \beta_2 = c + \gamma_2, \text{ etc.}$$

To these add the group $k, \quad = a_1 + \beta_2 = a_2 + \beta_1 = \text{etc.}$ which does

not contain a or b . Select two marks, c, d , common to g, h, k , then the fourth group, if any, must be $q_1 + \delta_2 + q_3 + \delta_4$, $\therefore f$, suppose,

$$\begin{aligned} j &= p_1 + q_2 & g &= a + q_1 & h &= a + q_2 & k &= a + p_2 & f &= q_1 + \delta_2 \\ &= p_3 + q_4 & &= b + p_1 & &= b + p_2 & &= a_2 + p_1 & &= q_3 + \delta_1 \\ &= a + u_2 & &= c + q_1 & &= c + q_2 & &= c + q_3 \\ & & &= d + p_1 & &= d + p_2 & &= d + \delta_4 \\ g &+ h &= c + d + f. \end{aligned}$$

f must contain either a or b but not both. Suppose a , then it contains p_1 and p_2 ,

$$f \text{ is non-confignous to } h + k = q_1 + q_2 + \delta_3 + \delta_4.$$

Since f contains either a or b but not both, this case is the same as the last. f is linked with g, h by N_{r-1} cross-pairs. Out of each cross-pair of k , f must contain one mark but not both; thus there are N_{r-1} marks common to g and h which belong to cross-pairs both of k and f , and $2N_{r-1}$ belonging to cross-pairs of k and f separately. Hence the number of marks common to g, h, k, f is $2^{r-1} + N_{r-1}$,

$$\begin{aligned} j &= p_1 + q_2 + p_3 + q_4 \\ &= a + u_2 \end{aligned}$$

must not contain q_1, q_3, c, k , but must contain a ,

$$f + g + h + j + k = 0.$$

*XL

ON THE DOUBLE THETA-FUNCTIONS.

THE θ -series of two variables u_1, u_2 is defined as follows:

$$\theta(u_1, u_2; a_{11}, a_{12}, a_{22}) = \sum_{n_1, n_2} e\{n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22} + 2\pi i n_1 + 2\pi i n_2\}^*,$$

where the whole numbers n_1, n_2 , in respect of which the summation takes place, may have all values from $-\infty$ to $+\infty$. The parameters a_{11}, a_{12}, a_{22} must be such that the real part of $n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22}$ is negative for all values of n_1, n_2 . This is necessary and sufficient for the convergency of the series.

The function θ is unaltered if either of the arguments be increased by any multiple of πi ; this follows from the fact that $e\{2\pi i\} = 1$. This is expressed by saying that u_1, u_2 have independently the period πi ; or that taken together they have the two periods $\pi i, 0$ and $0, \pi i$.

As the number n_1 may have all integer values, the function will not be altered if we write $n_1 + q_1$ for it (q_1 being any integer) and then give n_1 all integer values. Thus we find

$$\begin{aligned} \theta(u_1, u_2) &= \sum_{n_1, n_2} e\{(n_1 + q_1)^2 a_{11} + 2(n_1 + q_1)n_2 a_{12} + n_2^2 a_{22} \\ &\quad + 2\pi i(n_1 + q_1)n_1 + 2\pi i n_2\} \\ &= e\{q_1^2 a_{11} + 2q_1 n_1\} \cdot \sum_{n_1, n_2} e\{n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22} \\ &\quad + 2\pi i n_1(n_1 + q_1) + 2\pi i(n_2 + q_2 n_1)\} \\ &= e\{q_1^2 a_{11} + 2q_1 n_1\} \theta(u_1 + q_1 a_{11}, u_2 + q_1 a_{12}); \end{aligned}$$

* [Or as it might be written

$$\sum_{n_1, n_2} \exp. \{i\{a_{11} n_1^2 + a_{12} n_1 n_2 + a_{22} n_2^2 + 2\pi i n_1 + 2\pi i n_2\}\}$$

or, which is the same thing,

$$\theta(u_1 + q_1 a_{11}, u_2 + q_1 a_{12}) = e^{\pi i (q_1^2 a_{11} + 2q_1 a_1)} \theta(u_1, u_2).$$

In the same manner we may show that

$$\theta(u_1 + q_2 a_{21}, u_2 + q_2 a_{22}) = e^{\pi i (q_2^2 a_{22} + 2q_2 a_2)} \theta(u_1, u_2).$$

From these equations it appears that when the arguments u_1, u_2 are simultaneously increased by the parameters a_{11}, a_{12} respectively, the function θ becomes multiplied by the exponential factor $e^{\pi i (a_{11}^2 + 2a_1)}$; and that when they are simultaneously increased by a_{21}, a_{22} respectively, the θ is multiplied by $e^{\pi i (a_{22}^2 + 2a_2)}$. On this account the arguments u_1, u_2 are said to have the *quasi-periods* a_{11}, a_{12} and a_{21}, a_{22} .

The effect of adding a *half* period to either argument is to change the sign of certain terms in the series. Thus if we write $u_1 + \frac{1}{2}\pi i$ for u_1 , each term becomes affected with the factor $e^{\pi i (a_1 \pi i)}$, that is $(-1)^{a_1}$. The series thus arising are conveniently distinguished by the numbers 0 or 1 affixed to the θ , as follows: $\phi(u_1, u_2)$ being for shortness written instead of

$$u_1^2 a_{11} + 2a_1 u_1 a_{11} + u_2^2 a_{12} + 2a_1 u_1 a_{12} + u_2^2 a_{22}.$$

$$\theta(u_1 + \frac{1}{2}\pi i, u_2) = \sum \sum (-1)^{a_1} \phi(u_1, u_2) = \theta^{(0)}(u_1, u_2),$$

$$\theta(u_1, u_2 + \frac{1}{2}\pi i) = \sum \sum (-1)^{a_2} \phi(u_1, u_2) = \theta^{(1)}(u_1, u_2),$$

$$\theta(u_1 + \frac{1}{2}\pi i, u_2 + \frac{1}{2}\pi i) = \sum \sum (-1)^{a_1 + a_2} \phi(u_1, u_2) = \theta^{(1)}(u_1, u_2).$$

We may include these in the formula

$$\theta^{(0)}(u_1 + \frac{1}{2}\pi i, u_2 + \frac{1}{2}\pi i) = \sum \sum (-1)^{(a_1 + a_2)(a_1 + a_2 + 2a_1)} \phi(u_1, u_2) \\ = \theta^{(0)}(a_1 + 1, a_2),$$

where, in $\theta^{(0)}$, we must suppose 0 or 1 substituted for each of the whole numbers a_1, a_2 according as it is even or odd.

The effect of adding a half quasi-period to the arguments is to add $\frac{1}{2}$ to one of the numbers a_1, a_2 and multiply the θ function by an exponential factor. The new series thus arising are conveniently distinguished by 0 or 1 *affixed* to the θ , as follows:

$$\theta(u_1 + \frac{1}{2}a_{11}, u_2 + \frac{1}{2}a_{12}) = e^{\pi i (a_{11} - a_1)^2} \sum \sum e^{\pi i \phi(u_1 + \frac{1}{2}a_{11}, u_2 + \frac{1}{2}a_{12})} \\ = e^{\pi i (a_{11} - a_1)^2} \theta_{\frac{1}{2}}(u_1, u_2).$$

$$\begin{aligned}\theta(u_1 + \tfrac{1}{2}a_{12}, u_2 + \tfrac{1}{2}a_{22}) &= \epsilon | -\tfrac{1}{4}a_{22} - u_2 | \cdot \Sigma \Sigma \epsilon | \phi(n_1, n_2 + \tfrac{1}{2}) | \\ &= \epsilon | -\tfrac{1}{4}a_{11} - u_1 | \cdot \theta_{01}(u_1, u_2),\end{aligned}$$

$$\begin{aligned}\theta(u_1 + \tfrac{1}{2}a_{11} + \tfrac{1}{2}a_{12}, u_2 + \tfrac{1}{2}a_{12} + \tfrac{1}{2}a_{22}) \\ = \epsilon | -\tfrac{1}{4}a_{11} - \tfrac{1}{2}a_{12} - \tfrac{1}{4}a_{22} - u_1 - u_2 | \cdot \Sigma \Sigma \epsilon | \phi(n_1 + \tfrac{1}{2}, n_2 + \tfrac{1}{2}) | \\ = \epsilon | -\tfrac{1}{4}a_{11} - \tfrac{1}{2}a_{12} - \tfrac{1}{4}a_{22} - u_1 - u_2 | \cdot \theta_{11}(u_1, u_2).\end{aligned}$$

These cases are included in the formula

$$\begin{aligned}\theta_{\alpha\beta}(u_1 + \tfrac{1}{2}\gamma a_{11} + \tfrac{1}{2}\delta a_{12}, u_2 + \tfrac{1}{2}\gamma a_{12} + \tfrac{1}{2}\delta a_{22}) \\ = \epsilon | -\phi(\tfrac{1}{2}\gamma, \tfrac{1}{2}\delta) | \cdot \theta_{\alpha+\gamma, \beta+\delta}(u_1, u_2),\end{aligned}$$

where, in the suffixes, we must suppose 0 or 1 written for each number according as it is even or odd.

In regard to the periods and quasi-periods of these θ , it is to be remarked that θ^{10} , θ^{01} and θ^{11} have each the period πi for each argument, like θ ; but that besides acquiring the exponential factor, θ^{10} changes sign for addition of the quasi-period a_{11} , a_{12} , θ^{01} for addition of a_{12} , a_{22} , and θ^{11} for addition of either. On the other hand, θ_{10} , θ_{01} , and θ_{11} resemble θ in regard to the quasi-periods, but change sign when u_1 , u_2 , or either of them is increased by πi respectively. A general formula for these cases will be given presently.

If we add both half-periods and half-quasi-periods to the arguments, we obtain nine new θ -series, which together with the preceding are all defined in the following formula:

$$\begin{aligned}\theta(u_1 + \tfrac{1}{2}\alpha\pi i + \tfrac{1}{2}\gamma a_{11} + \tfrac{1}{2}\delta a_{12}, u_2 + \tfrac{1}{2}\beta\pi i + \tfrac{1}{2}\gamma a_{12} + \tfrac{1}{2}\delta a_{22}) \\ = \epsilon | -\phi(\tfrac{1}{2}\gamma, \tfrac{1}{2}\delta) | \Sigma \Sigma (-)^{\alpha n_1 + \beta n_2} \epsilon | \phi(n_1 + \tfrac{1}{2}\gamma, n_2 + \tfrac{1}{2}\delta) | \\ = \epsilon | -\phi(\tfrac{1}{2}\gamma, \tfrac{1}{2}\delta) | \theta_{\gamma\delta}^{\alpha\beta}(u_1, u_2).\end{aligned}$$

Here each of the letters α , β , γ , δ stands for either 0 or 1.

This formula may be generalized into the following, which contains the entire theory of the transformation of the θ into one another by addition of half-periods and half-quasi-periods to the arguments.

$$\begin{aligned}\theta_{\alpha\delta}^{a\beta}(u_1 + \tfrac{1}{2}\alpha\pi i + \tfrac{1}{2}\gamma a_{11} + \tfrac{1}{2}\delta a_{12}, u_2 + \tfrac{1}{2}\beta\pi i + \tfrac{1}{2}\gamma a_{12} + \tfrac{1}{2}\delta a_{22}) \\ = \epsilon | \tfrac{1}{2}(\alpha c + \beta d) \pi i - \phi(\tfrac{1}{2}\gamma, \tfrac{1}{2}\delta) | \theta_{\alpha+\gamma, \delta+\delta}^{a+\alpha, b+\beta}(u_1, u_2).\end{aligned}$$

Here α , β , γ , δ may be any whole numbers, and the affixes and suffixes are to be understood as before.

Of the sixteen functions θ_{ca}^{ab} , six are odd functions of u_1, u_2 , namely those for which $ac + bd$ is an odd number; and these of course vanish for the values $u_1 = 0, u_2 = 0$. The other ten are even functions, and it is easy by means of the preceding formulæ to assign for each the six pairs of values which make it zero. Namely

$$\theta_{ca}^{ab} (\tfrac{1}{2}a\pi i + \tfrac{1}{2}\gamma a_{11} + \tfrac{1}{2}\delta a_{12}, \tfrac{1}{2}\beta\pi i + \tfrac{1}{2}\gamma a_{12} + \tfrac{1}{2}\delta a_{22}) = 0,$$

whenever $(a + \alpha)(c + \gamma) + (b + \beta)(d + \delta)$

is an odd number.

The Product-Theorem.

If we multiply together term by term two θ -series with different arguments, we shall obtain a quadruply infinite series, the exponent in each term of it being the sum of the exponents in those terms of the θ -series from which it is derived. Thus we shall have

$$\begin{aligned} \theta(u_1, u_2) \theta(v_1, v_2) &= \sum \sum \epsilon | \phi(m_1, m_2) | \cdot \sum \sum \epsilon | \psi(n_1, n_2) | \\ &= \sum \sum \sum \sum \epsilon | \phi(m_1, m_2) + \psi(n_1, n_2) |, \end{aligned}$$

where

$$\begin{aligned} \phi(m_1, m_2) &= m_1^2 a_{11} + 2m_1 m_2 a_{12} + m_2^2 a_{22} + 2m_1 u_1 + 2m_2 u_2, \\ \psi(n_1, n_2) &= n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22} + 2n_1 v_1 + 2n_2 v_2. \end{aligned}$$

Now if we write

$$m_1 + n_1 = p_1, \quad m_1 - n_1 = q_1,$$

$$m_2 + n_2 = p_2, \quad m_2 - n_2 = q_2,$$

we shall have

$$2(m_1^2 + n_1^2) = p_1^2 + q_1^2, \quad 2(m_2^2 + n_2^2) = p_2^2 + q_2^2,$$

$$2(m_1 m_2 + n_1 n_2) = p_1 p_2 + q_1 q_2,$$

and therefore

$$\begin{aligned} &\phi(m_1, m_2) + \psi(n_1, n_2) \\ &= \tfrac{1}{2}(p_1^2 + q_1^2) a_{11} + (p_1 p_2 + q_1 q_2) a_{12} + \tfrac{1}{2}(p_2^2 + q_2^2) a_{22} \\ &\quad + p_1(u_1 + v_1) + q_1(u_1 - v_1) + p_2(u_2 + v_2) + q_2(u_2 - v_2) \\ &= \tfrac{1}{2}p_1^2 a_{11} + p_1 p_2 a_{12} + \tfrac{1}{2}p_2^2 a_{22} + p_1(u_1 + v_1) + p_2(u_2 + v_2) \\ &\quad + \tfrac{1}{2}q_1^2 a_{11} + q_1 q_2 a_{12} + \tfrac{1}{2}q_2^2 a_{22} + q_1(u_1 - v_1) + q_2(u_2 - v_2). \end{aligned}$$

Now the p and q are not all numbers indiscriminately, but p_1 and q_1 must be both odd or both even, being the sum and difference of two integers, and so also p_2 and q_2 must be both odd or both even. In the quadruply infinite series, therefore, there are four kinds of terms; those in which the p, q are all odd, in which they are all even, in which p_1, q_1 are odd and p_2, q_2 even, and in which p_1, q_1 are even and p_2, q_2 odd. We shall sum these separately.

First let all four numbers be even; and let

$$p_1 = 2s_1, \quad p_2 = 2s_2, \quad q_1 = 2t_1, \quad q_2 = 2t_2,$$

then

$$\begin{aligned} \phi(m_1, m_2) + \psi(n_1, n_2) \\ = 2s_1^2 u_{11} + 4s_1 s_2 u_{12} + 2s_2^2 u_{22} + 2s_1 \{u_1 + v_1\} + 2s_2 \{u_2 + v_2\} \\ + 2t_1^2 u_{11} + 4t_1 t_2 u_{12} + 2t_2^2 u_{22} + 2t_1 \{u_1 - v_1\} + 2t_2 \{u_2 - v_2\}. \end{aligned}$$

If we sum the exponential of this quantity for all integer values of s and t , we shall clearly obtain

$$\theta(u_1 + v_1, u_2 + v_2; 2u_{11}, 2u_{12}, 2u_{22}) \cdot \theta(u_1 - v_1, u_2 - v_2; 2u_{11}, 2u_{12}, 2u_{22}),$$

that is, the product of two θ -series whose parameters are the doubles of those we have been considering. To avoid the trouble of expressing these parameters in every case, it is convenient to denote such θ -series by a capital Θ ; thus

$$\Theta(u_1, u_2) = \theta(u_1, u_2; 2u_{11}, 2u_{12}, 2u_{22}).$$

Next let p_1, q_1 be odd, while p_2, q_2 are even; and let

$$p_1 = 2s_1 + 1, \quad p_2 = 2s_2, \quad q_1 = 2t_1 + 1, \quad q_2 = 2t_2,$$

then

$$\begin{aligned} \phi(m_1, m_2) + \psi(n_1, n_2) \\ = 2(s_1 + \tfrac{1}{2})^2 u_{11} + 4(s_1 + \tfrac{1}{2})s_2 u_{12} + 2s_2^2 u_{22} + 2(s_1 + \tfrac{1}{2})\{u_1 + v_1\} \\ + 2s_2\{u_2 + v_2\} \\ + 2(t_1 + \tfrac{1}{2})^2 u_{11} + 4(t_1 + \tfrac{1}{2})t_2 u_{12} + 2t_2^2 u_{22} + 2(t_1 + \tfrac{1}{2})\{u_1 - v_1\} \\ + 2t_2\{u_2 - v_2\}, \end{aligned}$$

and the sum of the exponentials of this quantity for all integer values of s and t is clearly

$$\Theta_{12}(u_1 + v_1, u_2 + v_2) \Theta_{12}(u_1 - v_1, u_2 - v_2)$$

Similarly if p_1, q_1 be even and p_2, q_2 odd, we shall obtain

$$\Theta_{01}(u_1 + v_1, u_2 + v_2) \Theta_{01}(u_1 - v_1, u_2 - v_2),$$

and for all four numbers odd,

$$\Theta_{11}(u_1 + v_1, u_2 + v_2) \Theta_{11}(u_1 - v_1, u_2 - v_2).$$

The product $\theta(u_1, u_2), \theta(v_1, v_2)$ is thus equal to the sum of these four products. To state the proposition with brevity and clearness, we shall mention only one of the two variables in each case, omitting the suffix; thus $\theta(u)$ will stand for $\theta(u_1, u_2)$, and $\Theta(u+v)$ for $\Theta(u_1 + v_1, u_2 + v_2)$. What we have proved, then, is that

$$\begin{aligned} \theta u \cdot \theta v &= \Theta(u+v) \Theta(u-v) \\ &\quad + \Theta_{10}(u+v) \Theta_{10}(u-v) \\ &\quad + \Theta_{01}(u+v) \Theta_{01}(u-v) \\ &\quad + \Theta_{11}(u+v) \Theta_{11}(u-v) \\ &= \Sigma \Theta_{ab}(u+v) \Theta_{ab}(u-v). \end{aligned} \quad (a, b = 0, 1).$$

From this formula we may, by adding half-periods and half-quasi-periods to the arguments u, v , obtain an expression for any such product as $\theta_{ca}^{ab} u \cdot \theta_{\gamma\delta}^{a\beta} v$. The number of such distinct products is 136, including the theorem just stated; the general formula including them all will be subsequently examined. But the case in which $\alpha, \beta, \gamma, \delta = a, b, c, d$ admits of simple treatment and leads to some important consequences.

First, let us adopt the abbreviation

$$\theta\left(u + \frac{ab}{cd}\right) \text{ for } \theta\left(u_1 + \frac{1}{2}a\pi\tau + \frac{1}{2}ca_{11} + \frac{1}{2}du_{12}, u_2 + \frac{1}{2}b\pi i + \frac{1}{2}ca_{12} + \frac{1}{2}da_{22}\right),$$

then we may write

$$\begin{aligned} \theta\left(u + \frac{ab}{cd}\right) &= \epsilon | - \phi\left(\frac{1}{2}c, \frac{1}{2}d\right) | \theta_{ca}^{ab}(u), \\ \theta\left(v + \frac{ab}{cd}\right) &= \epsilon | - \psi\left(\frac{1}{2}c, \frac{1}{2}d\right) | \theta_{ca}^{ab}(v), \\ \Theta\left(u + v + \frac{ab}{cd}\right) &= \epsilon | - \phi\left(\frac{1}{2}c, \frac{1}{2}d\right) | \Theta_{ca}^{ab}(u+v), \end{aligned}$$

where

$$\begin{aligned}\phi\left(\frac{1}{2}c, \frac{1}{2}d\right) &= 1(c^2a_{11} + 2cd a_{12} + d^2 a_{22}) + ca_1 + d a_2, \\ \psi\left(\frac{1}{2}c, \frac{1}{2}d\right) &= 1(c^2a_{11} + 2cd a_{12} + d^2 a_{22}) + ca_1 + d a_2, \\ \Phi\left(\frac{1}{2}c, \frac{1}{2}d\right) &= \frac{1}{2}(c^2a_{11} + 2cd a_{12} + d^2 a_{22}) + c(a_1 + a_2) + d(a_1 + a_2) \\ &= \phi\left(\frac{1}{2}c, \frac{1}{2}d\right) + \psi\left(\frac{1}{2}c, \frac{1}{2}d\right).\end{aligned}$$

Hence by substitution we obtain*

$$\begin{aligned}c &= \phi\left(\frac{1}{2}c, \frac{1}{2}d\right) + \psi\left(\frac{1}{2}c, \frac{1}{2}d\right) \theta_{11}^2(u) \theta_{22}^2(v) \\ &= \Sigma (a_{11} + a_{22} + \frac{2a_{12}}{c-d}) \theta_{11}(u) \theta_{22}(v) \\ &= \Sigma c (ap + bq) \pi i = \Phi\left(\frac{1}{2}c, \frac{1}{2}d\right) \theta_{11}(u) \theta_{22}(v) + \theta_{12}(u) \theta_{21}(v),\end{aligned}$$

and consequently

$$\theta_{11}^2(u) \theta_{22}^2(v) = \Sigma (-1)^{p+q} \theta_{11}(u) \theta_{22}(u) \theta_{12}(v) \theta_{21}(v) \quad (p, q=0, 1).$$

In this equation write u for v ; then

$$\{\theta_{11}^2(u)\}^2 = \Sigma (-1)^{p+q} \theta_{11}(u) \theta_{22}(u) \theta_{12}(u) \theta_{21}(u).$$

It appears from this result that the square of each of the θ -functions is a linear function of the four quantities

$$\theta_{11}(2u), \theta_{12}(2u), \theta_{21}(2u), \theta_{22}(2u);$$

the coefficients being the quantities

$$\theta_{11}(0), \theta_{12}(0), \theta_{21}(0), \theta_{22}(0).$$

It follows therefore that a linear relation may be found between the squares of any five of the θ -functions.

In certain cases, however, it may be shown that such a relation holds between the squares of four of them. Just as each θ vanishes for each of six pairs of values of the arguments, so each such pair of values annuls six θ -functions. Consider four of these six having the same zero, and one other θ which is not annulled by that pair of values. By what we have just proved, there is a linear relation between the squares of these five; but this is impossible unless the coefficient of the fifth be zero, because we can give to the arguments such values that

* On the right-hand side it is to be observed that c, d may really be instead of $2c, 2d$, because of the double parameters of α .

the first four functions shall vanish, but not the fifth. It follows therefore that

If four θ functions vanish for the same pair of values of the variables, their squares are connected by a linear relation.

For example, the four functions

$$\theta(u), \theta^1(u), \theta_{\alpha}^1(u), \theta_{\alpha}^0(u)$$

are all reduced to zero by the values $u = \frac{0, 1}{1, 1}$, that is to say, $u_1 = \frac{1}{2}u_{11} + \frac{1}{2}u_{12}$, $u_2 = \frac{1}{2}u_{11} + \frac{1}{2}u_{12}$. To find the coefficients in the linear relation, let us assume

$$x\theta(u)^2 + y\theta^1(u)^2 + z\theta_{\alpha}^1(u)^2 + w\theta_{\alpha}^0(u)^2 = 0;$$

then we have only to observe that every two of these functions have one other zero in common, so that by giving to the u this pair of values we can find the ratio of two of the coefficients. Thus for

$$u = \frac{10}{11}, \text{ we have } x\theta_{11}(0)^2 + w\theta_{12}(0)^2 = 0,$$

$$u = \frac{11}{10}, \quad \quad \quad y\theta_{11}^1(0)^2 + z\theta_{12}^1(0)^2 = 0,$$

$$u = \frac{01}{01}, \quad \quad \quad x\theta_{11}^1(0)^2 + y\theta_{12}^1(0)^2 = 0,$$

by successive applications of the formula

$$\theta_{\frac{a}{b}}^{\frac{c}{d}}\left(\frac{u}{p}\right) = (-1)^{(c+d)(a+b)}\pi i - \phi\left(\frac{1}{2}\gamma, \frac{1}{2}\delta\right) + \theta_{\frac{c+1}{2}, \frac{d+1}{2}}^{\frac{a+1}{2}, \frac{b+1}{2}}(0).$$

The result is that

$$x + y + z + w = \theta_{11}^1(0)^2 + \theta_{12}^1(0)^2 + \theta_{11}^0(0)^2 + \theta_{12}^0(0)^2,$$

so that the sought relation is

$$x\theta_{11}^1(0)^2 + y\theta_{12}^1(0)^2 + z\theta_{11}^0(0)^2 + w\theta_{12}^0(0)^2 + \theta_{11}^1(u)^2 + \theta_{12}^1(u)^2 + \theta_{11}^0(u)^2 + \theta_{12}^0(u)^2 = 0.$$

With a view to further investigation of the group of six functions $\theta^1, \theta_{\alpha}^1, \theta_{\alpha}^0, \theta_{\alpha}^1, \theta_{\alpha}^0, \theta_{\alpha}^0$, whose characteristics are the sums in pairs of the four characteristics $\frac{10}{00}, \frac{01}{00}, \frac{00}{10}, \frac{00}{01}$, it will be useful to set down here three other relations which

connect triads of their squares with $\theta(u)^2$. They are obtained in precisely the same manner as the one already written down.

$$\begin{aligned} & \dots \theta^{00}(0)^2, \theta^{11}(u)^2 + \theta^{01}(0)^2, \theta(u)^2 - \theta_{11}(0)^2, \theta_{11}^1(u)^2 - \theta_{11}(0)^2, \theta_{11}^1(u)^2 = 0, \\ & \dots \theta^{00}(0)^2, \theta_{11}^{00}(u)^2 + \theta^{01}(0)^2, \theta_{11}^{01}(u)^2 - \theta_{11}^{00}(0)^2, \theta(u)^2 + \theta_{11}(0)^2, \theta_{11}(u)^2 = 0, \\ & \dots \theta^{00}(0)^2, \theta_{11}^{00}(u)^2 + \theta^{01}(0)^2, \theta_{11}^{01}(u)^2 - \theta_{11}^{00}(0)^2, \theta_{11}(u)^2 + \theta_{11}(0)^2, \theta(u)^2 = 0. \end{aligned}$$

These four equations, it will be observed, are not independent; for if they be multiplied respectively by $\theta^{11}(0)^2$, $\theta^{01}(0)^2$, $\theta_{11}(0)^2$ and $\theta_{11}(0)^2$, the result is $\theta(u)^2$ multiplied by

$$-\theta^{11}(0)^4 + \theta^{01}(0)^4 + \theta_{11}(0)^4 + \theta_{11}(0)^4,$$

which vanishes, as may be proved by writing $u = \frac{10}{100}$ in the first equation.

Eliminating the constant multipliers, we obtain

$$\left\{ \begin{aligned} & \dots \theta_{11}^2 + \theta^{12} + \theta_{11}^{12} + \theta_{11}^{12} \\ & \dots \theta^{02} + \theta^2 + \theta_{11}^{02} + \theta_{11}^{02} \\ & \dots \theta_{11}^{02} + \theta_{11}^{02} + \theta_{11}^2 + \theta_{11}^{02} \\ & \dots \theta_{11}^{02} + \theta_{11}^{02} + \theta_{11}^{02} + \theta^2 \end{aligned} \right\} = 0.$$

[Pages of MS. up to this point are numbered 14 to 16 in CHIFFARD's own hand writing; then comes the *Fluction Theorem* not numbered, and some more pages "too incomplete for printing." (C.)]

The Fluction Theorem.

A θ -series may be differentiated in regard to either of the variables; or, as CHIFFARD suggested, the question may be left open, if we write $\theta = x\theta_{11} + y\theta_{11}^1$, where x and y are quantities to be determined at the end of the investigation. This being so, we find

$$\theta(u), \partial\theta(v) = \theta(v), \partial\theta(u) = 2\Sigma\theta_{11}(u+v), \partial\theta_{11}(u-v)$$

$$(x, y = 0, 1),$$

by the same process as that which was used for the product theorem. It is necessary for our subsequent investigation to

extend this theorem so that for $\theta(u)$ we may write $\theta_{cd}^{ab}(u)$. To this end we proceed as follows. We have

$$\theta\left(u + \frac{ab}{cd}\right) = \epsilon \mid -\phi\left(\frac{1}{2}c, \frac{1}{2}d\right) \mid \theta_{cd}^{ab}(u),$$

therefore

$$\partial\theta\left(u + \frac{ab}{cd}\right) = \epsilon \mid -\phi\left(\frac{1}{2}c, \frac{1}{2}d\right) \mid \{\partial\theta_{cd}^{ab}(u) - \theta_{cd}^{ab}(u) \partial\phi\}.$$

Again,

$$\Theta_{pq}\left(u \pm v + \frac{a}{\frac{1}{2}c}, \frac{b}{\frac{1}{2}d}\right) = \epsilon \mid \frac{1}{2}(ap + bq) \pi i \mid \Theta_{pq}^{ab}\left(u \pm v + \frac{1}{2}c, \frac{1}{2}d\right),$$

therefore

$$\partial\Theta_{pq}\left(u - v + \frac{a}{\frac{1}{2}c}, \frac{b}{\frac{1}{2}d}\right) = \epsilon \mid \frac{1}{2}(ap + bq) \pi i \mid \cdot \partial\Theta_{pq}^{ab}\left(u - v + \frac{1}{2}c, \frac{1}{2}d\right).$$

By substitution of these values, the fluxion-theorem becomes when we write in it $u + \frac{ab}{cd}$ for u ,

$$\begin{aligned} & \epsilon \mid -\phi\left(\frac{1}{2}c, \frac{1}{2}d\right) \mid \{\theta_{cd}^{ab}(u) \cdot \partial\theta(v) - \theta(v) \partial\theta_{cd}^{ab}(u) - \theta_{cd}^{ab}(u) \theta(v) \partial\phi\} \\ &= 2\Sigma (-)^{ap+bq} \Theta_{pq}^{ab}\left(u + v + \frac{1}{2}c, \frac{1}{2}d\right) \partial\Theta_{pq}^{ab}\left(u - v + \frac{1}{2}c, \frac{1}{2}d\right) \\ & \quad (p, q = 0, 1). \end{aligned}$$

But by the product-theorem we have also

$$\begin{aligned} & \epsilon \mid -\phi\left(\frac{1}{2}c, \frac{1}{2}d\right) \mid \theta_{cd}^{ab}(u) \theta(v) \\ &= \Sigma (-)^{ap+bq} \Theta_{pq}^{ab}\left(u + v + \frac{1}{2}c, \frac{1}{2}d\right) \Theta_{pq}^{ab}\left(u - v + \frac{1}{2}c, \frac{1}{2}d\right). \end{aligned}$$

Moreover

$$\begin{aligned} & 2\partial \cdot \epsilon^{1\phi} \Theta_{pq}^{ab}\left(u - v + \frac{1}{2}c, \frac{1}{2}d\right) \\ &= \epsilon^{1\phi} \{2\partial\Theta_{pq}^{ab}\left(u - v + \frac{1}{2}c, \frac{1}{2}d\right) + \Theta_{pq}^{ab}\left(u - v + \frac{1}{2}c, \frac{1}{2}d\right) \partial\phi\}, \end{aligned}$$

therefore finally

$$\begin{aligned} & \epsilon \mid -\frac{1}{2}\phi\left(\frac{1}{2}c, \frac{1}{2}d\right) \mid \{\theta_{cd}^{ab}u \cdot \partial\theta v - \theta v \cdot \partial\theta_{cd}^{ab}u\} \\ &= 2\Sigma (-)^{ap+bq} \Theta_{pq}^{ab}\left(u + v + \frac{1}{2}c, \frac{1}{2}d\right) \partial \cdot \epsilon^{1\phi} \Theta_{pq}^{ab}\left(u - v + \frac{1}{2}c, \frac{1}{2}d\right). \end{aligned}$$

*XII.

MOTION OF A SOLID IN ELLIPTIC SPACE*.

I.

THE *second moment* of a solid body in regard to a plane is the sum obtained by multiplying the mass of each particle into the squared sine of its distance from the plane and adding together the products thus formed. The body being referred to a quadrantal tetrahedron, let the equation of the plane be

$$\Sigma \xi x = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 = 0,$$

then the coordinates of any particle p of mass P , being $p_1 p_2 p_3 p_4$, the second moment of the plane is

$$\int P (\Sigma \xi p)^2 = k^2, \text{ suppose,}$$

it being assumed that $\Sigma p^2 = 1$, $\Sigma \xi^2 = 1$, always. Moreover, as only one body will be here considered, its mass may conveniently be taken as the unit of mass; so that the quantity k is to be called the *swing-radius* of the body in regard to the plane ξ .

We may write also

$$k^2 = (1, 1) \xi_1^2 + (2, 2) \xi_1^2 + \dots \\ + 2(1, 2) \xi_1 \xi_2 + \dots$$

$$\text{where } (1, 1) = \int P p_1^2, \text{ etc.; } (1, 2) = \int P p_1 p_2, \text{ etc.}$$

* [At the Meeting of the London Mathematical Society held February 12, 1871, Prof. CHUBB gave an account of a paper on "the Free motion of a Solid in Elliptic Space": I take XII. to be this paper.]

The quantities $(1, 1)$, $(1, 2)$ are the moments and products of inertia of the body in respect of the coordinate planes.

Those planes whose second moment vanishes envelop a surface of the second order $L^2 = 0$ (the null-surface), which determines all the dynamical relations of the body. The six axes of this surface (being the edges of the self-conjugate tetrahedron common to it and the absolute) are called the principal axes of the body. We may for all dynamical purposes substitute for the body a system of four particles having appropriate masses placed at the vertices of any tetrahedron self-conjugate to the null-surface.

If two planes at right angles be drawn through any axis, having k_1 , k_2 for their swing radii, then $k^2 = k_1^2 + k_2^2$ is a constant independent of the orientation of the two planes, and k is called the swing-radius of the body in regard to that axis. Those axes whose swing-radius vanishes are therefore such that the tangent planes drawn through them to the null-quadratic are at right angles; or the lines are harmonically related to the null-quadratic and the absolute. Hence, if the coordinates of an axis are λ_{11} , λ_{22} , etc., we shall have

$$k^2 = \frac{1}{4}(1, 1) + \frac{1}{2}(2, 2)(\lambda_{11}^2 + \text{etc.}) + \frac{3}{4}(1, 2)(\lambda_{11}\lambda_{22} + \lambda_{11}\lambda_{33} + \text{etc.}),$$

which is easily verified.

It must be observed that we have

$$(1, 1) + (2, 2) + (3, 3) + (4, 4) = 1,$$

which gives three relations among the coefficients in the expression for k^2 .

Suppose now that the instantaneous velocity of the body is the twist

$$V = \alpha + \eta i + \zeta j + k\zeta + \omega(\eta i + \zeta j + k\omega).$$

Here α , η , ζ are the component rotations about axes through the origin, and ω , η , ζ component velocities along them. Let a , b , c be the swing radii of the body about the axes i , j , k ; then $\sqrt{1} = a^2$, $\sqrt{1} = b^2$, $\sqrt{1} = c^2$ will be the swing-radii about

ad , oj , nh . Let also r^2 , p^2 , h^2 be the products of inertia in regard to the planes through the origin, P , m^2 , a^2 the products of inertia of the m with the planes of the origin. Supposing for the moment that m is not a vector, the point becomes a mere rotation, and we can find the energy in terms of its axis. Namely, it is $\frac{1}{2}I\omega^2$, where

$$\begin{aligned} I &= (a^2 + o^2 h^2 + h^2 o^2) + h^2 p^2 + p^2 o^2 + (a^2 + o^2 + h^2) m^2 \\ &\quad + f^2 (2a^2 + o^2 + p^2) + (a^2 + o^2 + h^2) g^2 + (a^2 + o^2 + h^2) m^2 \\ &\quad + P^2 (2a^2 + o^2 + p^2) + (a^2 + o^2 + h^2) m^2 + (p^2 + o^2 + h^2) g^2. \end{aligned}$$

Now this expression is evidently twice the kinetic energy of the body, since P is twice its angular velocity about the axis. But it is easy to see that this same expression represents twice the kinetic energy in the general case when m is not a vector. For, assuming it to be a vector, we can express it in the sum of two polar vectors, viz.,

$$\begin{aligned} m &= m_1 + m_2 = \frac{1}{2} \{ (P + p) + (p + q) \} + \frac{1}{2} \{ (P + p) + (p + q) \} \\ &\quad + \frac{1}{2} \{ (P + p) + (p + q) \} + \frac{1}{2} \{ (P + p) + (p + q) \} \\ &= m_1 + m_2, \text{ say, where} \end{aligned}$$

$$m_1 = \frac{1}{2} (P + p) + \frac{1}{2} (p + q) + \frac{1}{2} (p + q) + \frac{1}{2} (p + q),$$

$$\text{then } m_2 = \frac{1}{2} (P + p) + \frac{1}{2} (p + q) + \frac{1}{2} (p + q) + \frac{1}{2} (p + q),$$

$$\text{and } K_1 = \frac{1}{2} (P + p) + \frac{1}{2} (p + q) + \frac{1}{2} (p + q) + \frac{1}{2} (p + q),$$

$$\text{because } m_1 + m_2 = m.$$

But these vectors m_1 and m_2 being polar, the velocities due to them are everywhere at right angles, so that the energy of their resultant is the sum of their energies, that is, it is represented by $K_1 + K_2$.

Now let $\mathbf{r} = r\mathbf{i} + p\mathbf{j} + h\mathbf{k}$, $\mathbf{p} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$, $\mathbf{h} = h\mathbf{i} + r\mathbf{j} + p\mathbf{k}$, then, P being the kinetic energy, $\mathbf{V}\mathbf{P}$ is a vector which we shall call the momentum of the moving body. Namely, it is

$$\begin{aligned} M &= (p^2 + h^2) \mathbf{i} + (p^2 + h^2) \mathbf{j} + (p^2 + h^2) \mathbf{k} \\ &\quad + (h^2 + p^2) \mathbf{j} + (p^2 + h^2) \mathbf{k} + (p^2 + h^2) \mathbf{i} \\ &\quad + (p^2 + h^2) \mathbf{j} + (p^2 + h^2) \mathbf{k} + (p^2 + h^2) \mathbf{i} \\ &\quad + (1 - p^2) \mathbf{i} + (1 - h^2) \mathbf{j} + (1 - h^2) \mathbf{k} \\ &\quad + (1 - h^2) \mathbf{i} + (1 - h^2) \mathbf{j} + (1 - h^2) \mathbf{k} \\ &\quad + (1 - p^2) \mathbf{i} + (1 - h^2) \mathbf{j} + (1 - h^2) \mathbf{k} \\ &\quad + (1 - p^2) \mathbf{i} + (1 - h^2) \mathbf{j} + (1 - h^2) \mathbf{k} \end{aligned}$$

If we write

$$\phi(ix + jy + kz) = \begin{bmatrix} a^2, & h^2, & g^2 \\ h^2, & b^2, & f^2 \\ g^2, & f^2, & c^2 \end{bmatrix} 0 \div yz 0 \div jk),$$

then the expression for M may be written

$$M = \phi(x + \omega t + \phi) / l + a (\dot{\phi} + \omega t),$$

where

$$a = l^2 + m^2 j + n^2 k,$$

or

$$\begin{aligned} M &= \frac{1}{2} (x + \omega t) + (\phi + \frac{1}{2}) (x + \omega t) + \omega a (x + \omega t) \\ &= \frac{1}{2} V + (\phi + \frac{1}{2} + \omega a) (x + \omega t) \\ &= \frac{1}{2} V + X (x + \omega t), \end{aligned}$$

where

$$\begin{aligned} X &= (a^2 + \frac{1}{2}) \div, & h^2 + \omega m^2, & g^2 + \omega m^2, \\ & h^2 + \omega m^2, & b^2 + \frac{1}{2} \div, & f^2 + \omega l^2, \\ & g^2 + \omega m^2, & f^2 + \omega l^2, & c^2 + \frac{1}{2} \div. \end{aligned}$$

The rate of variation of M is equal to the resultant impressed wrench; now in the case where the impressed forces have a potential V , this resultant wrench is $-XV$. Hence the equation of motion is

$$\dot{M} + XV = 0,$$

wherein M is understood to be expressed as a linear function of V . The kinetic energy T satisfies the equation

$$S, VM + 2T = 0,$$

II,

In the case where there are no forces, V is a constant, and we have

$$\dot{M} = 0,$$

or

$$M = \text{a constant motion.}$$

Hence the expression for M as a linear function of V gives us six first integrals of the equation of motion. We have also $S, VM + 2T = 0$, which, now that T and M are constant, gives us a seventh relation among the components of V . This may, however, be deduced from the other six.

$$\begin{aligned}
\frac{d\langle l^2 \rangle}{dt} &= \frac{d}{dt} \int P p_1 p_4 = \int P (p_1 \dot{p}_4 + \dot{p}_1 p_4) \\
&= u \int P (p_4^2 - p_1^2) + y \int P p_1 p_4 - z \int P p_2 p_4 \\
&\quad - v \int P p_1 p_3 + w \int P p_1 p_3 \\
&= (1 - b^2 - c^2) u + n^2 y - m^2 z - h^2 v - g^2 w,
\end{aligned}$$

and from these the other six may be written down by symmetry.

Now let the moving axes be taken to be the principal axes of the body. Then all the products of inertia vanish, and we have

$$\begin{aligned}
M &= a^2 i + b^2 j + c^2 k + \omega \{ (1 - a^2) u + (1 - b^2) v + (1 - c^2) w \} \\
\frac{1}{2} \frac{d\langle a^2 \rangle}{dt} &= 0, \\
\frac{d\langle f^2 \rangle}{dt} &= (c^2 - b^2) x, \quad \frac{d\langle g^2 \rangle}{dt} = (a^2 - c^2) y, \quad \frac{d\langle h^2 \rangle}{dt} = (b^2 - a^2) z, \\
\frac{d\langle l^2 \rangle}{dt} &= (1 - b^2 - c^2) u, \quad \frac{d\langle m^2 \rangle}{dt} = (1 - c^2 - a^2) v, \quad \frac{d\langle n^2 \rangle}{dt} = (1 - a^2 - b^2) w.
\end{aligned}$$

Now by differentiating the coefficients of i and ωi in the value of M on page 380, we find

$$\frac{d}{dt} (a^2 x + h^2 y + g^2 z + n^2 v - m^2 w) = 0,$$

$$\frac{d}{dt} \{ (1 - a^2) \cdot u - h^2 v - g^2 w - n^2 y + m^2 z \} = 0,$$

that is to say

$$\begin{aligned}
0 &= a^2 \dot{x} + (b^2 - a^2) yz + (a^2 - c^2) yz + (1 - a^2 - b^2) vw - (1 - c^2 - a^2) vw \\
&= a^2 \dot{x} + (b^2 - c^2) (yz - vw),
\end{aligned}$$

and

$$\begin{aligned}
0 &= (1 - a^2) \dot{u} - (b^2 - a^2) zv - (a^2 - c^2) yw - (1 - a^2 - b^2) yw \\
&\quad + (1 - c^2 - a^2) zv \\
&= (1 - a^2) \dot{u} + (1 - c^2 - b^2) (zv - yw).
\end{aligned}$$

These equations take the place of Euler's.

From these we get, in the first place,

$$\begin{aligned} a^2\dot{x}\dot{x} + b^2\dot{y}\dot{y} + c^2\dot{z}\dot{z} &= (b^2 + c^2).xrv + (c^2 + a^2).ywu + (a^2 + b^2).zuv, \\ (1 + a^2).u\dot{u} + (1 + b^2).v\dot{v} + (1 + c^2).w\dot{w} \\ &= -(1 + b^2 + c^2).(xuv + ywu) \\ &\quad + (1 + c^2 + a^2).(xrv + zuv) \\ &\quad + (1 + a^2 + b^2).(ywu + xrv) \\ &= (c^2 + b^2).xrv + (a^2 + c^2).ywu + (b^2 + a^2).zuv; \end{aligned}$$

$\therefore a^2\dot{x}\dot{x} + b^2\dot{y}\dot{y} + c^2\dot{z}\dot{z} + (1 + a^2).u\dot{u} + (1 + b^2).v\dot{v} + (1 + c^2).w\dot{w} = 0$,
whence $a^2\dot{x}^2 + \text{etc.} + (1 + a^2).u^2 + \text{etc.} = 2T$, the equation of energy.

*XLII.

[FURTHER NOTE ON BICQUATERNIONS.]

I.

THE two expressions "twice three are six" and "six is the product of two and three," represent two different views of multiplication, although they are written down by the same shorthand formula

$$2 \times 3 = 6.$$

Accordingly they give two different interpretations of this formula.

In the first interpretation 3 is a concrete number of things, say three marbles, while 2 is not a number but an operation, namely the operation of doubling; and we may read the equation "doubling three marbles makes six marbles."

The second interpretation regards 2 and 3 as abstract numbers, and affirms the existence of a third number 6 having a definite relation to them which it is convenient to study, this third number so related being called their product; and various meanings given to the numbers 2 and 3 may lead to various concrete interpretations of the formula. Each of these views of multiplication may be extended to other things besides numbers; I propose at present to consider certain extensions of the first view.

In this we have regarded 2 as a symbol of operation, 3 as a concrete number, and 6 as a concrete number. But we may also regard all three symbols as symbols of operation, and so read the formula "doubling the triple of anything makes the

nextuple of it." Such an equation as *above* will then have two meanings:

1. *a* times *b* times *c* things makes *d* things;
2. *a* times *b* times *c* times anything makes *d* the things.

That is to say, we may regard the *last* symbol in each the equation as either a concrete number or a set operation; but all the others must be regarded as syn operations.

To extend this from concrete numbers to steps of \pm or subtraction is not difficult, but it requires care to give a meaning to the signs \pm and $-$, as well as to all our symbols. The first meaning is to indicate the direction step; thus ± 3 means a step of 3 forward, i. e. an addition; -3 means a step of 3 backward, i. e. a subtraction. To these symbols are attached to the operations performed steps, they mean *retaining* and *reversing* respectively; the equation

$$23 \pm 3 = 26$$

has two meanings:

1. Doubling a step of 3 forward and retaining it; a step of 3 backward.
2. To triple a step and retain its direction, then to add reverse it, is the same as four steps and reverse it.

These steps of addition and subtraction may be regarded changes of position, or vectors, on a straight line, along all numbers are supposed to be ranged; and by each number be continuous operation we may deal in this of all vectors in a straight line. In every equation we may the *last* symbol in every term as either a vector or an op but all the others must be regarded as operations.

Assuming the law of addition of vectors in $A + B = C$, we find at once the interpretation of *imaginary* or *impossible* quantities in the operations add vector one vector into another. Thus [Fig. 32] if A is a

a vector turns it counter-clockwise through a right angle, so that $I \cdot OA = OA'$, and if

$$a = \frac{OM}{OA}, b = \frac{MB}{OA'},$$

a and b being ratios of vectors in a line as just previously defined, then

$$OB = OM + MB = a \cdot OA + b \cdot OA' = (a + bI) OA,$$

and it is clear that $I^2 = -1$. Thus every expression of the form $a + bI$ is the ratio of two vectors.

Every vector in the plane may be represented by $a \cdot OA + b \cdot OA'$ by giving to a and b proper values. For shortness we may write $OA = j$, $OA' = k$; then $Ij = k$, $Ik = -j$, and we shall have

$$(a + bI)(cj + dk) = (ac - bd)j + (ad + bc)k.$$

In this way we may have to consider two classes of expressions, those in which the last symbol in every product is a vector, and all the others are ratios of vectors; and those in which all the symbols represent ratios of vectors. But, observing that

$$cj + dk = (c + dI)j;$$

we may make the useful convention that j is, if convenient, to be understood as written after every term; so that the complex symbol $c + dI$ will now mean *either* a ratio of two vectors as above, *or* the vector $cj + dk$. And then every expression will have a double meaning as before, and we shall have only one kind to deal with.

This artifice amounts to taking a definite vector as the unit, and representing all others by means of their ratios to the unit. The success of the artifice depends on the fact that the product of two such ratios is another ratio of the same kind.

Passing now to vectors in space, we shall find again that the operation which makes one into another is of the form $a + bQ$, where Q turns through a right angle *in the plane of the two vectors*. It will not, therefore, operate on any vector out of that plane; and the variety of these operators Q is the same as the variety of planes, or is doubly infinite. We may represent Q

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as a sort of handle or axis of unit length perpendicular to the plane; and the compound operation bQ , which turns through a right angle and increases in the ratio $1 : b$, may be represented by an axis of length b . This being so, let Q, R be two such compound operations; there is *one* vector α on which they will both operate, namely, the intersection of their planes. It is found that $Q\alpha + R\alpha$ is a vector at right angles to α , and that if S is the rectangular versor which converts α into $Q\alpha + R\alpha$, so that $S\alpha = Q\alpha + R\alpha$, then the axis of S is got by adding the axes of Q and R as if they were vectors. So we write $S = Q + R$, and we have the equation

$$(Q + R)\alpha = Q\alpha + R\alpha.$$

By the law of formation of $Q + R$ it is clear that

$$P + (Q + R) = (P + Q) + R = (P + R) + Q = \text{etc.},$$

this being a rectangular versor whose axis is the vector-sum of the axes of P, Q, R . Accordingly it is called $P + Q + R$; although the equation

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does not admit of interpretation in general, because there is no vector α which is capable of being operated on by P, Q , and R .

Thus every rectangular versor may be represented by the form $xI + yJ + zK$, where IJK are the three rectangular versors whose axes are the unit-vectors ijk . And we have two kinds of complex quantities to consider; *vectors*, of the form $\rho = ai + bj + ck$, and *quaternions*, of the form $q = w + xI + yJ + zK$. The product of any number of quaternions is itself a quaternion, the units IJK being multiplied by the rules

$$IJ = K = -JI, \quad KI = J = -IK, \quad JK = I = -KJ, \\ I^2 = J^2 = K^2 = -1.$$

But we cannot multiply a vector by a quaternion in general; $q\rho$ will only have a meaning if ρ is perpendicular to the axis of q , or, which is the same thing, if $ax + by + cz = 0$. And even then, although we have the formulæ

$$Ij = k = -Ji, \quad Ki = j = -Ik, \quad Kk = i = -Kj,$$

we cannot find the value of qp by direct multiplication, for the symbols Ii , Jj , Kk are unmeaning. If, however, we assume that they have the *same* value, the result of direct multiplication will come out right whenever qp is interpretable.

The artifice, by which in the geometry of two dimensions the two kinds of complex quantities were reduced to one, is not applicable here. For although we may write

$$ai + bj + ck = (a + bK - cJ) i,$$

and so represent every vector by its ratio to the unit i , yet it no longer remains true that the product of two such ratios is another ratio of the same sort. But we may attain the desired reduction by a simpler method; viz., by using the symbols ijk in a double sense, as vectors and as versors. Thus in the product qp , if p be regarded as a rectangular versor, the true value may be obtained by direct multiplication

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But in complex numbers, which deal with two dimensions, the artifice is essentially different; and that which, by a convenient inaccuracy, may be called the *product of two vectors*, has very different geometrical relations to its components. It will be shown further on that all geometric algebras dealing with an *odd* number of dimensions resemble scalars and quaternions in this respect; while those dealing with an *even* number of dimensions resemble complex numbers.

It is clear that the versors IIK may be represented on great circles of a sphere whose centre is the origin; and if these be regarded as steps on the surface of the sphere, it will be found that the consideration of their ratios leads to the whole theory of quaternions. In this interpretation *vectors* can only be represented by points on the sphere supposed to have definite *weights* attached to them, proportional to the length of the corresponding vectors. Here then we have the geometric algebra of three dimensions interpreted by means of a space of two dimensions which has constant positive curvature, namely the surface of a sphere.

In the same way we may interpret the algebra of a space of four dimensions, which cannot be imaged, by means of a space of three dimensions having constant positive curvature, of which a clear mental picture may be formed.

Consider any vertical line, and a series of horizontal planes cutting it at right angles. In ordinary or Euclidian geometry these planes intersect on the *horizon*, which is a straight line infinitely distant. In the geometry of a space of constant positive curvature, or *elliptic* geometry, the horizon is at a certain finite distance in all directions from the vertical line with which we started; it belongs to that particular line, which is called its *polar*, and is not the same for all vertical lines. Although it appears to be a great circle when viewed from the neighbourhood of its polar, yet if we were to go to it and examine it we should find it straight. Points of it which are in opposite directions from a point on the polar are really identical; and every straight line in this space resembles a circle in being of finite length, so that if we travel far enough along it we shall arrive at our starting point. Every straight line

has a polar line, which is the intersection of all planes at right angles to it.

Let us take a very small circle on a sphere, and suppose it to expand, keeping always the same centre. At the beginning the circle will be concave inside and convex outside; but when the expansion has gone on far enough it will become a great circle of the sphere, which is of the same shape on both sides, or is *straight* so far as the surface of the sphere is concerned. So if in Euclidian space we take a sphere and suppose it to expand, keeping always the same centre, it will continue to be concave inside and convex outside so long as it is finite; but when the radius has become infinite, the inside in one direction is the same as the outside in the opposite direction, opposite points being identical; thus the sphere is of the same shape on both sides, or is a *plane*, viz., the plane at infinity. In elliptic space, just as in geometry on the surface of a sphere, this takes place for a *finite* length of the radius, not for an infinite length; for every point there is a sphere having its centre at that point, which is also a plane. Or, which is the same thing, every point has a polar plane which is the locus of all points situate at a certain distance from it; this distance is called a *quadrant*. So also every plane has a certain point, called its *pole*, which is distant a quadrant from every point in the plane. All lines and planes perpendicular to the plane pass through its pole, and conversely. The polar lines of all lines in the plane pass through its pole, and so do the polar planes of all points in the plane.

When two lines are polars of one another, every point of one is distant a quadrant from every point of the other; hence the polar planes of all points on one pass through the other. Every line which is at right angles to one meets the other, and conversely.

In the *Proceedings of the London Mathematical Society*, [Vol. IV. p. 381—395*] I have given a sketch of a geometric algebra adapted to this elliptic geometry of space, which I have there called Biquaternions.

In the interpretation of quaternions on the surface of a sphere, rectangular versors are represented by quadrants of

* [XX. *supra*.]

great circles. We may represent such a vector as represented by a tensor, for example, *et*, by an arc AB measured on the great circle *i*, so that $\tan AB = e$. This being so, AB differs from a vector in a plane in a most important way, for while a vector in a plane is unaltered by being moved parallel to itself in any direction, AB can only be slid along the great circle, and must not be moved out of it. We shall leave the construction of similar quantities in the elliptic geometry of three dimensions, namely quantities represented by a length measured off on a certain straight line, which are unaltered when the length is slid along the line but not in any other way. They are as it were *vectors having position*. A vector represents the translational velocity of a rigid body, which is everywhere the same, these quantities will represent the rotation velocity of a rigid body, which is about a certain stationary axis. For this reason I have called them *rotors* (short for *rotation*) by analogy with Hamilton's word *vector*. They are added together according to the law of composition of forces and rotations. That is to say, if a rotor P [fig. 54] along OA added to a rotor Q along OB gives a rotor R along OC making angles α, β with the lines, then $\cos \theta$ is in the plane of OA, OB , and

$$P : (Q : R = \sin \theta) : \sin \theta \text{ (see fig. 55),}$$

this determines the position and magnitude of the resultant. We cannot use the parallelogram construction on the addition of vectors, for the elliptic geometry of the plane ABH is the same as that of the surface of an sphere whose opposite points are regarded as identical, and any parallelogram can be drawn on it.

Since any two great circles of a sphere meet at two antipodes (in one point, according to our present conventions) at least one that any two rotors have a single rotor which is their resultant or sum. But in three dimensions this is not the case; if the axes of two rotors do not meet there will be no vector equal to any one rotor. We may however find two other rotors which have the same sum, and that in an infinite number of ways. Of these ways one is of the greatest importance, namely, that in which the axes of the two rotors are perpendicular to each other. If

we regard each rotor as representing rotation about its axis, each of these rotations is equivalent to a translation along the other axis. Thus rotation about a vertical line is translation along the horizon, and *vice versa*. Hence the resultant of the two rotations may be regarded as a screw motion about either of the axes. As this describes the most general motion of a rigid body, I have proposed to call the quantity which represents it a *motor*. We shall say then that the sum of two rotors which do not meet is a motor, and that every motor has two axes which are polars of one another.

This being so, let three planes at right angles be drawn through a point O , and let unit rotors along their intersections be denoted by ijk . Then any rotor through the point O is denoted by $ai + bj + ck$. The ratio of two such rotors is a quantity of the form $x + yi + zj + wk$, if we let ijk mean also rectangular versors whose axes are the rotors ijk . In fact we are merely applying the results of quaternions to vectors passing through a fixed point and their ratios.

Let now ω be the operation which converts any rotor into an equal rotor along the polar line of its axis. Then $\omega i, \omega j, \omega k$ [fig. 55] will be rotors along the lines of intersection of the polar plane of O with the three rectangular planes through O . And since (as it is easy to see) any rotor may be resolved into two, one passing through O , and the other lying in the plane PQL , whereof the former is compounded of ijk , and the latter of $\omega i, \omega j, \omega k$, it follows that the expression for the sum of any number of rotors (i.e. for a motor) is of the form

$$ai + bj + ck + \omega (fi + gj + hk) = \alpha + \omega \beta, \text{ say.}$$

Suppose now that the *versors* ijk are allowed to operate, not only on rotors through O which meet their axes, but on any rotors which meet them at right angles. Then we shall have

$$i(\omega j) = \omega k = \omega ij; j(\omega k) = \omega i = \omega jk; k(\omega i) = \omega j = \omega ki,$$

which equations show that ω is *commutative* with the symbols i, j, k .

Moreover, if ω turns a rectangular versor into a similar

NOTES ON BICQUATERNIONS.

1. AXES of motor $\alpha + \omega\beta$, or $\xi\gamma + \eta\delta$.

If μ be an axis of a motor, the motor is a multiple of the rotor μ together with a multiple of the polar rotor $\omega\mu$. Thus we may write

$$\xi\gamma + \eta\delta = (h + \omega k) \mu.$$

Operating by ξ, η , we have

$$\xi\gamma = (h + k) \xi\mu,$$

$$\eta\delta = (h - k) \eta\mu,$$

whence

$$\mu = \frac{\xi\gamma}{h+k} + \frac{\eta\delta}{h-k}.$$

Thus we have an indeterminate representation of any motor as the sum of two polar motors. But if μ is a *rotor*, we have

$$\frac{T\gamma}{h+k} = \frac{T\delta}{h-k},$$

whence

$$\frac{h}{k} = \frac{T\gamma + T\delta}{T\gamma - T\delta},$$

so that

$$\mu = \xi \cdot U\gamma + \eta \cdot U\delta,$$

and

$$\omega\mu = \xi \cdot U\gamma - \eta \cdot U\delta.$$

But the component rotors along the axes are

$$(T\gamma \pm T\delta) (\xi \cdot U\gamma \pm \eta \cdot U\delta).$$

For the sum of these is easily seen to be equal to the original motor $\xi\gamma + \eta\delta$.

* [These "Notes" were apparently written when Prof. Clifford was in the enjoyment of vigorous health, and occupy three pages of MS. pinned together. They are, I should say, intended to be supplementary to his *London Mathematical Society* paper, XX. supra.]

2. Sum of rotors of different pitches whose axes meet at right angles.

Let the rotors be $(a + ab)j$ and $(c + cd)k$; the sum is of course

$$aj + ck + a(lj + dk)$$

or

$$\frac{1}{2} \xi (a + b, j + c + d, k) + \frac{1}{2} \eta (a - b, j + c - d, k).$$

Thus an axis of the sum is

$$\xi \frac{a + b, j + c + d, k}{\sqrt{(a + b)^2 + c^2 + d^2}} + \eta \frac{a - b, j + c - d, k}{\sqrt{(a - b)^2 + c^2 + d^2}}.$$

Now this is a rotor at right angles to i and at an angular distance θ from the origin such that

$$\cos 2\theta = \frac{a^2 + b^2 + c^2 + d^2}{\sqrt{(a + b)^2 + c^2 + d^2} \sqrt{(a - b)^2 + c^2 + d^2}}$$

$$\tan 2\theta = \frac{2(b - ad)}{a^2 + b^2 + c^2 + d^2}.$$

*XLIII.

ON THE CLASSIFICATION OF GEOMETRIC ALGEBRAS*.

- 1806 Argand, *Méthode de représenter les quantités imaginaires*,
Paris, *Mém. sur les qu. imag.*
- 1827 Möbius, *Barycentrischer Calcul*.
- 1831 Grassman,
- 1834 Peacock, *Doctrine of Operations in Algebra*.
- 1843 Hamilton, *Quaternions*.
- 1844 Grassmann, *Lineale Ausdehnungslehre*.
- 1845 Babinet-Venant, *Multiplication of vectors*.
- 1846 Kirkman, *Plückerian and Hamilton Products*.
- 1853 Cauchy, *Chiffre Algébriques*.
- 1863 Grassmann, *Ausdehnungslehre*.
- 1870 Peirce, *Linear Associative Algebra*.

In the Barycentric Calculus a point is represented by a complex number which is a linear assemblage of symbols each representing a fixed point; the coefficients are coordinates. By regarding *ab* the ordinary symbol for a line joining two points, *ac* of the nature of a product, and *ca* distributive, we arrive at Grassmann's extensive quantities. For then we must put *aa*=0, i.e., $(a_1, \dots, a_n)^2 = 0$, which requires a_1, \dots, a_n , and then *abca*=*abc* always. If there are *n* independent units, we may consider in such an algebra scalars or quantities of order 0, *a* of order 1, $\frac{1}{2}n$, (*n*-1) of order 2, etc., 1 of order *n*; in all 2^n *bits*, to borrow Peirce's term. Every intelligible expression is however homogeneous; if a product of points, the coefficients are determinants.

In the theory of Quaternions the symbols *ijk*, when used as multipliers, represent not things but operations of turning; thus $i^2 = -1$ and not 0. But regarding them as vectors, we may use them to represent the geometry of the plane passing through the ends of *ijk*, on the principles of the barycentric calculus. Thus $px + (y + z)$ will represent the point where it cuts the plane, with a weight $x + y + z$. The Grassmann algebra will be reproduced if we attend only to the vector part of binary products, and the scalar part of ternary. Physical considerations however lead us to regard i^2 as a scalar (not zero) even

* [The "forewords" are the abstract which Prof. Clifford communicated to the *London Mathematical Society*, on March 10th, 1876 (*Proceedings*, Vol. vii, p. 115). The paper, which is unfinished, was found amongst his MSS.]

when i is regarded as a vector (not a versor). For these purposes it does not matter whether it is put $= -1$ or $+1$.

I propose here to extend this assumption to the Grassmann representation in general: i.e., I take n units $i_1 i_2 \dots i_n$ such that $i_i^2 = +1$, and $i_i i_j = -i_j i_i$. All products of linear factors must therefore contain either terms of odd order only, or terms of even order only. There can be no term of higher order than the n^{th} , and the whole number of terms is 2^n as with Grassmann; i.e. in both cases we have a linear associative algebra of 2^n units, but the Grassmann algebra is *nilpotent*, and only homogeneous forms occur; while this is *idempotent*, and admits of *odd* forms and *even* forms, which are not in general homogeneous. It is convenient to use *selective* symbols $V_0, V_1 \dots V_n$ analogous to Hamilton's S and V , for picking out those parts of any expression which are of order $0, 1 \dots n$ respectively.

The Quaternion symbols satisfy the equation $ijk = -1$, and this together with the assumption $\rho^2 = \text{scalar}$ gives all the laws of their multiplication. If we put $\omega = i_1 i_2 \dots i_n$, this means that in the case $n=3$ we may take ω to be a scalar. There is here a very important distinction between the cases n odd and n even; in the former case ω is commutative with the symbols i , or $\omega i = i\omega$, in the latter case $\omega i = -i\omega$. Hence when n is odd ω acts as a scalar, when n is even it acts as a vector. In putting $\omega =$ to a scalar in the former case we are conveniently representing two different things by the same symbols, because they have the same laws of combination. We thus reduce the algebra to 2^{n-1} units when n is odd. When n is even the symbol ω is of even order. In all cases we may if we like consider separately the even units and the odd units; the former make an algebra by themselves, and by restricting ourselves to these we get the same result as by putting $\omega = \pm 1$. Thus, $n=3$, the even algebra gives quaternions at once. When n is even we may still further simplify; for the symbol ω belongs to the even algebra, and this consists of 2^{n-2} terms (i.e. the even terms up to order $n-2$ and half of these) together with the products of these terms by ω , which is commutative with them. Thus $n=4$, the even algebra gives biquaternions, and the general expression is $q + \omega r$ where q, r are quaternions. For n odd, we may represent the odd algebra by the even algebra; this amounts to making $\omega = \pm 1$.

THE extensive quantities of Grassmann, which Hankel has called alternate numbers, namely symbols which have the property of *polar* multiplication $ab = -ba$ and whose square vanishes, $a^2 = 0$, serve to conveniently represent the projective geometry of n dimensions. In plane geometry, for example, let the symbols t_1, t_2, t_3 represent three points; then

$$a = a_1 t_1 + a_2 t_2 + a_3 t_3$$

will represent a point which is the centre of inertia of masses a_1, a_2, a_3 placed at the fundamental points respectively. If the

products t_2t_3, t_3t_1, t_1t_2 be taken to mean the lines joining the fundamental points, then the product

$$ab = (a_1t_1 + a_2t_2 + a_3t_3)(b_1t_1 + b_2t_2 + b_3t_3) \\ = (a_2b_3 - a_3b_2)t_1t_2 + (a_3b_1 - a_1b_3)t_2t_3 + (a_1b_2 - a_2b_1)t_1t_3$$

represents the line joining the points a, b ; for the coefficients of the binary products t_it_j, \dots are clearly the coefficients in the equation of that line referred to the fundamental triangle.

In like manner the ternary product

$$abc = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} t_1t_2t_3$$

is proportional to the area of the triangle abc , and vanishes when the three points are collinear. And so on.

The system of quaternions differs from this, first in that the squares of the units, instead of being zero, are made equal to -1 ; and secondly in that the ternary product $t_1t_2t_3$ is made equal to -1 . The interpretation is at the same time extended to three dimensions, but with this restriction: that whereas the alternate units represent *any* three points in a plane, and the system deals primarily with projective relations, Hamiltonian units represent three vectors at right angles, and the system is the natural language of metrical geometry and of physics.

I shall now examine the consequence of making, in a system of n alternate numbers t_1, t_2, \dots, t_n , the *first* of the modifications just named; namely I shall suppose that the square of each of the units is -1 . I shall then enquire whether any assumption can be made which is analogous to the second Hamiltonian law, $t_1t_2t_3 = -1$.

In the system of Grassmann every product of linear factors is homogeneous in the units; namely the product of m linear factors is a linear function of the m -ary products when m is not greater than n , and is zero when $m > n$. But if we make the square of every unit equal to -1 , it is clear that the product of m factors (m not greater than n) may contain terms of the

order $m, m-2, m-4, \dots, m-2r, \dots$, since the order is reduced by 2 for every substitution of -1 for ϵ^2 . Consequently every such product consists of terms which are either all of odd order or all of even order. The same thing is true when m is greater than n , except that the highest order which can occur is either n or $n-1$, according as $m-n$ is even or odd. The products of an even number of the units will therefore form an algebra complete in itself. We have altogether one term of order 0, n of order 1, $\frac{1}{2}n(n-1)$ of order 2, ... one of order n ; that is to say $1 + n + \frac{1}{2}n(n-1) + \dots + n + 1 = 2^n$ terms. Moreover we have

$$\begin{aligned} & \text{terms of even order} - \text{terms of odd order} \\ &= 1 - n + \frac{1}{2}n(n-1) - \dots + (-1)^n = (1-1)^n = 0, \end{aligned}$$

or the number of terms in the even algebra is equal to the number of terms in the odd algebra, namely 2^{n-1} .

Let us now write ω for the product of all the units, $\omega = \epsilon_1 \epsilon_2 \dots \epsilon_n$, and enquire into the value of ω^2 . This is

$$\omega^2 = \epsilon_1 \epsilon_2 \dots \epsilon_n \epsilon_1 \epsilon_2 \dots \epsilon_n,$$

which after p interchanges of contiguous symbols can be transformed into

$$\omega^2 = (-)^p \epsilon_1^2 \epsilon_2^2 \dots \epsilon_n^2 = (-)^{n(n-1)/2}.$$

To find the value of p , we observe that it takes $n-1$ changes to bring the second ϵ_1 between the first ϵ_1 and ϵ_n , then $n-2$ changes to bring the second ϵ_2 between the first ϵ_n and ϵ_n , and so on. Therefore $p + n = 1 + 2 + \dots + n = \frac{1}{2}n(n+1)$; and it follows that $\omega^2 = \pm 1$ according as $\frac{1}{2}n(n+1)$ is even or odd, or according as n belongs to the forms $4m, 4m+3$ or $4m+1, 4m+2$.

Next let us observe that $\epsilon_1 \omega = -\epsilon_2 \epsilon_3 \dots \epsilon_n$, but $\omega \epsilon_n = (-)^n \epsilon_2 \epsilon_3 \dots \epsilon_n$; that is to say, $\epsilon_1 \omega = \pm \omega \epsilon_1$ according as n is odd or even. Thus the multiplication of ω with the units is polar or commutative according as n is even or odd.

We are thus led to distinguish four classes of geometric algebra, which are characterized by the sign of ω^2 and the nature of the multiplication $\omega \epsilon$.

Class I. $n \equiv 0 \pmod{4}$. $\omega^2 = 1$, $\omega\iota = -\iota\omega$. The even algebra contains the symbol ω , which is commutative with all even terms. Since every term of order $r > \frac{1}{2}n$ can be expressed as the product of ω into a term of order $n - r$, while the terms of order $\frac{1}{2}n$ divide themselves into complementary pairs, each of which is ω multiplied by the other, this algebra contains 2^{n-2} symbols, together with their products by ω . Thus for $n = 4$, the most general expression in the even algebra is

$$w + w\iota_3\iota_2 + y\iota_3\iota_1 + z\iota_1\iota_2 + \omega(w' + w'\iota_2\iota_3 + y'\iota_3\iota_1 + z'\iota_1\iota_2) = q + \omega r,$$

which is precisely what I have elsewhere called a *biquaternion*, because the products $\iota_3\iota_2$, $\iota_3\iota_1$, $\iota_1\iota_2$, satisfy the laws of the Hamiltonian symbols i , j , k , and therefore the quantities q , r are quaternions in the ordinary senso.

The most general expression in the odd algebra consists also of eight terms, namely it is

$$w\iota_1 + y\iota_2 + z\iota_3 + w'\iota_4 - \omega(x'\iota_1 + y'\iota_2 + z'\iota_3 + w\iota_4).$$

The coefficients are purposely so arranged as to bring out the fact that this expression may be derived from the preceding by multiplying it by $\iota_4\omega$. It does not, however, follow that we may always represent the vector $w\iota_1 + y\iota_2 + z\iota_3 + w'\iota_4$ by the proportional quantity of the even algebra

$$w\iota_3\iota_2 + y\iota_3\iota_1 + z\iota_1\iota_2 + w'\omega.$$

For if α , β be any two vectors, the quantity $\iota_4\omega\alpha \cdot \iota_4\omega\beta$ is not in general proportional to $\alpha\beta$. This would require that the multiplication of α or β with $\iota_4\omega$ should be either polar or commutative; whereas in general it is neither.

Class II. $n \equiv 1 \pmod{4}$ $\omega^2 = -1$, $\omega\iota = \iota\omega$. The odd and even algebras may be included in the same formulæ by putting ω equal to the scalar $\sqrt{-1}$; the case is thus closely analogous to that of $n \equiv 3 \pmod{4}$.

Class III. $n \equiv 2 \pmod{4}$, $\omega^2 = -1$, $\omega\iota = -\iota\omega$. Here ω has clearly the properties of a unit vector, and the system may always be treated as a degenerate case of the next. The general expression of the even algebra may be got from the general expression of the odd algebra by multiplying it by $\omega\iota_n$.

*XLIV

ON THE THEORY OF SCREWS IN A SPACE OF CONSTANT POSITIVE CURVATURE

Polar Lines.

THE word *line* will be here used to mean *great circle* except where ambiguity might occur. To every line in a space of constant positive curvature corresponds a polar line such that every point on it is distant a quadrant from every point on the line. This is in fact its polar line in regard to the *absolute*, a quadric whose equation referred to a quadrantal tetrahedron is

$$x^2 + y^2 + z^2 + w^2 = 0.$$

Translation = rotation about polar (Klein).

The rotation of a rigid body about any line is a sliding of the body along the polar line; and conversely a translation or sliding without twist along any line is a rotation about the polar line. A translation therefore in such a space is not merely a vector having magnitude and direction, but a quantity having also a definite position.

Axis of Screw.

The most general motion of a rigid body is made up of translation along a certain axis and rotation about it; or we may say that it is made of rotations about two polar lines. These factors in the motion may be combined in any order without affecting the result. Let a and a' be two polar rect-

angular versors (rotations through a right angle about two polar lines), then the product

$$\alpha^p \alpha_1^q$$

is the most general representation of the motion of a rigid body.

This may be regarded either as a twist $p \frac{\pi}{2}$ about a screw whose

axis is α and pitch $\frac{q}{p}$, or as a twist $\frac{q\pi}{2}$ about a screw whose axis

is α_1 and pitch $\frac{p}{q}$. Thus a screw has two axes which are polars

of each other, and its pitch is only completely defined when we have picked out one of these axes for attention.

Rotor Sum.

A *rotor* has magnitude, direction, and position. The rotor AB will equal the rotor CD when they are in the same line, of equal length, and of like sense. Two rotors therefore will not in general meet; and the sum of two rotors will in general be a quantity of the nature of a twist or a wrench, which I call a *motor*. If we suppose a rigid body to have rotation velocities about these rotors proportional to their lengths, the resultant of these will be a twist velocity about the screw which is their sum.

$$\text{Rotor Ratio} = \text{Twist} \times \text{Tensor}.$$

The *ratio* of two rotors is a twist multiplied by a tensor. For there are two polar lines which cut the two rotors at right angles; hence one vector can be made into the other by rotations about these lines combined with an alteration of length.

Scale of quantities.

We have thus a certain scale of quantities, each of which is obtained by connecting the notion of magnitude with a certain geometrical form. The *ratios* of these quantities are the subject-matter of successive algebras.

| Geometrical Form. | Quantity. | Examples. | Ratio |
|-------------------------------|-------------------------|------------------------------------|----------------------------|
| <i>Sense</i> on straight line | Vector on straight line | | Real quantity (of algebra) |
| <i>Direction</i> in plane | Vector in plane | | Complex notation |
| <i>Direction</i> in space | Vector in space | Couple; transl. vel. of rigid body | Quaternion |
| Axis | Rotor | Force; rotation velocity | Twist |
| Screw | Motor | System of forces; twist velocity | Biquaternion |

It is to be observed that this is not an uniformly ascending series, but that the theoretic order has been broken to meet requirements of practical application. After Hamilton's quaternion should come the ratio of two vectors in four dimensions; now a *rotor* may be regarded as the *logarithm* of such a ratio when the vectors are at right angles.

Ratio of polar rotors.

Let α and β be two rotors; it is clear that the sum which converts α into β converts also the polar of α into the polar of β . Let that twist which converts any rotor into its polar be called ω ; {this is a rectangular twist about any screw of pitch 1 whose axis meets the rotor perpendicularly} then the result may be written

$$\frac{\beta}{\alpha} = \frac{\omega\beta}{\omega\alpha} \equiv q, \text{ supposito.}$$

Polar motor.

The sum of any number of rotors is a *motor*; and it is known that every motor may be expressed as the sum of two polar rotors, that is to say in the form

$$\alpha + k.\omega\alpha \equiv A,$$

where k is the pitch of the motor in regard to the axis α of the motor

$$k\alpha + \omega\alpha \equiv \omega A$$

will be called the polar of A . In regard to the axis α it has a pitch reciprocal of the former one.

Every motor may be regarded as the sum of two, one having any perfectly arbitrary pitch and the other the pitch reciprocal to it. For the equation

$$\alpha + k\omega\alpha = x(\alpha + h\omega\alpha) + y(h\alpha + \omega\alpha)$$

gives

$$1 = x + hy,$$

$$k = hx + y,$$

and thence

$$x = \frac{1 - kh}{1 - h^2}, y = \frac{k - h}{1 - h^2}.$$

Ratio of motors.

If two motors have the same pitch their ratio is a twist. For we have seen above that

$$\beta = q\alpha, \quad \omega\beta = q\omega\alpha,$$

from which it follows that

$$\beta + k\omega\beta = q(\alpha + k\omega\alpha).$$

Let the ratio of $\omega\beta$ to α be called the *polar twist* of q ; so that we shall write

$$\omega q = \frac{\omega\beta}{\alpha} = \frac{\beta}{\omega\alpha}.$$

{Observe that q is a *finite twist*, and not a *motor*, or *twist velocity*; so that this must not be confounded with the previous meanings of *polar* and of ω .}

The ratio of any two motors is the sum of two polar twists.

Let $\alpha + h\omega\alpha$, $\beta + k\omega\beta$ be the two motors; and let $\frac{\beta}{\alpha} = q$. Then

$$\begin{aligned} \frac{\beta + k\omega\beta}{\alpha + h\omega\alpha} &= \frac{(1 - kh)(\beta + h\omega\beta) + (k - h)(h\beta + \omega\beta)}{(1 - h^2)(\alpha + h\omega\alpha)} \\ &= \frac{1 - kh}{1 - h^2} \cdot q + \frac{k - h}{1 - h^2} \cdot \omega q. \end{aligned}$$

REMARKS ON A THEORY OF THE EXPONENTIAL FUNCTION DERIVED FROM THE EQUATION $\frac{du}{dt} = pu$.

AFTER showing that a quantity which is equally multiplied in equal times must always increase at a rate proportional to itself, or which is the same thing, satisfy the equation $\frac{du}{dt} = pu$, we may call the ratio p of the growth to the growing quantity, the *intrinsic rate*. If we then define as follows: e^{px} , u is the result of making u grow at the intrinsic rate p for x seconds. This applies so far to pure quantity only; but if we regard u as

- (a) a vector on a line, i.e., a signed magnitude,
- (b) a vector in a plane, i.e., a complex magnitude, or
- (c) a vector in space,

then p being the ratio of two such vectors is respectively

- (a) a signed ratio; justifying e^{px} ,
- (b) a complex ratio; from which it becomes obvious that $e^{i\theta} = \cos \theta + i \sin \theta$,

- (c) a quaternion; from which we get HAMILTON'S theory of exponential functions of quaternions.

A semi-justification of the syncholic form of Taylor's Theorem (independently of the series) is also obtained by this method.

The series for e^{px} , u in all cases where it holds good may then be either established by Taylor's theorem, or shown to be a solution and the only one of the equation $\frac{du}{dt} = pu$ between values 0 and x of t .

[From *Proceedings of London Mathematical Society*, Vol. ix, No. 42, p. III.]

XLVI.

NOTES ON VORTEX-MOTION, ON THE TRIPLE GENERATION OF THREE-BAR CURVES, AND ON THE MASS-CENTRE OF AN OCTAHEDRON*.

(i) ON VORTEX-MOTION.

Let v be the velocity, and ω the rotation, at any point of a moving substance. It is known that $2\omega = V \nabla \sigma$; viz., this is equivalent to the three equations ordinarily written thus :

$$\begin{aligned} 2\xi &= \delta_y v - \delta_x w, \\ 2\eta &= \delta_z w - \delta_x u, \\ 2\xi &= \delta_y u - \delta_z v. \end{aligned}$$

If, moreover, k be the *expansion*, or the logarithmic rate of change of the volume, we have $k \text{ vol} = S \nabla \sigma$; viz., this is $\delta_x u + \delta_y v + \delta_z w$. Hence the quaternion $q, \text{ vol} = k + 2\omega$, is simply $\nabla \sigma$. The problem solved by Stokes as a general question of Analysis, and subsequently by Helmholtz for the special case of fluid motion, may be stated as follows:—Given the expansion and the rotation at every point of a moving substance, it is required to find the velocity at every point. In symbols, it being known that $q = \nabla \sigma$, and q being given, it is required to find σ .

The solution of the problem is exhibited in a very simple form if we consider it a little more generally.

A quaternion q is given at every point of space; it is required to find a quaternion r so that $q = \nabla r$. The solution is that $\nabla q = \nabla^2 r$, and therefore r is the potential of ∇q ; that is,

$$r_s = \int \nabla q_s \frac{dv_s}{D_{ss}},$$

* [From the *Proceedings of the London Mathematical Society*, Vol. ix. No. 195, pp. 20–29.]

(III) ON THE MASS-CENTRE OF AN OCTAHEDRON.

Let af , bg , ch [Fig. 67] be three finite lines not meeting. By an *octahedron* I mean the solid whose eight faces are abc , def , agh , alb , fde , fgh , ghb , ghc . If this solid figure be filled with matter of uniform density, its mass-centre may be found by a very simple construction.

The solid is girdled by three skew quadrilaterals $bchg$, $cagh$, $abfg$. Now the middle points of the sides of any skew quadrilateral are in one plane. Draw, then, three planes bisecting the sides of these quadrilaterals, and let them meet in a point k ; which, following Sylvester in a paper to be presently mentioned, I will call the *mass-centre*. Let also m be the mean point of the six vertices a , b , c , d , e , f , g , h ; it is the mass-centre of the triangle formed by the middle points of af , bg , ch . The line km is the mass-centre of the solid, perpendicularly produce it to n so that $mn = km$.

The proof is that the solid is the sum of the four tetrahedra $afbc$, $afgd$, $afgh$, $afde$. Now the mass-centre of a tetrahedron is the mean point of its vertices; consequently the line joining the mass-centre of $afbc$ to the middle point of gh is divided by the point m in the ratio 1 : 2. The same is true of the other three tetrahedra and the middle points of hg , bf , cg . Hence the mass-centres of the four tetrahedra are in one plane passing through the point m found by the above construction, and therefore the mass-centre of the whole solid is in this plane. So also it is in the other two planes determined by dividing the solid into tetrahedra having the common edge bg and the common edge ch respectively. Therefore it coincides with the point n .

This remark was suggested to me by Sylvester's construction for the mass-centre of a tetrahedral frustum, of which it is a simple extension. In fact, by making the pairs of faces abh , ahf ; acg , cfh ; cbf , bfg to be respectively equilateral, we pass at once to that particular case.

GEOMETRICAL THEOREMS.

The following is a proof by Pure Geometry of the proposition given by Mr. Ferrers, in Vol. I. p. 149, as the reciprocal of Prof. Cayley's theorem about rectangular hyperbolas.

The other corollaries are well known propositions. This form of proof suggests the corresponding propositions in Geometry of three dimensions.

1. *If a series of conics be inscribed in the same quadrilateral, their directors will all have the same radical axis.*

The tangents drawn from any point to a series of conics inscribed in the same quadrilateral form a pencil in involution. If therefore the point be such that any two of the conics subtend right angles at it, all the conics will subtend right angles. But the director of a conic is the locus of intersection of tangents at right angles to each other. Therefore the intersections of any two directors are points on all the others, $\therefore Q. E. D.$

2. *The foci of the quadrilaterals formed by five lines lie on a circle.*

For they all lie on the director of the conic touching the five lines.

3. *The circles whose diameters are the diagonals of a quadrilateral have a common radical axis.*

For the diagonals may be regarded as any three ellipses inscribed in the quadrilateral.

* [From the *Great Cambridge and Dublin Mathematical Dictionary*, Vol. III. pp. 21, 2, 1831.]

4. *Every conic through the intersection of two rectangular hyperbolas is a rectangular hyperbola.*

This is the reciprocal of (1) with regard to either of the points of intersection. This is Prof. Chylo's theorem of Vol. I., p. 77.

5. *The directrix of a parabola passes through the polar centre of every circumscribed triangle.*

This follows from (1) by sending one side of the quadrilateral to infinity. For the circles on the diagonals of the quadrilateral as diameters become then the perpendiculars of the triangle.

6. *The polar centres of the triangles formed by four straight lines lie on the line joining the foci of the quadrilateral.*

This line is the directrix of the inscribed parabola, by (1).

XLVIII

ON TRIANGULAR SYMMETRY *

WE make the properties of a cone intuitive by studying it under the form of a circle; or those of a spherulabral, by studying it under the form of a square. This simplification depends upon the projective property of a right angle, viz., that it divides harmonically the chord at infinity of a circle. By means of this property we interpret as general propositions whose truth we see intuitively through the symmetry of the figure. This kind of symmetry (that of a circle or square) I call the *symmetry of the right angle*, or *rectangular symmetry*.

From the symmetry of two lines we ascend immediately to the symmetry of three lines, or of the equilateral triangle. This is exemplified in the Rhombohedral System of Crystals, just as Rectangular Symmetry is exemplified in the Pyramidal System. I want in this note to call attention to the use of Triangular Symmetry in presenting general propositions under an intuitive form.

The projective property of an equilateral triangle is this: *it determines on the line at infinity a point whose Hessian is the circular points*. Given four lines, we may project one of them to infinity so that the other three shall form an equilateral triangle; for we have only to construct the Hessian of the point-circle determined upon that one by the other three, and then project this Hessian into the circular points. Similar problems are

B. To project at once a cone and triangle into a circle and an equilateral triangle,

* [From *Mathematics from the Educational Point of View*, Vol. 1, p. 10, 21.]

C. To project at once two conics and a triangle into two rectangular hyperbolæ and an equilateral triangle.

D. To project at once two triangles into equilateral triangles.

In each of these we have the problem of finding the line which has to be projected to infinity; this problem admits, in the three cases respectively, of 43, 76, and 108 solutions. The *triangle* might of course be replaced by a *cubic*, to be so projected that its asymptotes should form an equilateral triangle; but this case is not particularly interesting. Every cubic may be projected into a perfectly symmetrical form in this way:—the three real inflexions of the cubic lie on a certain straight line, and determine a point-cubic upon it; let the Hessian of this point-cubic be projected into the circular points at infinity. Then the cubic is symmetrically situated in respect of an equilateral asymptotic triangle.

The form in Fig. [58] may be called *inscribed*, that in Fig. [59] *escribed*. The inscribed cubic may have no oval nor double point, and then will lie entirely without the triangle.

Cubics with a cunode cannot be thus symmetrized; but their reciprocals can. In fact, every three-cusped quartic can be projected into a hypocycloid of three branches. For, any four points can be projected into any other four points; if then the three cusps and the intersection of the cuspidal tangents be projected respectively into the vertices and centre of an equilateral triangle, the thing is done. This proof is Prof. Cayley's. We learn in this way that, in any three-cusped quartic, the cuspidal tangents and the lines joining the cusps determine on the double tangent two point-cubics, whose common Hessian is the points of contact of that tangent.

Trinodal quartics have four double tangents, which are all real as lines when the nodes are real, but one is ideal (I borrow this convenient expression from Poncelet) or has imaginary contacts. If the three pairs of nodal tangents divide the ideal double tangent in the same anharmonic ratio, the quartic can be projected into a hypotrochoid, the contacts of the ideal

tangent being then the circular points at infinity. The curve has many remarkable properties, which can be recognized at once from the asymmetry of the projected figure.

Higher orders of asymmetry are special cases. Thus quartic asymmetry (as of a regular octagon) requires that the point-quartic shall be an harmonic system, so that its cubinvariant vanishes. Quintic Symmetry (regular pentagon) requires a point-quintic of the form $ax^5 + fy^5$. The conditions that the quintic may be reduced to this form are that the invariants K of the eighth degree and L of the twelfth shall separately vanish (see Professor Sylvester's admirable *Trilogy, Phil. Trans.* 1864, p. 619).

The cube and sphere are examples of the Symmetry of the *cubangle*, whose projective property is that it determines on the plane at infinity a conjugate triad of the imaginary circle. One is naturally led to seek for the projective property of a regular tetrahedron. It determines on the plane at infinity four straight lines x, y, z, w ; and if we assume the identical relation

$$x + y + z + w = 0,$$

then the equation of the imaginary circle is

$$x^2 + y^2 + z^2 + w^2 = 0,$$

as is easily shown by the consideration that each face is an equilateral triangle. Many interesting properties of this conic will be found proved in the solution to Question 1620*.

* (*Mathematics from the Educational Times*, Vol. III, p. 32, 1881)

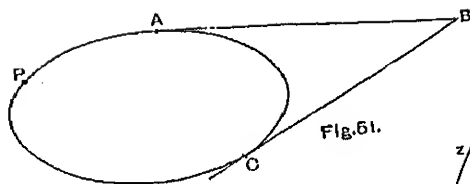


Fig. 51.

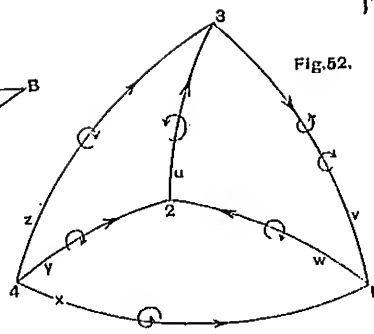


Fig. 52.

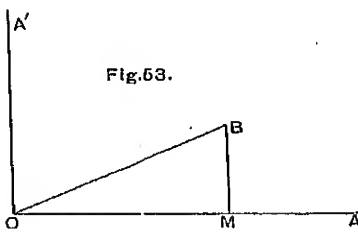


Fig. 53.

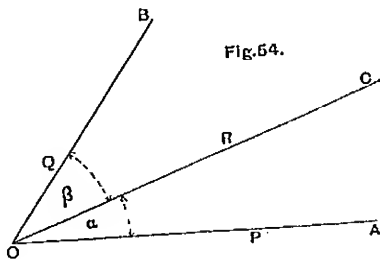


Fig. 54.

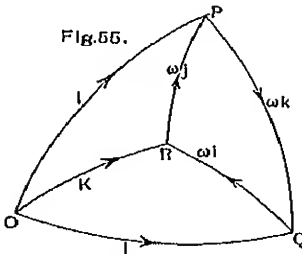


Fig. 55.

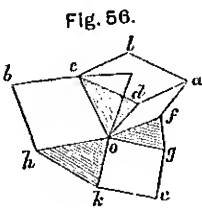


Fig. 56.

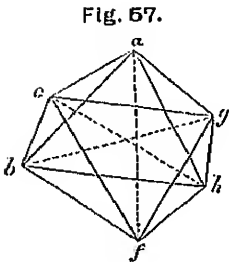


Fig. 57.

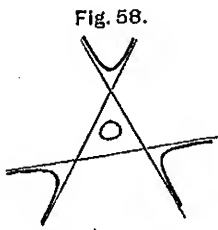


Fig. 58.

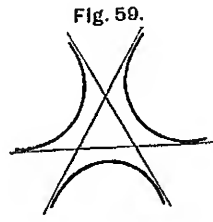


Fig. 59.

XLIX.

ON SOME EXTENSIONS OF THE FUNDAMENTAL PROPOSITION IN M. CHASLES'S THEORY OF CHARACTERISTICS*.

I MEAN by the "fundamental proposition" the following, viz.:—
 "If a variable system of two points on a right line be so related that when the second point is taken arbitrarily the first has a positions, and when the first point is taken arbitrarily the second has b positions; then there are $a + b$ points on the right line at which the system of two points coalesces into one point."

This principle has been admirably extended by Dr Salmon to the case of two dimensions, thus:—"If a variable system of two points in a plane be so related that when the second point is taken arbitrarily the first has a positions, and when the first point is taken arbitrarily the second has b positions, and that p pairs of points, each constituting a position of the system, may be found upon an arbitrary right line; then there are $a + b + p$ points in the plane at which the system of two points coalesces into one point."

The principle admits of further extension in two directions. First, we may consider a system of more than two points; and secondly, we may consider the systems as subject to a less number of relations than is sufficient to determine a single point. We are thus led to the following propositions:—

If a variable system of n points in space be so related that when all but the first point are taken arbitrarily the first point

* [From *Mathematics from the Educational Times*, Vol. v. pp. 49, 50.]

is determined to lie on a surface of order a , and when all but the second point are taken arbitrarily the second point is determined to lie on a surface of order b , and so on; then there are points in space at which the system of n points coalesces into one point, and the locus of such points is a surface of order $a + b + \dots$.

If a variable system of n points in space be so related that when all but the first point are taken arbitrarily the first point is determined to lie on a curve of order a , and when all but the second point are taken arbitrarily the second point is determined to lie on a curve of order b , and so on, and that when all but the first two points are taken arbitrarily there are on an arbitrary right line p pairs of points each constituting a position of the first two points, and that $q, r \dots$ are the corresponding numbers for the other pairs of points of the system; then there are points in space at which the system of n points coalesces into one point, and the locus of such points is a curve of order $a + b + \dots + p + q + r + \dots$.

It is not worth while to state the analogous propositions for Geometry of one and two dimensions, or the correlative propositions for lines and planes. I go on to exemplify the application of these propositions.

Let us begin with Mr Thomson's cubic (*Reprint*, Vol. II. p. 57). A conic is inscribed in a triangle so that the normals at the points of contact meet in a point; it is required to find the locus of this point. Consider now a variable system of three points, subject only to this condition; that if perpendiculars be drawn from them respectively to the three sides of a triangle, a conic may be drawn touching the sides of the triangle at the feet of those perpendiculars. Then, if we take two of the points arbitrarily, we determine two of the points of contact of the inscribed conic; that is we determine the conic itself uniquely, and therefore the third point of contact; and the normal at this point is therefore the locus of the third point of the system. That is to say, we have a variable system of three points so related that when any two of the points are taken arbitrarily, the locus of the third is a straight line;

consequently there are points in the plane at which the system of three points coalesces into one point (that is, where the three normals meet in a point), and the locus of such points is a curve of order $(1 + 1 + 1 =) 3$.

To complicate the question, let us suppose that a conic is drawn to touch three given *conics*, so that the normals at the points of contact may meet in a point. Here, as before, we take our variable system of three points, one on each of the three normals. Take two of the points arbitrarily; from each of these we can draw four normals to the corresponding conic. Pairing these together, we have 16 pairs of points of contact. Now when we have given two tangents and their points of contact, the number of conics of the system which can be drawn to touch a given conic is 4. By determining two points of our variable system we have, therefore, determined 64 conics; on the third *given* conic, these determine 64 points of contact, and the normals through these may be held to constitute a curve of the 64th order. Thus we have a variable system of three points so related that when any two of them are taken arbitrarily, the third is determined to lie on a curve of the 64th order; consequently the locus of those points at which the system coalesces into one point, or the three normals meet in a point, is a curve of the order $(3 \times 64) = 192$. More generally, if we substitute for the variable conic a curve of order m , class n , and deficiency D , or say a curve of *species* (m, n, D) , and for the three fixed conics $\frac{1}{2}(m + n - D + 2)$ curves of orders m_1, m_2, \dots and of classes n_1, n_2, \dots , then the corresponding locus will be of the order $3\phi(m, n, D)(m_1 + n_1)(m_2 + n_2) \dots$, where $\phi(m, n, D)$ is the number of curves of species (m, n, D) which can be drawn through $(m + n - D + 1)$ points, or touching $(m + n - D + 1)$ lines.

For another example, let us find the locus of those points the feet of the perpendiculars from which to four lines or planes in space are coplanar. In both these cases the locus comes out primarily of the fourth order; but the plane at infinity is evidently a part of the locus, the remainder of which is thus of

the third order. In both cases the envelope of the plane through the feet of the perpendiculars is of the fourth class, and touches the plane at infinity. I conjecture that the imaginary circle is a curve of contact.

If a conicoid be drawn to touch five straight lines, so that the normal planes at the points of contact meet in a point, the locus of this point is of the tenth order. And as our *altilatum*,

L*.

INSTRUMENTS USED IN MEASUREMENT.

By *Measurement*, for scientific purposes, is meant the measurement of *quantities*. In each special subject there are quantities to be measured; and these are very various, as may be seen from the following list of those belonging to geometry and dynamics.

Geometrical Quantities.

Lengths
Areas
Volumes
Angles (plane and solid)
Curvatures (plane and solid)
Strains (elongation, torsion, shear).

Circumstances of Motion. Properties of Bodies.

| | |
|--------------|---------------------------------|
| Time | Mass |
| Velocity | Weight |
| Momentum | Density |
| Acceleration | Specific gravity |
| Force | Elasticity (of form and volume) |
| Work | Viscosity |
| Horse-power | Diffusion |
| Temperature | Surface tension |
| Heat | Specific heat. |

* [*Handbook to the Special Loan Collection of Scientific Apparatus*, 1876, L. pp. 55—59, LI. pp. 60—77. The object of L., LI. and other introductory Notices in the "Handbook" was to "make the Exhibition as useful and interesting as possible."]

Notwithstanding the very different characters of these quantities, they are all measured by reducing them to the same kind of quantity, and estimating that in the same way. Every quantity is measured by finding a *length* proportional to the quantity, and then measuring this length. This will, perhaps, be better understood if we consider one or two examples.

The measurement of *angles* occurs in a very large majority of scientific instruments. It is always effected by measuring the *length of an arc* upon a graduated circle; the circumference of this circle being divided not into inches or centimeters, but into degrees and parts of a degree—that is, into aliquot parts of the whole circumference.

As a step towards their final measurement, some quantities, of which work is a good instance, are represented in the form of *areas*; and there seems reason to believe that this method is likely to be extended. Instruments for measuring areas are called Planimeters; and one of the simplest of these is Amodeo's, consisting of two rods joined together, the end of one being fixed and that of the other being made to run round the area which is to be measured. The moving rod rests on a wheel, which turns as the rod moves; and it is proved by geometry that the area is proportional to the distance through which the wheel turns. Thus the measurement of an area is reduced to the measurement of a length.

Volumes are measured in various ways, but all depending on the same principle. Quantities of earth excavated for engineering purposes are estimated by a rough determination of the shape of the cavity, and the measurement of its dimensions, namely, certain lengths belonging to it. The contents of a vessel are sometimes gauged in the same way; but the more accurate method is to fill it with liquid and then pour the liquid into a cylinder of known section, when the quantity is measured by the height of the liquid in the cylinder, that is, by a length. The volumes of irregular solids are also measured by immersing them in liquid contained in a uniform cylinder, and observing the height to which the liquid rises; that is, by measuring a length. An apparatus for this purpose is called a

Stereometer. The liquid must be so chosen that no chemical action takes place between it and the solid immersed, and that it wets the solid so that no air bubbles adhere to the surface. Thus mercury is used in the case of metals by the Standards Department.

Time is measured for ordinary purposes by the length of the arc traced out by a moving hand on a circular clock-face. For astronomical purposes it is sometimes measured by counting the ticks of a clock which beats seconds, and estimating mentally the fractions of a second; and in cases where the period of an oscillation has to be found, it is determined by counting the number of oscillations in a time sufficient to make the number considerable, and then dividing that time by the number. But by far the most accurate way of measuring time is by means of the line traced by a pencil on a sheet of paper rolled round a revolving cylinder, or a spot of light moving on a sensitive surface. If the pencil is made to move along the length of the cylinder so as to indicate what is happening as time goes along, the time of each event will be found when the cylinder is unrolled by measuring the distance of the mark recording it from the end of the unrolled sheet, provided that the rate at which the cylinder goes round is known. In this way Helmholtz measured the rate of transmission of nerve-disturbance.

A very common case of the measurement of *force* is the barometer, which measures the pressure of the atmosphere per square inch of surface. This is determined by finding the height of the column of mercury which it will support (mercurial barometer), or the strain which it causes in a box from which the air has been taken out (aneroid barometer). The height in the former case may be measured directly, or it may first be converted into the quantity of turning of a needle, and then read off as length of arc on a graduated circle; in the latter case the strain is always indicated by a needle turning on a graduated circle.

The *mass*, and (what is proportional to it) the *weight*, of different bodies at the same place, are measured by means of a balance; and at first sight this mode of measurement seems different from those which we have hitherto considered. For we

But in order to estimate a length in terms of these very small quantities, it must be *magnified*; and this is done in three ways. First, geometrically, by what is called a vernier scale. This is a movable scale, which gains on the fixed one by one-tenth in each division. To measure any part of a division, we find how many divisions it takes the vernier to gain so much as that part; this is how many tenths the part is. The quantity to be measured is here geometrically multiplied by ten. Next, optically, by looking at the length and scale with a microscope or telescope. Third, mechanically, by a screw with a disc on its head, on which there is a graduated rim, called a micrometer screw. If the pitch of the screw is one-tenth and the radius of the disc ten times that of the screw, the motion is multiplied by one hundred. The two latter modes are combined together in an instrument called a micrometer-microscope. Another mechanical multiplier is a mirror which turns round and reflects light on a screen at some distance, as in Thomson's reflecting galvanometer.

Properly speaking, however, any description of a length by counting of standard lengths is imperfect and merely approximate. The true way of indicating a length is to draw a straight line which represents it on a fixed scale. And this is done by means of self-recording instruments, which measure lengths from time to time on a cylinder in the manner described above. It is only by this graphical representation of quantities that the laws of their variation become manifest, and that higher branch of measurement becomes possible which determines the nature of the connection between two simultaneously varying quantities.

II.

INSTRUMENTS ILLUSTRATING KINEMATICS, STATICS, AND DYNAMICS.

Science of Motion.

GEOMETRY teaches us about the sizes, the shapes, and the distances of things; to know sizes and distances we have to measure *lengths*, and to know shapes we have to measure *angles*. The science of *Motion*, which is the subject of the present sketch, tells us about the changes in these sizes, shapes, and distances which take place from time to time. A body is said to move when it changes its place or position; that is to say, when it changes its distance from surrounding objects. And when the parts of a body move relatively to one another, *i.e.* when they alter their distance from one another, the body changes in size, or shape, or both. All these changes are considered in the science of motion.

Kinematics.

It is divided into two parts; the accurate description of motion, and the investigation of the circumstances under which particular motions take place. The description of motion may again be divided into two parts, namely, that which tells us *what* changes of position take place, and that which tells us *when* and *how fast* they take place. We might, for example, describe the motion of the hands of a clock, and say that they turn round on their axes at the centre of the clock-face in such a way that the minute-hand always moves twelve times as much as the hour-hand; this is the first part of the description of the motion. We might go on to say that when the clock is going correctly this motion takes place uniformly, so that the minute-

hand goes round once in each hour; and this would be the second part of the description. The first part is what was called Kinematics by Ampère; it tells us how the motions of the different parts of a machine depend on each other in consequence of the machinery which connects them. This is clearly an application of geometry alone, and requires no more measurements than those which belong to geometry, namely, measurements of lines and angles. But the name Kinematics is now conveniently made to include the second part also of the description of motion—when and how fast it takes place. This requires in addition the measurement of *time*, with which geometry has nothing to do. The word Kinematic is derived from the Greek *kinema*, "motion;" and will therefore serve equally well to bear the restricted sense given it by Ampère, and the more comprehensive sense in which it is now used. And since the principles of this science are those which guide the construction not only of scientific apparatus, but of all instruments and machines, it may be advisable to describe in some detail the chief topics with which it deals.

Dynamics.

That part of the science which tells us about the circumstances under which particular motions take place is called *Dynamics*. It is found that the change of motion in a body depends on the position and state of surrounding bodies, according to certain simple laws; when considered as so depending on surrounding bodies, the rate of change in the quantity of motion is called *force*. Hence the name Dynamic, from the Greek *dynamis*, "force." The word *force* is here used in a technical sense, peculiar to the science of motion; the connection of this meaning with the meaning which the word has in ordinary discourse will be explained further on.

Statics and Kinetics.

Dynamics are again divided into two branches; the study of those circumstances in which it is possible for a body to remain at rest is called *Statics*, and the study of the circum-

stances of actual motion is called Kinetics. The simplest part of Statics, the doctrine of the Lever, was successfully studied before any other part of the science of motion, namely, by Archimedes, who proved that when a lever with unequal arms is balanced by weights at the ends of it, these weights are inversely proportional to the arms. But no real progress could be made in determining the conditions of rest, until the laws of actual motion had been studied.

Translation of Rigid Bodies.

Returning, then, to the description of motion, or Kinematics, we must first of all classify the different changes of position, of size, and of shape, with which we have to deal. We call a body *rigid* when it changes only its position, and not its size or shape, during the time in which we consider it. It is probable that every actual body is constantly undergoing slight changes of size and shape, even when we cannot perceive them; but in Kinematics, as in most other matters, there is a great convenience in talking about only one thing at a time. So we first of all investigate changes of position on the assumption that there are no changes of size and shape; or, in technical phrase, we treat of the motion of rigid bodies. Here an important distinction is made between motion in which the body merely travels from one place to another, and motion in which it also turns round. Thus the wheels of a locomotive engine not only travel along the line, but are constantly turning round; while the coupling-bar which joins two wheels on the same side remains always horizontal, though its changes of position are considerably complicated. A change of place in which there is no rotation is called a *translation*. In a rotation the different parts of the body are moving different ways, but in a translation all parts move in the same way. Consequently, in describing a translation we need only specify the motion of any one particle of the moving body; where by a *particle* is meant a piece of matter so small that there is no need to take account of the differences between its parts, which may therefore be treated for purposes of calculation as a point.

We are thus brought down to the very simple problem of describing the motion of a point. Of this there are certain cases which have received a great deal of attention on account of their frequent occurrence in nature ; such as Parabolic Motion, Simple Harmonic Motion, Elliptic Motion. We propose to say a few words in explanation of each of these.

Parabolic Motion.

The motion of a *projectile*, that is to say, of a body thrown in any direction and falling under the influence of gravity, was investigated by Galileo ; and this is the first problem of Kinetics that was ever solved. We must confine ourselves here to a description of the motion, without considering the way in which it depends on the circumstance of the presence of the earth at a certain distance from the moving body. Galileo found that the path of such a body, or the curve which it traces out, is a parabola ; a curve which may be described as the shadow of a circle cast on a horizontal table by a candle which is just level with the highest point of the circle.

It is convenient to consider separately the vertical and the horizontal motion, for in accordance with a law subsequently stated in a general form by Newton, these two take place in complete independence of one another. So far as its horizontal motion is concerned, the projectile moves uniformly, as if it were sliding on perfectly smooth ice ; and, so far as its vertical motion is concerned, it moves as if it were falling down straight. The nature of this vertical motion may be described in two ways, each of which implies the other. First, a falling body moves faster and faster as it goes down ; and the rate at which it is going at any moment is strictly proportional to the number of seconds which has elapsed since it started. Thus its downward velocity is continually being added to at a uniform rate. Secondly, the whole distance fallen from the starting-point is proportional to the *square* of the number of seconds elapsed ; thus, in three seconds a body will fall nine times as far as it will fall in one second. The latter of these statements was experimentally proved by Galileo ; not, however, in the case of bodies falling vertically, which move too quickly for the

time to be conveniently measured, but in the case of bodies falling down inclined planes, the law of which he at first assumed, and afterwards proved to be identical with that of the other. The former statement, that the velocity increases uniformly, is directly tested by an apparatus known as Attwood's machine, consisting essentially of a pulley, over which a string is hung with equal weights attached to its ends. A small bar of metal is laid on one of the weights, which begins to descend and pull the other one up; after a measured time the bar is lifted off, and then, both sides pulling equally, the motion goes on at the rate which had been acquired at that instant. The distance travelled in one second is then measured, and gives the velocity; this is found to be proportional to the time of falling with the bar on.

The second statement, that the space passed over is proportional to the square of the number of seconds elapsed, is verified by Morin's machine, which consists of a vortical cylinder which revolves uniformly while a body falling down at the side marks it with a pencil. The curve thus described is a record of the distance the body had fallen at every moment of time.

Fluxions.

This investigation of Galileo's was in more than one aspect the foundation of dynamical science; but not the least important of these aspects is the proof that either of the two ways of stating the law of falling bodies involves the other. Given that the distance fallen is proportional to the square of the time, to show that the velocity is proportional to the time itself; this is a particular case of the problem. Given where a body is at every instant, to find how fast it is going at every instant. The solution of this problem was given by Newton's Method of Fluxions. When a quantity changes from time to time, its *rate* of change is called the *fluxion* of the quantity. In the case of a moving body the quantity to be considered is the distance which the body has travelled; the fluxion of this distance is the rate at which the body is going. Newton's method solves the problem, Given *how big* a quantity is at

any time, to find its fluxion at any time. The method has been called on the Continent, and lately also in England, the Differential Calculus; because the difference between two values of the varying quantity is mentioned in one of the processes that may be used for calculating its fluxion. The inverse problem, Given that the velocity is proportional to the time elapsed, to find the distance fallen, is a particular case of the general problem, Given how fast a body is going at every instant, to find where it is at any instant; or, Given the fluxion of a quantity, to find the quantity itself. The answer to this is given by Newton's Inverse Method of Fluxions; which is also called the Integral Calculus, because in one of the processes which may be used for calculating the quantity, it is regarded as a whole (integer) made up of a number of small parts. The method of Fluxions, then, or Differential and Integral Calculus, takes its start from Galileo's study of parabolic motion.

Harmonic Motion.

The ancients, regarding the circle as the most perfect of figures, believed that circular motion was not only *simple*, that is, not made up by putting together other motions, but also *perfect*, in the sense that when once set up in perfect bodies it would maintain itself without external interference. The moderns, who know nothing about perfection except as something to be aimed at, but never reached, in practical work, have been forced to reject both of these doctrines. The second of them, indeed, belongs to Kinetics, and will again be mentioned under that head. But as a matter of Kinematics it has been found necessary to treat the uniform motion of a point round a circle as compounded of two oscillations. To take again the example of a clock, the extreme point of the minute-hand describes a circle uniformly; but if we consider separately its vertical position and its horizontal position, we shall see that it not only oscillates up and down, but at the same time swings from side to side, each in the same period of one hour. If we suppose a button to move up and down in a slit between the figures XII. and VI., in such a way as to be always

same height at the end of the minute hand, this button will have only one of the two oscillations which are combined in the motion of that point; and the other oscillation would be exhibited by a button constrained to move in a similar manner between the figures III. and IX., so as always to be either vertically above or vertically below the extreme point of the minute-hand. The laws of these two motions are identical, but they are so timed, that each is at its extreme position when the other is crossing the centre. An oscillation of this kind is called a *simple harmonic motion*; the name is due to Mr William Thomson, and was given on account of the intimate connection between the laws of such motions and the theory of vibrating strings. Indeed, the harmonic motion, simple or compound, is the most universal of all forms; it is a simplified not only in the motion of every particle of a vibrating solid, such as the string of a piano or violin, a tuning fork, or the membrane of a drum, but in those minute excursions of particles of air which carry sound from one place to another, in the waves and tides of the sea, and in the amazingly rapid tremor of the luminiferous ether which, in its varying action on different bodies, makes itself known as light or radiant heat or chemical action. Simple harmonic motions differ from one another in three respects; in the extent or amplitude of the swing, which is measured by the distance from the middle-point to either extremity; in the period or interval of time between two successive passages through an extreme position; and in the time of starting, or epoch, as it is called, which is named by saying what particular stage of the vibration was being executed at a certain instant of time. One of the most astonishing and fruitful theorems of mathematical science is this; that every *periodic motion* whatever, that is to say, every motion which exactly repeats itself again and again at definite intervals of time, is a compound of simple harmonic motions, whose periods are successively smaller and smaller aliquot parts of the original period, and whose amplitudes (after a certain number of them) are less and less as these periods are more rapid. The "harmonic" tones of a string, which are always heard along with the fundamental tone, are a particular case of

these constituents. The theorem was given by Fourier in connection with the flow of heat, but its applications are innumerable, and extend over the whole range of physical science.

The laws of combination of harmonic motions have been illustrated by some ingenious apparatus of Messrs Tisley and Spiller, and by a machine invented by Mr Donkin; but the most important practical application of these laws is to be found in Sir W. Thomson's Tidal Clock, and in a more elaborate machine which draws curves predicting the height of the tide at a given port for all times of the day and night with as much accuracy as can be obtained by direct observation. One special combination is worthy of notice. The union of a vertical vibration with a horizontal one of half the period gives rise to that figure of 8 which M. Marey has observed by his beautiful methods in the motion of the tip of a bird's or insect's wing.

Elliptic Motion.

The motion of the sun and moon relative to the earth was at first described by a combination of circular motions; and this was the immortal achievement of the Greek astronomer Hipparchus and Ptolemy. Indeed, in so far as these motions are periodic, it follows from Fourier's theorem mentioned above that this mode of description is mathematically sufficient to represent them; and astronomical tables are to this day calculated by a method which practically comes to the same thing. But this representation is not the simplest that can be found; it requires theoretically an infinite number of component motions, and gives no information about the way in which these are connected with one another. We owe to Kepler the accurate and complete description of planetary or elliptic motion. His investigations applied in the first instance to the orbit of the planet Mars about the sun, but it was found true of the orbits of all planets about the sun, and of the moon about the earth. The path of the moving body in each of these motions, is an ellipse, or oval shadow of a circle, a curve having various properties in relation to two internal points or foci, which replace as it were the one centre of a circle. In the

case of the ellipse described by a planet, the sun is in one of these foci; in the case of the moon, the earth is one focus. No much for the geometrical description of the motion. Kepler further observed that a line drawn from the sun to a planet, or from the earth to the moon, and supposed to move round with the moving body, would sweep out equal areas in equal times. These two laws, called Kepler's first and second laws, complete the kinematic description of elliptic motion; but to obtain formulae fit for computation, it was necessary to calculate from these laws the various harmonic components of the motion to and from the sun, and round it; this calculation has much occupied the attention of mathematicians.

The laws of rotatory motion of rigid bodies are somewhat difficult to describe without mathematical symbols, but they are thoroughly known. Examples of them are given by the apparatus called a gyroscope, and the motion of the earth, and an application of the former to prove the nature of the latter, made by Foucault, is one of the most beautiful experiments belonging entirely to dynamics.

Rotation

Next in simplicity after the translation of a rigid body, come two kinds of motion which are at first sight very different, but between which a closer observation discovers very striking analogies. These are the motions of rotation about a fixed point, and the motion of sliding on a fixed plane. The first of these is most easily produced in practice by what is well known as a ball-and-socket joint; that is to say, a body ending in a portion of a spherical surface which can move about in a spherical cavity of the same size. The motion of the spherical surface is then a fixed point, and the motion is reduced to the sliding of one sphere inside another. In the same way, if we consider, for instance, the motion of a flat iron on an ironing-board, we may see that this is not a pure translation, for the iron is frequently turned round as well as moved about; but the motion may be described as the sliding of one plane upon another. Thus in each case the matter to be considered is the sliding of one surface on another which is exactly like. For

two surfaces to fit one another exactly, in all positions, they must be either both spheres of the same size, or both planes; and the latter case is really included under the former, for a plane may be regarded as a sphere whose radius has increased without limit. Thus, if a piece of ice be made to slide about on the frozen surface of a perfectly smooth pond, it is really rotating about a fixed point at the centre of the earth; for the frozen surface may be regarded as part of an enormous sphere, having that point for centre. And yet the motion cannot be practically distinguished from that of sliding on a plane.

In this latter case it is found that, excepting in the case of a pure translation, there is at every instant a certain point which is at rest, and about which as a centre the body is turning. This point is called the instantaneous centre of rotation; it travels about as the motion goes on, but at any instant its position is perfectly definite. From this fact follows a very important consequence; namely, that every possible motion of a plane sliding on a plane may be produced by the *rolling* of a curve in one plane upon a curve in the other. The point of contact of the two curves at any instant is the instantaneous centre at that instant. The problems to be considered in this subject are thus of two kinds: Given the curves of rolling to find the path described by any point of the moving plane; and, Given the paths described by *two* points of the moving plane (enough to determine the motion) to find the curves of rolling and the paths of all other points. An important case of the first problem is that in which one circle rolls on another, either inside or outside; the curves described by points in the moving plane are used for the teeth of wheels. To the second problem belongs the valuable and now rapidly increasing theory of *link-work*, which, starting from the wonderful discovery of an exact parallel motion by M. Peaucellier, has received an immense and most unexpected development at the hands of Professor Sylvester, Mr Hart, and Mr A. H. Kempe.

Passing now to the spherical form of this motion, we find that the instantaneous centre of rotation (which is clearly equivalent to an instantaneous axis perpendicular to the plane) is replaced by an instantaneous axis passing through the com-

mon centre of the moving spheres. In the same way the rolling of one curve on another in the plane is replaced by the rolling of one *cone* upon another, the two cones having a common vertex at the same centre.

Analogous theorems have been proved for the most general motion of a rigid body. It was shown by M. Chasles that this is always similar to the motion of a corkscrew descending into a cork; that is to say, there is always a rotation about a certain instantaneous axis, combined with translation along this axis. The amount of translation per unit of rotation is called the *pitch* of the screw. The instantaneous screw moves about as the motion goes on, but at any given instant it is perfectly definite in position and pitch. And any motion whatever of a rigid body may be produced by the rolling and sliding of one surface on another, both surfaces being produced by the motion of straight lines. This crowning theorem in the geometry of motion is due to Professor Cayley. The laws of combination of screw motions have been investigated by Dr Ball.

Thus, proceeding gradually from the more simple to the more complex, we have been able to describe every change in the position of a body. It remains only to describe changes of size and shape. Of these there are three kinds, but they are all included under the same name—*strains*. We may have, first, a change of size without any change of shape, a uniform dilatation or contraction of the whole body in all directions, such as happens to a sphere of metal when it is heated or cooled. Next, we may have an elongation or contraction in one direction only, all lines of this body pointing in this direction being increased or diminished in the same ratio; such as would happen to a rod six feet long and an inch square, if it were stretched to seven feet long, still remaining an inch square. Thirdly, we may have a change of shape produced by the sliding of layers over one another, a mode of deformation which is easily produced in a pack of cards; this is called a *shear*. By appropriate combinations of these three, every change of size and shape may be produced; or we may even leave out the second element, and produce any strain whatever by a dilatation or contraction, and two shears.

Dynamics.

We have already said that the change of motion of a body depends upon the position and state of surrounding bodies. To make this intelligible it will be necessary to notice a certain property of the three kinds of motion of a point which we described.

The combination of velocities may be understood from the case of a body carried in any sort of cart or vehicle in which it moves about. The whole velocity of the body is then compounded of the velocity of the vehicle and of its velocity relative to the vehicle. Thus, if a man walks across a railway carriage his whole velocity is compounded of the velocity of the railway carriage and of the velocity with which he walks across.

When the velocity of a body is changed by adding to it a velocity in the same direction or in the opposite direction, it is only altered in amount; but when a transverse velocity is compounded with it, a change of direction is produced. Thus, if a man walks fore and aft on a steamboat, he only travels a little faster or slower; but if he walks across from one side to the other, he slightly changes the direction in which he is moving.

Now, in the parabolic motion of a projectile, we found that while the horizontal velocity continues unchanged, the vertical velocity increases at a uniform rate. Such a body is having a downwards velocity continually poured into it, as it were. This gradual change of the velocity is called *acceleration*; we may say that the acceleration of a projectile is always the same, and is directed vertically downwards.

In a simple harmonic motion it is found that the acceleration is directed towards the centre, and is always proportional to the distance from it. In the case of elliptic motion it was proved by Newton that the acceleration is directed towards the focus, and is inversely proportional to the square of the distance from it.

Let us now consider the circumstances under which these motions take place. To produce a simple harmonic motion we may take a piece of elastic string, whose length is equal to the

height of a smooth table; then fasten one end of the string to a bullet and the other end to the floor, having passed it through a hole in the table, so that the bullet just rests on the top of the hole when the string is unstretched. If the bullet be now pulled away from the hole so that the string is stretched, and then let go, it will oscillate to and fro on either side of the hole with a simple harmonic motion. The acceleration (or rate of change of velocity) is here proportional to the distance from the hole; that is, to the *amount of elongation of the string*. It is directed towards the hole; that is, in the direction of this elongation. In the case of the moon moving round the earth, the acceleration is directed towards the earth, and is inversely proportional to the square of the distance from the earth.

In both these cases, then, the change of velocity depends upon surrounding circumstances; but in the case of the bullet, this circumstance is the strained condition of an adjoining body, namely, the elastic string; while in the case of the moon the circumstance is the position of a distant body, namely, the earth. The motion of a projectile turns out to be only a special case of the motion of the moon; for the parabola which it describes may be regarded as one end of a very long ellipse, whose other end goes round the earth's centre.

There is a remarkable difference between the two cases. The swing of the bullet depends upon its size; a large bullet will oscillate more slowly than a small one. This leads us to modify the rule. If a large bullet is equivalent to two small ones, then when it is going at the same rate it must contain twice as much motion as one of the small ones; or, as we now say, with the same velocity it has twice the *momentum*. Now the change of momentum is found to be the same for all bullets, when the momentum is reckoned as proportional to the quantity of matter in the bullet as well as to the velocity. The quantity of matter in a body is called its *mass*; for bodies of the same substance it is, of course, simply the quantity of that substance; but for bodies of different substances it is so reckoned as to make the rule hold good. The rule for this case may then be stated thus; the change of momentum of a body (that is, the change of velocity multiplied by the mass), depends

on the state of strain of adjoining bodies. Regarded as so depending, this change of momentum is called the *pressure* or *tension* of the adjoining body, according to the nature of the strain; both of these are included in the name *stress*, introduced by Rankine.

But in the case of projectiles, the acceleration is found to be the same for all bodies at the same place; and this rule holds good in all cases of planetary motion. So that it seems as if the change of velocity, and not the change of momentum, depended upon the position of distant bodies. But this case is brought under the same rule as the other by supposing that the mass of the moving body is to be reckoned among the "circumstances." The change of momentum is in this case called the attraction of gravitation, and we say that the attraction is proportional to the mass of the attracted body. And this way of representing the facts is borne out by the electrical and magnetic attractions and repulsions, where the change of momentum depends on the position and state of the attracting thing, and upon the electric charge or the induced magnetism of the attracted thing.

Force, then, is of two kinds; the stress of a strained adjoining body, and the attraction or repulsion of a distant body. Attempts have been made with more or less success to explain each of these by means of the other. In common discourse the word "force" means muscular effort exerted by the human frame. In this case the part of the human body which is in contact with the object to be moved is in a state of strain, and the force, dynamically considered, is of the first kind. But this state of strain is preceded and followed by nervous discharges, which are accompanied by the sensations of effort and of muscular strain; a complication of circumstances which does not occur in the action of inanimate bodies. What is common to the two cases is, that the change of momentum depends on the strain.

Having thus explained the law of Force, which is the foundation of Dynamics, we may consider the remaining laws of motion. It is convenient to state them first for particles, or bodies so small that we need take account only of their position.

Every particle, then, has a rate of change of momentum due to the position or state of every other particle, whether adjoining it or distant from it. These are compounded together by the law of composition of velocities, and the result of the whole is the actual change of momentum of the particle. This statement, and the law of Force stated above, amount together to Newton's first and second laws of motion. His third law is, that the change of momentum in one particle, due to the position or state of another, is equal and opposite to the change of momentum in the other, due to the position or state of the first.

By the help of these laws D'Alembert showed how the motion of rigid bodies, or systems of particles, might be dealt with. It appears from his method that two stresses, acting on a rigid body, may be equivalent, in their effect on the body as a whole, to a single stress, whose direction and position will be totally independent of the shape and nature of the body considered. The law of combination of stresses acting on a system of particles is, in fact, the same as the law of combination of velocities, so far as regards the motion of the system as a whole. This beautiful but somewhat complex result of Dynamics has been used in some text-books as the independent foundation of Statics, under the name of the parallelogram of forces; a singular inversion of the historical order and of the methods of the great writers.

When the result of all the circumstances surrounding a body is that there is no change of momentum, the body is said to be in equilibrium. In this case, if the body is at rest, it will remain so; and on this account the study of such conditions is called Statics. In dealing with the statics of rigid bodies, we have only to examine those cases in which the resultant of the external stresses and attractions acting on the body amounts to nothing. But the most important part of statics is that which finds the stresses acting in the interior of bodies between contiguous parts of them; for upon this depends the determination of the requisite strength of structures which have to bear given loads. It is found that the way in which the stress due to a given strain depends on the strain, varies according to the physical nature of the body; for bodies, however, which are

not crystalline or fibrous, but which have the same properties in all directions, there are two quantities which, if known, will enable us always to calculate the stress due to a given strain. These are, the elasticity of volume, or resistance to change of size; and the rigidity, elasticity of figure, or resistance to change of shape. Problems relating to the interior state of bodies are far more difficult than those which regard them as rigid. Thus, if a beam is supported at its two ends, it is very easy to find the portion of its weight which is borne by each support; but the determination of the state of stress in the interior is a problem of great complexity.

There is one theorem of kinetics which must be mentioned here. If we multiply half the momentum of every particle of a body by its velocity, and add all the results together, we shall get what is called the kinetic energy of the body. When the body is moved from one position to another, if we multiply each force acting on it—whether attraction or stress—by the distance moved in the direction opposite to the force, and add the results, we shall get what is called the work done against the forces during the change of position. It does not at all depend on the rate at which the change is made, but only on the two positions. If a body moves, and loses kinetic energy, it does an amount of work equal to the kinetic energy lost. If it gains kinetic energy, an amount of work equal to this gain must be done to take it back from the new position to the old one. The amount of work which must be done to take a body from a certain standard position to the position which it has at present is called the potential energy of the body. The theorem may be stated in this form; the sum of the potential and kinetic energies is always the same, provided the surrounding circumstances do not alter. Hence the theorem is called the Conservation of Energy. It is one fact out of many that may be deduced from the equations of motion; it is not sufficient to determine the motion of a body, but it is exceedingly useful as giving a general result in cases where it might be difficult or undesirable to investigate all the particulars; and it is especially applicable to machines, the important question in regard to which is the amount of work which they can do.

It will have been seen that the science of motion depends on a few fundamental principles which are easily verified, and consists almost entirely of mathematical deductions and calculations based on those principles. It is no longer therefore an experimental science in the same sense as those are in which the fundamental facts are still being discovered. The apparatus connected with it may be conveniently classified under three heads :

(a) Apparatus for illustrating theorems or solving problems of kinematics, such as those mentioned above for compounding harmonic motions. There is reason to hope for great extension of our powers in this direction.

(b) Apparatus for measuring the dynamical quantities, such as weight, work, and the elasticities of different substances. These are more fully classified under Measurements.

(c) Apparatus designed for purposes belonging to other sciences, but illustrating by its structure and functions the results of kinematics or dynamics. In this class the remainder of the collection is included.

APPENDIX.

[Among Clifford's Papers three *Cahiers* have been found relating to the theory of Elliptic functions. The most complete of these is entitled "Algebraic Introduction to Elliptic Functions;" the second is "A Tract on Elliptic Functions;" the third is without any title. In all three, the elliptic functions are treated according to what may be termed the second method of Jacobi; viz. the properties of the theta functions are first investigated and the properties of the elliptic functions are deduced from them. The three cahiers appear to have been intended either as notes for a course of Lectures on Elliptic functions, or as drafts for a treatise. They contain no new results, and perhaps no original methods of investigation. But as the mode of treatment adopted by Clifford is not employed in any English treatise, and as information respecting it would have to be sought by the student in scattered original memoirs, it has been thought advisable to print, in this collection, nearly the whole of the "Algebraic Introduction," and one section of the "Tract." It is unnecessary to say that Clifford is never a mere copyist, and that these fragments possess an independent value of their own, even when they relate to elementary parts of the subject.

H. J. S.]

ALGEBRAIC INTRODUCTION TO ELLIPTIC FUNCTIONS.

I.

Definitions and Elementary Properties of the Theta Functions.

The geometric series $a + ar + ar^2 + \dots$ may be easily expressed as a sum of exponentials. If $\log a = \alpha$, $\log r = \beta$, the series in fact becomes

$$e^{\alpha} + e^{\alpha+\beta} + e^{\alpha+2\beta} + \dots$$

in which the general term is $e^{\alpha+n\beta}$; and we may then conveniently write it $\Sigma e^{\alpha+n\beta}$. Here the exponent of the n^{th} term is of the first order in n . It seems natural to extend the conception of this series by considering exponents of the second and higher orders in n , which lead to the series

$$\Sigma e^{\alpha+n\beta+n^2\gamma}, \Sigma e^{\alpha+n\beta+n^2\gamma+n^3\delta}, \text{ etc.}$$

We shall in fact now occupy ourselves with the series $\Sigma e^{\alpha+n\beta+n^2\gamma}$ which is called a Theta-series.

All these series may be regarded as extending in both directions; i.e. n may have all integer values both positive and negative; but there is an important difference between the cases in which the exponent is of an odd order in n and those in which it is of even order. The geometric series for example may be written

$$\dots + e^{\alpha-3\beta} + e^{\alpha-2\beta} + e^{\alpha-\beta} + e^{\alpha} + e^{\alpha+\beta} + e^{\alpha+2\beta} + e^{\alpha+3\beta} + \dots$$

but, as is well known, the series is always convergent towards one end and divergent towards the other, unless $\beta=0$. But the series $\Sigma e^{\alpha+n\beta+n^2\gamma}$ is always convergent towards both ends or divergent towards both ends, according as the real part of γ is negative or positive. And a similar thing is true of all those series in which the exponent is of even order in n .

For simplicity we shall suppose $\alpha=0$, which is equivalent to dividing the whole series by e^{α} . We shall also now use the symbols $2u$, a instead of β and γ . This being so, the series

(the terms being alternately $+$ and $-$, instead of all $+$ as before). This is distinguished as $\Theta(u, a)$; viz., we have $\Theta(u + \frac{1}{2}\pi i, a) = \Theta(u, a)$ and moreover

$$\Theta(u + \frac{1}{2}\pi i, a) = \Theta(u, a).$$

Next, adding half the quasi-period, we find

$$\begin{aligned} \Theta(u + \frac{1}{2}\pi i, a) &= \sum_{n=-\infty}^{\infty} e^{\pi i n^2 a} e^{2\pi i n u} = e^{-\frac{\pi}{4a}} \sum_{n=-\infty}^{\infty} e^{-\pi i n^2/a} e^{2\pi i n u} \\ &= e^{-\frac{\pi}{4a}} \sum_{n=-\infty}^{\infty} \left(\frac{1}{2} \right) \left(e^{\pi i n^2/a} + e^{-\pi i n^2/a} \right) e^{2\pi i n u} = e^{-\frac{\pi}{4a}} \Theta(u, a). \end{aligned}$$

The series in the brackets is distinguished as $\Theta_1(u, a)$, so that we have

$$\Theta(u + \frac{1}{2}\pi i, a) = e^{-\frac{\pi}{4a}} \Theta_1(u, a), \quad \Theta_1(u + \frac{1}{2}\pi i, a) = e^{-\frac{\pi}{4a}} \Theta(u, a).$$

We now calculate the result of adding $\frac{1}{2}\pi i$ to u . It is

$$\begin{aligned} \Theta(u + \frac{1}{2}\pi i + \frac{1}{2}\pi i) &= e^{-\frac{\pi}{2a}} \left(e^{\frac{1}{2}\pi i} e^{2\pi i u} + e^{-\frac{1}{2}\pi i} e^{2\pi i u} + 2e^{\frac{1}{2}\pi i} e^{2\pi i u} + e^{-\frac{1}{2}\pi i} e^{2\pi i u} \right) \\ &= 2e^{-\frac{\pi}{2a}} \cos \frac{1}{2}\pi u. \end{aligned}$$

(since $e^{\frac{1}{2}\pi i} = i$, $e^{-\frac{1}{2}\pi i} = -i$). The series in the brackets is distinguished as $\Theta_2(u, a)$; thus

$$\Theta(u + \frac{1}{2}\pi i + \frac{1}{2}\pi i) = e^{-\frac{\pi}{2a}} \Theta_2(u, a), \quad \Theta_2(u + \frac{1}{2}\pi i + \frac{1}{2}\pi i) = e^{-\frac{\pi}{2a}} \Theta(u, a).$$

It will be observed that this last function vanishes when $u = 0$ and changes sign with u .

For most physical applications it is convenient to convert the a series into a trigonometrical form, by the substitution of x for u . These are then regarded as functions of x will be denoted by a small θ instead of a large one. Thus, we shall write θ standing for Θ ,

$$\theta(x, a) = \Theta(x, a) = 1 + 2\theta_1 \cos 2\pi x + 2\theta_2^2 \cos 4\pi x + 2\theta_3^3 \cos 6\pi x + \dots$$

$$\theta_1'(x, a) = \Theta_1(x, a) = 1 - 2\theta_1 \cos 2\pi x + 2\theta_2^2 \cos 4\pi x - 2\theta_3^3 \cos 6\pi x + \dots$$

$$\theta_2(x, a) = \Theta_2(x, a) = 2\theta_1^2 \cos 2\pi x + 2\theta_2^2 \cos 4\pi x + 2\theta_3^3 \cos 6\pi x + \dots$$

$$\theta_3(x, a) = \Theta_3(x, a) = i \{ 2\theta_1^2 \sin 2\pi x + 2\theta_2^2 \sin 4\pi x + 2\theta_3^3 \sin 6\pi x + \dots \}.$$

The three latter functions differ from θ in the matter of the period and $\frac{1}{2}$ -quasi-period. θ has the period a and the quasi-period $\frac{1}{2}a$, all have the period $\frac{1}{2}a$ and are multiplied by an exponential factor when the argument is increased by $\frac{1}{2}a$. But θ_1 and θ_2 change sign when the argument is increased by $\frac{1}{2}a$, and θ_3 when it is increased by $\frac{1}{2}a$. These changes are indicated in the following table, (m, n any two positive integers),

$$\theta(x + m\pi + n\pi i) = e^{-\frac{\pi}{2a}(2m + 2n + 1)x} \theta(x, a),$$

$$\theta_1'(x + m\pi + n\pi i) = (-1)^m e^{-\frac{\pi}{2a}(2m + 2n + 1)x} \theta_1'(x, a),$$

$$\theta_2(x + m\pi + n\pi i) = (-1)^m e^{-\frac{\pi}{2a}(2m + 2n + 1)x} \theta_2(x, a),$$

$$\theta_3(x + m\pi + n\pi i) = (-1)^{m+n} e^{-\frac{\pi}{2a}(2m + 2n + 1)x} \theta_3(x, a).$$

The quantities $\frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \frac{7}{2}\pi$ are conveniently called *quadrants*. When we increase the argument of the θ by quadrants, they pass into one another according to the following table:

$$\begin{array}{l|l} \theta(x + \frac{1}{2}\pi) = \theta_1 x & \theta(x + \frac{3}{2}\pi) = e^{i\frac{1}{2}\pi} \theta_1 x \\ \theta(x + \frac{3}{2}\pi) = \theta_2 x & \theta(x + \frac{5}{2}\pi) = e^{i\frac{3}{2}\pi} \theta_1 x \\ \theta(x + \frac{5}{2}\pi) = \theta_3 x & \theta(x + \frac{7}{2}\pi) = e^{i\frac{5}{2}\pi} \theta_1 x \\ \theta(x + \frac{7}{2}\pi) = \theta_4 x & \theta(x + 2\pi) = e^{i\frac{7}{2}\pi} \theta_1 x \end{array}$$

where it will be observed that the factor e has been reduced to θ_1 for the sake of symmetry; the rule is then that the addition of $\frac{1}{2}\pi$ multiplies θ_1 and θ_2 by i (besides the transformation) and the addition of $\frac{3}{2}\pi$ multiplies all the θ by $e^{i\frac{3}{2}\pi}$.

Putting these results together we obtain the following table. (Kronecker's theorem)

| increase of argument | θ_1 | θ_2 | θ_3 | θ_4 | Exponential factor |
|---|--------------|--------------|--------------|--------------|-----------------------------|
| $m\pi + n\pi$ | θ_1 | θ_2 | θ_3 | θ_4 | $e^{i(m+n)\pi}$ |
| $(m + \frac{1}{2})\pi + n\pi$ | $i\theta_1$ | $i\theta_2$ | $i\theta_3$ | $i\theta_4$ | $e^{i(m+n+\frac{1}{2})\pi}$ |
| $m\pi + (n + \frac{1}{2})\pi$ | $i\theta_1$ | $i\theta_2$ | $i\theta_3$ | $i\theta_4$ | $e^{i(m+n+\frac{1}{2})\pi}$ |
| $(m + \frac{1}{2})\pi + (n + \frac{1}{2})\pi$ | $-i\theta_1$ | $-i\theta_2$ | $-i\theta_3$ | $-i\theta_4$ | $e^{i(m+n+1)\pi}$ |

II.

Product of two Theta Functions. Differential equation satisfied by the elliptic functions.

We shall now return for simplicity to the exponential form θ , and establish the theorem of great importance.

To prove that

$$(i) \quad \theta(x, \theta) \theta(x + \frac{1}{2}\pi, \theta) \theta(x + \pi, \theta) \theta(x + \frac{3}{2}\pi, \theta) = \theta(x, \theta) \theta(x + \pi, \theta) \theta(x + \frac{3}{2}\pi, \theta) \theta(x + 2\pi, \theta),$$

we have $\theta(x, \theta) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$, $\theta(x + \frac{1}{2}\pi, \theta) = \sum_{n=-\infty}^{\infty} a_{n+\frac{1}{2}} e^{2\pi i (n+\frac{1}{2})x}$, where $a_n = 0$ for all but integral values of n . Let us multiply these series together term by term; the result will be

$$\sum_{m,n} a_m a_{n+\frac{1}{2}} e^{2\pi i (m+n+\frac{1}{2})x},$$

where a double summation has to be effected, namely in regard to m and in regard to n . But now if we write $m+n = \mu$, $m-n = \nu$, this becomes

$$2 \sum_{\mu, \nu} a_{\frac{\mu+\nu}{2}} a_{\frac{\mu-\nu}{2}} e^{2\pi i \mu x},$$

and we find

$$\theta(x, \theta) \theta(x + \frac{1}{2}\pi, \theta) \theta(x + \pi, \theta) \theta(x + \frac{3}{2}\pi, \theta) = \theta(x, \theta) \theta(x + \pi, \theta) \theta(x + \frac{3}{2}\pi, \theta) \theta(x + 2\pi, \theta),$$

where μ, ν must not take all values independently, but must be either both even or both odd, because they are the sum and difference of two integers. Putting

them first equal to $2p, 2q$, and then to $2p+1, 2q+1$ we obtain two parts of the sum, which are respectively

$$\sum_{n=0}^{\infty} p^{2n} (p+q)(p+q+2n) e^{-\pi n} (p+q) = \sum_{n=0}^{\infty} p^{2n} (p+q)(p+q) e^{-\pi n} \sum_{n=0}^{\infty} 2q^{2n} (2q+2n) e^{-\pi n}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} p^{2n} (p+\frac{1}{2}+q+\frac{1}{2})(p+2p+\frac{1}{2}+q+\frac{1}{2}+2n) e^{-\pi n} \\ = \sum_{n=0}^{\infty} p^{2n} (\frac{1}{2}+q+\frac{1}{2})(p+\frac{1}{2}+q+\frac{1}{2}) e^{-\pi n} \sum_{n=0}^{\infty} 2q^{2n} (\frac{1}{2}+q+\frac{1}{2}+2n) e^{-\pi n}. \end{aligned}$$

But the products on the right are simply

$$O(u+v, 2a) O(u-v, 2a) \text{ and } O_1(u+v, 2a) O_1(u-v, 2a)$$

and therefore

$$O(u, 0) O(u+v, 2a) O(u-v, 2a) + O_1(u+v, 2a) O_1(u-v, 2a)$$

is easy to be proved.

We shall apply this theorem to obtain formulae relating to the squares and the products of the O . Writing $v = u$, and then $u = 0$, we have

$$O(u) = O(2u, 2a) O(0, 2a) + O_1(2u, 2a) O_1(0, 2a) \text{ and } O(0) = O^2(0, 2a) + O_1^2(0, 2a),$$

and now adding to u successively $\frac{1}{2}a, \frac{1}{2}a, \frac{1}{2}a, \frac{1}{2}a$, we get

$$O^2(u) = O^2(2u, 2a) O^2(0, 2a) + O_1^2(2u, 2a) O_1^2(0, 2a), \quad O(u) = O^2(0, 2a) + O_1^2(0, 2a),$$

$$O_1^2(u) = O_1^2(2u, 2a) O_1^2(0, 2a) + O^2(2u, 2a) O^2(0, 2a), \quad O_1(u) = O^2(0, 2a) + O_1^2(0, 2a),$$

$$O_1^2(u) = O_1^2(2u, 2a) O_1^2(0, 2a) + O^2(2u, 2a) O^2(0, 2a).$$

From these we obtain the following important equations (O_1^2 written for $O_1^2(0)$, etc.)

$$O_1^2 = O^2 O + O^2 + O_1^2 O + O^2 O_1,$$

$$O_1^2 + O_1 = O + O^2 + O^2 O + O^2 O_1,$$

$$O^2 + O = O_1 + O_1^2 O + O^2 + O^2 O_1.$$

And, either by putting $u = 0$ in the last, or directly from the equation on the right, we get

$$O^2 + O_1^2 = O_1.$$

It is convenient to write k for $\frac{O_1^2}{O^2}$; if $k^2 + k^2 = 1$, the last equation shows us that $k = \frac{O_1}{O}$.

To obtain expressions for the products of the O , substitute $u + \frac{1}{2}a$ for u in the formula above; thus we find

$$O(u), O(u) = O^2(2u, 2a) O^2(0, 2a), \text{ since } O_1^2(0, 2a) = 0.$$

Next put $v = u + \frac{1}{2}a$ then

$$e^{-\pi u} O(u), O_1(u) = O^2(2u + \frac{1}{2}a, 2a) O_1^2(\frac{1}{2}a, 2a) + O_1^2(2u + \frac{1}{2}a, 2a) O_1^2(\frac{1}{2}a, 2a).$$

Lastly put $v = u + \frac{1}{2}a + \frac{1}{2}a$ thus

$$e^{-2\pi u} O_1(u) = O^2(2u + \frac{1}{2}a, 2a) O^2(\frac{1}{2}a, 2a) + O_1^2(2u + \frac{1}{2}a, 2a) O_1^2(\frac{1}{2}a, 2a).$$

Hence finally

$$e^{-\pi u/\omega} \{ \hat{O}_1 u, \hat{O}' u - \hat{O}' u, \hat{O}_1 u \} = 2\hat{O}_1 e^{\frac{1}{2}\pi u} \{ \hat{O}'(u + \frac{1}{2}\pi, 2a) \{ \hat{O}'(2u + \frac{1}{2}\pi, 2a) + \hat{O}_1'(2u + \frac{1}{2}\pi, 2a) \}, \\ \therefore \hat{O}_1 u, \hat{O}' u - \hat{O}' u, \hat{O}_1 u = \frac{2\hat{O}_1 e^{\frac{1}{2}\pi u} \hat{O}'(u + \frac{1}{2}\pi, 2a)}{\hat{O}'(\frac{1}{2}\pi, 2a)} \hat{O} u, \hat{O}_1' u.$$

Similarly, we find by putting $u + \frac{1}{2}\pi i, v + \frac{1}{2}\pi i$ in the original formula

$$\hat{O}' u, \hat{O}' v - \hat{O}' u, \hat{O}' v = 2\hat{O}'(v - u, 2a) \hat{O}(v + u, 2a) = 2\hat{O}_1(v - u, 2a) \hat{O}_1(v + u, 2a).$$

Now write $v = u + \frac{1}{2}\pi$; we have in the first place

$$\hat{O}' u, \hat{O}'(u + \frac{1}{2}\pi) - \hat{O}' u, \hat{O}'(u + \frac{1}{2}\pi) = e^{-\pi u/\omega} \{ \hat{O}' u, \hat{O}_1' u - \hat{O}' u, \hat{O}_1' u - \hat{O}' u, \hat{O}_1' u \},$$

$$\text{but also} \quad = 2\hat{O}'(\frac{1}{2}\pi, 2a) \hat{O}(2u + \frac{1}{2}\pi, 2a) = 2\hat{O}_1(\frac{1}{2}\pi, 2a) \hat{O}_1(2u + \frac{1}{2}\pi, 2a).$$

Moreover we have

$$e^{-\pi u/\omega} \{ \hat{O}' u, \hat{O}_1' u - \hat{O}' u, \hat{O}_1' u \} = \hat{O}(\frac{1}{2}\pi, 2a) \hat{O}(2u + \frac{1}{2}\pi, 2a) = \hat{O}_1(\frac{1}{2}\pi, 2a) \hat{O}_1(2u + \frac{1}{2}\pi, 2a).$$

Now

$$\hat{O}(u + \frac{1}{2}\pi, 2a) = e^{-\pi u/\omega} \hat{O}_1(u - \frac{1}{2}\pi, 2a) \text{ or } e^{\frac{1}{2}\pi u} \hat{O}(u + \frac{1}{2}\pi, 2a) = e^{-\frac{1}{2}\pi u} \hat{O}_1(u - \frac{1}{2}\pi, 2a),$$

whence

$$\hat{O} e^{\frac{1}{2}\pi u} \hat{O}(u + \frac{1}{2}\pi, 2a) + e^{\frac{1}{2}\pi u} \hat{O}(u + \frac{1}{2}\pi, 2a) = \hat{O} e^{-\frac{1}{2}\pi u} \hat{O}_1(u - \frac{1}{2}\pi, 2a) + e^{-\frac{1}{2}\pi u} \hat{O}_1(u - \frac{1}{2}\pi, 2a),$$

or $(u = 0)$

$$\hat{O}(\frac{1}{2}\pi, 2a) + 2\hat{O}(\frac{1}{2}\pi, 2a) = \hat{O}_1(\frac{1}{2}\pi, 2a) + 2\hat{O}_1(\frac{1}{2}\pi, 2a) = \hat{O}_1 e^{\frac{1}{2}\pi u} \hat{O}(u + \frac{1}{2}\pi, 2a).$$

Hence finally

$$e^{-\pi u/\omega} \{ \hat{O}' u, \hat{O}_1' u - \hat{O}' u, \hat{O}_1' u \} = \hat{O}_1 e^{\frac{1}{2}\pi u} \hat{O}(u + \frac{1}{2}\pi, 2a) \{ \hat{O}(2u + \frac{1}{2}\pi, 2a) + \hat{O}_1(2u + \frac{1}{2}\pi, 2a) \},$$

$$\text{and} \quad \hat{O}' u, \hat{O}_1' u - \hat{O}' u, \hat{O}_1' u = \frac{\hat{O}_1 e^{\frac{1}{2}\pi u} \hat{O}(u + \frac{1}{2}\pi, 2a)}{\hat{O}(\frac{1}{2}\pi, 2a)} \hat{O} u, \hat{O}_1 u.$$

The three formulae thus arrived at may be written as follows:—

$$\hat{O}' u, \hat{O} u - \hat{O}' u, \hat{O} u = \alpha, \hat{O}' u, \hat{O}_1' u,$$

$$\hat{O}' u, \hat{O}_1 u - \hat{O}' u, \hat{O}_1 u = \beta, \hat{O}_1' u, \hat{O} u,$$

$$\hat{O}' u, \hat{O}_1' u - \hat{O}' u, \hat{O}_1' u = \gamma, \hat{O} u, \hat{O}_1 u,$$

and the quantities α, β, γ may now be determined in terms of $\hat{O}, \hat{O}', \hat{O}_1$ and \hat{O}_1' by substituting particular values of u . Thus putting $u = \frac{1}{2}\pi + \frac{1}{2}\pi i, \frac{1}{2}\pi i$, and 0 successively, we find

$$\hat{O}_1 \hat{O}_1' = \alpha \hat{O} \hat{O}',$$

$$\hat{O} \hat{O}_1' = \beta \hat{O}' \hat{O}_1,$$

$$\hat{O} \hat{O}_1 = \gamma \hat{O} \hat{O}_1.$$

If we now write

$$f u = \frac{\hat{O}}{\hat{O}_1}, \quad \frac{\hat{O}_1' u}{\hat{O}' u} = \mu u = \frac{\hat{O}'}{\hat{O}_1}, \quad \frac{\hat{O}_1 u}{\hat{O}' u} = h u = \frac{\hat{O}'}{\hat{O}_1}, \quad \frac{\hat{O} u}{\hat{O}' u} =$$

(the multipliers in the two latter being each chosen to make them $\rightarrow 1$ for $u = 0$), and in the former to simplify the following formulae, which were derived from those on p. 447, viz.:

$$1 + fu^2 + gu^3 = 1 + h^2(u^2 + 4u^3) = (h^2 + 4u^3)u^2 = 2u^2g,$$

then we have for the fluxions of these two factors

$$fu = Ngu, \quad hu = Nhu, \quad hu = h^2u^2/gu, \quad \text{where } N = \frac{u^2h^2}{u^2g^2}.$$

By substitution from the three equations we have, taking the squares of f, g, h we obtain

$$\begin{aligned} fu &= N\sqrt{(1 + fu^2)(1 + h^2u^2)}, & fu &= \int_0^u \frac{dx}{\sqrt{(1 + x^2)(1 + h^2x^2)}} \\ gu &= N\sqrt{(gu^2 - 4)(h^2u^2 + 4)}, & gu &= \int_0^u \frac{dx}{\sqrt{(g^2x^2 - 4)(h^2x^2 + 4)}} \\ hu &= N\sqrt{(1 + h^2u^2)(1 + f^2u^2)}, & hu &= \int_0^u \frac{dx}{\sqrt{(1 + h^2x^2)(1 + f^2x^2)}} \end{aligned}$$

when $u = \frac{1}{2}sh$, $fu = r$; writing in the latter the last and

$$\frac{1}{2}shN = \int_0^{\frac{1}{2}sh} \frac{dx}{\sqrt{(1 + x^2)(1 + h^2x^2)}} = R, \quad N = \frac{2R}{sh}.$$

To put the trigonometrical form, we write $\sin \phi = shu$, then $\sin \phi = \frac{1}{2}$, which in the same thing as saying that

$$\frac{1}{2}sh = \int_0^{\frac{1}{2}sh} \frac{dx}{\sqrt{(1 + x^2)(1 + h^2x^2)}} = R, \quad \text{then } h = \frac{2R}{sh}.$$

which gives N an explicit function of h .

In passing now to the trigonometrical form, we shall denote by N , the expression by N , because $f\left(\frac{y}{h}\right)$ becomes 1 when $x = y$. Writing $shu = \frac{y}{h} = \frac{shu}{sh}$, we shall have in Jacobi's notation,

$$\begin{aligned} f\left(\frac{yx}{h}\right) &= \sin \sin^{-1} x = \frac{\sin\left(\frac{\pi x}{sh}\right)}{\sin\left(\frac{\pi y}{sh}\right)}, \\ g\left(\frac{yx}{h}\right) &= \cos \sin^{-1} x = \sqrt{1 - \frac{\sin^2\left(\frac{\pi x}{sh}\right)}{\sin^2\left(\frac{\pi y}{sh}\right)}}, \\ h\left(\frac{yx}{h}\right) &= N \sin x = \frac{N\left(\frac{\pi x}{sh}\right)}{N\left(\frac{\pi y}{sh}\right)}. \end{aligned}$$

III.

The Addition-Theorem.

Starting from the formula

$$\partial u, \partial v = \overline{\partial u + v, 2a}, \overline{\partial u - v, 2a} + \overline{\partial_1 u + v, 2a}, \overline{\partial_1 u - v, 2a} \dots \dots \dots (1),$$

we derive by writing $u + \frac{1}{2}a$ for u and $v + \frac{1}{2}a + \frac{1}{2}\pi$ for v ,

$$\partial_1 u, \partial_1 v = \overline{\partial_1' u + v, 2a}, \overline{\partial_1' u - v, 2a} - \overline{\partial' u + v, 2a}, \overline{\partial_1' u - v, 2a} \dots \dots \dots (2),$$

$$\text{and so } \partial_1' u, \partial_1 v = \overline{\partial_1' u + v, 2a}, \overline{\partial' u - v, 2a} + \overline{\partial' u + v, 2a}, \overline{\partial_1' u - v, 2a} \dots \dots \dots (3).$$

Write $v=0$ in this formula; then

$$\partial_1' u, \partial_1 = \overline{2\partial' u, 2a}, \overline{\partial_1' u, 2a} \dots \dots \dots (4).$$

$$\text{So from } \partial_1 u, \partial_1 v = \overline{\partial_1 u + v, 2a}, \overline{\partial u - v, 2a} + \overline{\partial u + v, 2a}, \overline{\partial_1 u - v, 2a} \dots \dots \dots (5),$$

$$\text{we got } \partial_1 u, \partial_1 = \overline{2\partial_1 u, 2a}, \overline{\partial u, 2a}, \text{ and } \partial_1^2 = \overline{2\partial_1(0, 2a)}, \overline{\partial(0, 2a)} \dots \dots \dots (6, 7).$$

Substituting in (2), (3), we got

$$\begin{aligned} \overline{\partial_1' u + v, 2a}, \overline{\partial' u - v, 2a} - \overline{\partial' u + v, 2a}, \overline{\partial_1' u - v, 2a} &= \overline{\partial_1 u, \partial_1' v} \\ &= \frac{\overline{4\partial_1 u, 2a}, \overline{\partial u, 2a}, \overline{\partial' v, 2a}, \overline{\partial_1' v, 2a}}{(\partial_1^2 =) \overline{2\partial_1(0, 2a)}, \overline{\partial(0, 2a)}}, \end{aligned}$$

or omitting the $2a$ throughout

$$\partial, \partial_1 (\overline{\partial_1' u + v}, \overline{\partial' u - v} - \overline{\partial' u + v}, \overline{\partial_1' u - v}) = \overline{2\partial_1 u}, \overline{\partial u}, \overline{\partial' v}, \overline{\partial_1' v},$$

$$\partial, \partial_1 (\overline{\partial_1' u + v}, \overline{\partial' u - v} + \overline{\partial' u + v}, \overline{\partial_1' u - v}) = \overline{2\partial_1 v}, \overline{\partial v}, \overline{\partial' u}, \overline{\partial_1' u}.$$

Divide throughout by $\partial_1^2, \overline{\partial' u + v}, \overline{\partial' u - v}$; thus

$$f(u+v) + f(u-v) = \frac{2\partial_1 v, \partial v, \overline{\partial' u}, \overline{\partial_1' u}}{\partial_1^2, \overline{\partial' u + v}, \overline{\partial' u - v}} = 2f u, g v, h v \frac{\overline{\partial' u^2}, \overline{\partial' v^2}}{\overline{\partial'^2}, \overline{\partial' u + v}, \overline{\partial' u - v}} \dots \dots \dots (11).$$

To find the value of the dexter side we proceed as follows. We have

$$\partial' x, \partial y = \overline{\partial' x + y, 2a}, \overline{\partial' x - y, 2a} - \overline{\partial_1' x + y, 2a}, \overline{\partial_1' x - y, 2a} \dots \dots \dots (12),$$

$$\partial x, \partial' y = \overline{\partial' x + y, 2a}, \overline{\partial' x - y, 2a} + \overline{\partial_1' x + y, 2a}, \overline{\partial_1' x - y, 2a} \dots \dots \dots (13),$$

$$\text{therefore } \partial x, \partial' x = \overline{\partial' 2x, 2a}, \overline{\partial' 0, 2a} \dots \dots \dots (14).$$

Multiply together (12) and (13), taking account of (14) on the left, and omit the $2a$ throughout; thus

$$\partial'^2, \partial' 2x, \partial' 2y = \overline{\partial' x + y}^2, \overline{\partial' x - y}^2 - \overline{\partial_1' x + y}^2, \overline{\partial_1' x - y}^2,$$

or writing

$$2x = u + v, \quad 2y = u - v, \quad x + y = u, \quad x - y = v,$$

$$\partial'^2, \overline{\partial' u + v}, \overline{\partial' u - v} = \overline{\partial' u^2}, \overline{\partial' v^2} - \overline{\partial_1' u^2}, \overline{\partial_1' v^2},$$

V.

Product of Four Theta functions. Smith's reconstruction of Jacobi's method.

Consider four integers n_1, n_2, n_3, n_4 , and let $n_1 + n_2 + n_3 + n_4 = s$, $\nu_1 = s - 2n_1$, $\nu_2 = s - 2n_2$, $\nu_3 = s - 2n_3$, $\nu_4 = s - 2n_4$. Then $\nu_1 + \nu_2 + \nu_3 + \nu_4 = 2s$, and $2s - 2\nu_1 = 4n_1$, etc.

Multiply now together the four θ -series.

$$\theta x = \sum e \{ n_1^2 a + 2n_1 x \}, \quad \theta y = \sum e \{ n_2^2 a + 2n_2 y \},$$

$$\theta z = \sum e \{ n_3^2 a + 2n_3 z \}, \quad \theta w = \sum e \{ n_4^2 a + 2n_4 w \},$$

and we get a quadruply infinite series in which the exponent of the general term is

$$a \sum n^2 + 2 \{ n_1 x + n_2 y + n_3 z + n_4 w \}.$$

Now $\sum \nu^2 = 4 \sum n^2$, and if we write $\sigma = x + y + z + w$,

then

$$2 \{ n_1 x + n_2 y + n_3 z + n_4 w \} = \nu_1 \xi + \nu_2 \eta + \nu_3 \zeta + \nu_4 \omega,$$

where

$$2\xi = \sigma - 2x, \quad 2\eta = \sigma - 2y, \quad 2\zeta = \sigma - 2z, \quad 2\omega = \sigma - 2w.$$

So the exponent becomes

$$\frac{1}{4} a \sum \nu^2 + (\nu_1 \xi + \nu_2 \eta + \nu_3 \zeta + \nu_4 \omega).$$

Here if s is odd, the ν are all odd, and if s is even, they are all even. Thus the ν and $\frac{1}{4} \sum \nu^2$ must be either all odd or all even together.

First let them be even, and $= 2p_1, 2p_2, 2p_3, 2p_4$ respectively. Then

$$p_1 + p_2 + p_3 + p_4$$

must be even. Substituting, the exponent becomes

$$a \sum p^2 + 2 \{ p_1 \xi + p_2 \eta + p_3 \zeta + p_4 \omega \},$$

and this has to be summed under the condition that $\sum p$ is even. Call the exponent P , twice the sum required is

$$\sum^4 e | P | + \sum^4 (-)^{\sum p} e | P |, \quad \text{or} \quad \theta \xi \cdot \theta \eta \cdot \theta \zeta \cdot \theta \omega + \theta' \xi \cdot \theta' \eta \cdot \theta' \zeta \cdot \theta' \omega.$$

Next let the ν be odd and equal to $2p_h + 1$ ($h=1, 2, 3, 4$). Then $\frac{1}{4} \sum \nu^2 = \sum p + 2$ must be odd, or $\sum p$ must be odd. Hence if P is the exponent

$$a \sum (p + \frac{1}{2})^2 + 2(p_1 + \frac{1}{2} \xi + \dots),$$

twice the sum is

$$\sum^4 e | P | - \sum^4 (-)^{\sum p} e | P |, \quad \text{or} \quad \theta_1 \xi \cdot \theta_1 \eta \cdot \theta_1 \zeta \cdot \theta_1 \omega - \theta_1' \xi \cdot \theta_1' \eta \cdot \theta_1' \zeta \cdot \theta_1' \omega.$$

Hence the Theorem

$$2\theta x \cdot \theta y \cdot \theta z \cdot \theta w = \theta \xi \cdot \theta \eta \cdot \theta \zeta \cdot \theta \omega + \theta' \xi \cdot \theta' \eta \cdot \theta' \zeta \cdot \theta' \omega + \theta_1 \xi \cdot \theta_1 \eta \cdot \theta_1 \zeta \cdot \theta_1 \omega - \theta_1' \xi \cdot \theta_1' \eta \cdot \theta_1' \zeta \cdot \theta_1' \omega$$

$$= \sum (-)^{a\beta} \theta_\beta^a(\xi) \cdot \theta_\beta^a(\eta) \cdot \theta_\beta^a(\zeta) \cdot \theta_\beta^a(\omega).$$

To generalize this, add to the arguments $x \ y \ z \ w$ the quadrants $\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{matrix}$

The Addition-Theorem.

We have found expressions for the product of two different θ functions of $x+y$, $x-y$; the following value for $\theta_b^a(x+y)\theta_b^a(x-y)$ is used subsequently for functions of the second kind, as well as here for the addition-theorem. Writing for the ab the values $x-y$ $x+y$ 0 0 we have for the $\xi\alpha\beta$, $y-y$ x x and so

$$\begin{array}{cccc} a & a & 1 & 1 \\ b & b & 0 & 0 \end{array} \qquad \begin{array}{cccc} 1 & 1 & a & a \\ 0 & 0 & b & b \end{array}$$

$$2\theta_b^a(x+y)\theta_b^a(x-y)\theta^a = \theta^a y^2 \cdot \theta_b^a x^2 + \theta y^2 \cdot \theta_b^{a+1} x^2 + (-)^a \theta_1^a y^2 \cdot \theta_{b+1}^a x^2 - (-)^a \theta_1 y^2 \cdot \theta_{b+1}^{a+1} x^2, \dots (B).$$

But now writing for the ab , $x-y$ $x+y$ 0 0 we got for the $\xi\alpha\beta$, $y-y$ x x

$$\begin{array}{cccc} a & a & 1 & 1 \\ b+1 & b+1 & 1 & 1 \end{array} \qquad \begin{array}{cccc} 1 & 1 & a & a \\ 1 & 1 & b+1 & b+1 \end{array}$$

so that

$$0 = \theta_1^a y^2 \cdot \theta_{b+1}^a x^2 + \theta_1 y^2 \cdot \theta_{b+1}^{a+1} x^2 + (-)^a \theta^a y^2 \cdot \theta_b^a x^2 - (-)^a \theta y^2 \cdot \theta_b^{a+1} x^2, \dots (B');$$

then $H.A. (-)^a H'$ gives

$$\begin{aligned} \theta_b^a(x+y)\theta_b^a(x-y)\theta^a &= \theta^a y^2 \cdot \theta_b^a x^2 + (-)^a \theta_1^a y^2 \cdot \theta_{b+1}^a x^2 \\ &= \theta y^2 \cdot \theta_b^{a+1} x^2 - (-)^a \theta_1 y^2 \cdot \theta_{b+1}^{a+1} x^2. \end{aligned}$$

For example, writing $a, b=1, 0$ we have the important formula

$$\theta'(x+y)\theta'(x-y)\theta^2 = \theta^2 y^2 \cdot \theta^2 x^2 - \theta_1^2 y^2 \cdot \theta_1^2 x^2, \text{ or } \frac{\theta'x+\theta'y \cdot \theta'x-\theta'y \cdot \theta^2}{\theta^2 x^2 \cdot \theta^2 y^2} = 1 - k^2 f x^2 \cdot f y^2.$$

Dividing by this equation the three formulæ

$$\theta'x - y \cdot \theta_1^2 x + y \cdot \theta \cdot \theta_1 = \theta_1 y \cdot \theta y \cdot \theta_1^2 x \cdot \theta'x + \theta_1^2 y \cdot \theta' y \cdot \theta_1 x \cdot \theta x,$$

$$\theta'x - y \cdot \theta_1 x + y \cdot \theta_1 \cdot \theta' = \theta_1 y \cdot \theta' y \cdot \theta_1 x \cdot \theta'x + \theta_1^2 y \cdot \theta y \cdot \theta_1^2 x \cdot \theta x,$$

$$\theta'x - y \cdot \theta x + y \cdot \theta' \cdot \theta = \theta' y \cdot \theta y \cdot \theta' x \cdot \theta x + \theta_1^2 y \cdot \theta_1 y \cdot \theta_1^2 x \cdot \theta_1 x,$$

we find from the first

$$\frac{\theta_1^2 x + y \cdot \theta \theta_1}{\theta' x + y \cdot \theta^2} = \frac{\theta_1 y \cdot \theta y \cdot \theta_1^2 x \cdot \theta' x + \theta_1^2 y \cdot \theta' y \cdot \theta_1 x \cdot \theta x}{\theta^2 x^2 \cdot \theta^2 y^2 - \theta_1^2 x^2 \cdot \theta_1^2 y^2},$$

or

$$f(x+y) = \frac{fx \cdot gy \cdot hy + fy \cdot gx \cdot hx}{1 - k^2 f x^2 \cdot f y^2},$$

and so from the second and third

$$g(x+y) = \frac{gx \cdot hy + fx \cdot fy \cdot hx \cdot hy}{1 - k^2 f x^2 \cdot f y^2},$$

$$h(x+y) = \frac{hx \cdot hy + k^2 fx \cdot fy \cdot gx \cdot gy}{1 - k^2 f x^2 \cdot f y^2}.$$

Functions of the Third Kind.

In the equation

$$\frac{\theta'_x(x+y, \theta^2) - \theta'_x(x-y, \theta^2)}{\theta'^2(x), \theta'^2(y)} = 1 + k^2 f x^2, f y^2$$

take logarithmic fluxion in respect of y ; this is

$$\partial_y \log \theta'_x(x+y) - \partial_y \log \theta'_x(x-y) + 3 \partial_y \log \theta'_y(y) = \frac{2k^2 f x^2, \partial y y, \partial y, f y}{1 + k^2 f x^2, f y^2}.$$

Integrate in respect of x from 0 to x ; thus

$$\frac{1}{2} \log \frac{\theta'_x(x+y)}{\theta'_x(x-y)} + \frac{\partial_y}{\theta'_y} x = \lambda f y, \partial y, \partial y \int_0^x \frac{k^2 f x^2 dx}{1 + k^2 f x^2, f y^2}$$

$$\therefore \frac{1}{2} \log \frac{\theta' \left(\frac{ix}{2K} u + v \right)}{\theta' \left(\frac{ix}{2K} u - v \right)} + u \frac{\theta' \left(\frac{ixv}{2K} \right)}{\theta' \left(\frac{ixv}{2K} \right)} = \int_0^u \frac{\sin v \cos v du \sin^2 \sin^2 u du}{1 + k^2 \sin^2 v \sin^2 u}.$$

VI.

Abelian form of the Addition-Theorem.

Consider the curve

$$y^2 = x(1-x)(1-k^2x)$$

If we write

$$x = \sin^2 u,$$

then

$$y = \sin u \cos u \sin u,$$

and every point on the curve may be denoted by its parameter u .

Let us now cut the curve by the straight line

$$\xi x + \eta y = 1;$$

the abscissae of the points of contact are given by the equation

$$u = \{ (1 - \xi^2)^2 - \eta^2 v (1 - x) (1 - k^2 x) \}, \text{ say } \phi(x) = 0.$$

If we vary ξ, η in this equation, we shall also vary the roots of it. Let us now slightly vary one of the roots, then we shall have

$$\phi'(x) dx + \eta v (1 - \xi x) d\xi + 3 \eta x (1 - x) (1 - k^2 x) d\eta = 0.$$

Now since $\phi(x) = 0$, $\eta \sqrt{x(1-x)(1-k^2x)} = 1 - \xi x$,

$$\therefore \frac{dx}{\sqrt{x(1-x)(1-k^2x)}} + 2 \frac{\eta d\xi + (1 - \xi x) \eta d\eta}{\phi'(x)} = 0.$$

Run this equation for the three roots $x_1, x_2, x_3 = \sin^2 u, \sin^2 v, \sin^2 w$; the sum of the second terms vanishes, and the first terms are $2du, 2dv, 2dw$. Hence inte-

thus: if $a \mp f \mp x = 0$, $b \mp y \mp y = 0$, $c \mp b \mp x = 0$, we know that $x + y + z = 0$, or therefore $A(Choch)$. Hence if a conic have six-point contact at v , $0v \in C$ this shows that the six points of contact of tangents from an inflexion, are there are 27 of them.

3. The points of contact of four tangents from u are $-\frac{1}{2}u$, $-\frac{1}{2}u + I$, $\frac{1}{2}u + iK$, $\frac{1}{2}u + K + iK'$. Hence the line joining two of them meets the line joining the other two on the cubic. [Theory of corresponding points.]

4. Goursat's construction; so, A , ABH , $acC = 0$ is equation to a cubic of connecting the triangles abc , ABH and passing through the three intersection bcA , caH , abC of their corresponding sides. For BC , CA , AB write a , β , γ then

$$b \mp c \mp \beta \mp \gamma, c \mp a \mp \gamma \mp a, a \mp b \mp a \mp \beta,$$

whence we find $a \mp a \mp b \mp \beta \mp a \mp \gamma = K$ or iK' or $K + iK'$.

Thus given abc there is one triangle abH on the same branch.

VII.

The Theta functions expressed as infinite products.

To expand the product of n factors

$$H^n(1+x) = (1+x)(1+rx)(1+r^2x) \dots (1+r^{n-1}x)$$

in powers of x , assume that the expansion is

$$P = 1 + p_1x + p_2x^2 + \dots + p_nx^n.$$

Now if in the product we change x into rx , it becomes multiplied by $1+rx$, and divided by $1+x$. But if in P we change x into rx , it becomes

$$Q = 1 + p_1rx + p_2r^2x^2 + \dots + p_nr^nx^n.$$

Consequently we must have $P \cdot (1+rx) = Q \cdot (1+x)$. But

$$P(1+rx) = 1 + (p_1+rx) + (p_2+rp_1)x^2 + \dots + (p_n+rp_{n-1})x^n,$$

$$Q(1+x) = 1 + (p_1r+1)x + (p_2r^2+p_1r)x^2 + \dots + (p_nr^n+p_{n-1}r^{n-1})x^n.$$

Equating coefficients of like powers, we find

$$p_1 + rx = p_1r + 1, \quad \text{or } p_1 = \frac{1-r^n}{1-r},$$

$$p_2 + rx^2p_1 = p_2r^2 + p_1r, \quad \text{or } p_2 = p_1 \cdot \frac{1-r^{n-1}}{1-r} = r,$$

$$p_3 + rx^3p_2 = p_3r^3 + p_2r^2, \quad \text{or } p_3 = p_2 \cdot \frac{1-r^{n-2}}{1-r} = r^2,$$

$$\text{etc.} \dots \text{etc.}$$

$$p_n + rx^n p_{n-1} = p_nr^n + p_{n-1}r^{n-1}, \quad \text{or } p_n = p_{n-1} \cdot \frac{1-r}{1-r^{n-1}} = r^{n-1}.$$

Let the product $(1-r)(1-r^2)\dots(1-r^n)$ be called $\mathfrak{R}(n)$. Then our result may be written

$$\frac{11^n(y-r^2x)}{\mathfrak{R}(n)} = \frac{y^n}{\mathfrak{R}(n)} + \frac{y^{n-1}x}{\mathfrak{R}(n-1)} + \frac{x}{\mathfrak{R}(1)} \cdot r^{n-1} + \frac{y^{n-2}x^2}{\mathfrak{R}(n-2)} + \frac{x^2}{\mathfrak{R}(2)} \cdot r^{n-2} + \dots + \frac{x^n}{\mathfrak{R}(n)} r^{n(n-1)}.$$

As r approaches the limit 1, the fraction $\frac{1-r^n}{1-r}$ approaches the limit $\frac{n}{r}$. Hence the series just obtained passes into the binomial theorem.

To expand the product of the two factorials

$$11^n(1+xr^{2s-1})(1+xr)(1+xr^3)(1+xr^5)\dots(1+xr^{2n-1}) \\ 11^n(1+x^{-1}r^{2s-1})(1+x^{-1}r)(1+x^{-1}r^3)(1+x^{-1}r^5)\dots(1+x^{-1}r^{2n-1})$$

in positive and negative powers of x , we proceed as follows. Let each factor $1+x^{-1}r^{2s-1}$ of the second be replaced by $x^{-1}r^{2s-1}(1+xr^{1+2s})$, to which it is equal; then the factorial becomes

$$11^n(1+x^{-1}r^{2s-1}) \cdot x^{-n}r^{nn}(1+xr^3)(1+xr^5)(1+xr^7)\dots(1+xr^{2n-1}),$$

and in this form it is seen to be a continuation of the former factorial backwards, for negative powers of r . To put them both together therefore we must begin with the last factor of the second, $(1+xr^{2n-1})$. Let this be called $1+y$, then $x^{-n}r^{nn}y^{n(n-1)}$, and we have

$$11^n(1+xr^{2s-1})(1+x^{-1}r^{2s-1}) \cdot y^{-n}r^{nn-n}(1+y)(1+y^3)(1+y^5)\dots(1+y^{2n-1}) \\ = y^{-n}r^{n(n-n)}11^n(1+y^{2s}).$$

But by our previous result

$$\frac{11^{2n}(1+y^{2s})}{\mathfrak{R}^2[2n]} = \frac{1}{\mathfrak{R}^2[2n]} + \frac{1}{\mathfrak{R}^2[2n-1]} \cdot \frac{y}{\mathfrak{R}^2[1]} \\ + \frac{1}{\mathfrak{R}^2[2n-2]} \cdot \frac{y^3}{\mathfrak{R}^2[2]} + \dots + \frac{y^{2n}}{\mathfrak{R}^2[2n]} r^{2n(2n-1)}; \\ \therefore \frac{11^n(1+xr^{2s-1})(1+x^{-1}r^{2s-1})}{\mathfrak{R}^2[2n]} = \frac{x^{-n}}{\mathfrak{R}^2[2n]} r^{nn} + \frac{x^{1-n}}{\mathfrak{R}^2[2n-1]} \cdot \frac{r^{(n-1)}}{\mathfrak{R}^2[1]} \\ + \frac{x^{2-n}}{\mathfrak{R}^2[2n-2]} \cdot \frac{r^{(n-2)}}{\mathfrak{R}^2[2]} + \dots + \frac{1}{\mathfrak{R}^2[n]} \cdot \frac{1}{\mathfrak{R}^2[n]} + \dots + \frac{x^{n-1}}{\mathfrak{R}^2[2n-1]} \cdot \frac{r^{(n-1)}}{\mathfrak{R}^2[1]};$$

therefore

$$\frac{\mathfrak{R}^2[n] \cdot \mathfrak{R}^2[n]}{\mathfrak{R}^2[2n]} 11^n(1+xr^{2s-1})11^n(1+x^{-1}r^{2s-1}) = 1 + r(x+x^{-1}) + \frac{1-r^{2n}}{1-r^{2n-2}} \\ + r^3(x^2+x^{-2}) + \frac{1-r^{2n}}{1-r^{2n-4}} + \frac{1-r^{2n-2}}{1-r^{2n-6}} + \dots \\ + r^{2s}(x^s+x^{-s}) + \frac{1-r^{2n}}{1-r^{2n-2s}} + \frac{1-r^{2n-2}}{1-r^{2n-2s+2}} + \dots + r^{2n}(x^n+x^{-n}) \cdot \frac{\mathfrak{R}^2[n] \mathfrak{R}^2[n]}{\mathfrak{R}^2[2n]}.$$

Suppose r less than 1, and let n increase indefinitely. Then

$$\frac{R^2(n)R^2(n)}{R^2(2n)} = \frac{1-r^{2n}, 1-r^{2n+2}, \dots, 1-r^{2n+2n}}{1-r^{2n+2}, 1-r^{2n+4}, \dots, 1-r^{2n+2n}} \text{ now each of the factors } \frac{1-r^{2n}}{1-r^{2n+2}}$$

approaches the value $1-r^{2n}$; therefore the limiting value of the expression is $R^2[1]$. On the right hand side the factors $\frac{1-r^{2n+2n}}{1-r^{2n+2}, 1-r^{2n+4}, \dots, 1-r^{2n+2n}}$ approach the limit 1; therefore we have

$$R^2[x] = H_0''(1+x^{2n+1})(1+x^{-1}r^{2n+1}) \dots (1+r(x+x^{-1})+r^2(x^2+x^{-2})+\dots+r^n(x^n+x^{-n})+\dots).$$

Write q for R and $e^{2i\phi}$ for x ; the formula becomes

$$q^2[x], H_0''(1+2q^{2n+1}\cos 2\phi+q^{4n+2}) \dots (1+2q\cos 2\phi+2q^4\cos 4\phi+2q^9\cos 6\phi+\dots+r^n(y, \log q)) \text{ if } \phi = i\phi.$$

Either putting $\phi = \frac{\pi}{2}$ for ϕ or changing the sign of q , we have

$$q^2[x], H_0''(1-2q^{2n+1}\cos 2\phi+q^{4n+2}) \dots (1-2q\cos 2\phi+2q^4\cos 4\phi-2q^9\cos 6\phi+\dots+r^n(y, \log q)).$$

In the original formula write x for x ; it becomes

$$R^2[x], H_0''(1+x^{2n+1})(1+x^{-1}r^{2n+2}) \dots (1+r^2x(x^{-1}+r^{2n+2}+r^{2n+2}x^2+r^{2n+2}x^4+r^{2n+2}x^6+\dots+r^{2n+2}r^{2n+2}x^{2n+2}+r^{2n+2}x^{2n+4}+\dots)).$$

Hence also write q for R and $e^{2i\phi}$ for x , then we find

$$q^2[x], 2q^{\frac{1}{2}}\cos \phi H_0''(1+2q^{2n+2}\cos 2\phi+q^{4n+4}) \dots (2q^{\frac{1}{2}}\cos \phi+2q^{\frac{5}{2}}\cos 3\phi+2q^{\frac{9}{2}}\cos 5\phi+\dots)$$

In this if we write $\phi = \frac{\pi}{2}$ for ϕ , it becomes

$$q^2[x], 2q^{\frac{1}{2}}\sin \phi H_0''(1-2q^{2n+2}\cos 2\phi+q^{4n+4}) \dots (2q^{\frac{1}{2}}\sin \phi-2q^{\frac{5}{2}}\sin 3\phi+2q^{\frac{9}{2}}\sin 5\phi-\dots)$$

Now observing that $q=e^{\theta}$, and that the factor

$$(1+2q^{2n+1}\cos 2\phi+q^{4n+2}) \dots (1+q^{2n+1}e^{2i\phi})(1+q^{2n+1}e^{-2i\phi}),$$

which becomes

$$(1+e^{2n+1}i\alpha+3i\beta)(1+e^{2n+1}i\alpha-2i\beta)$$

$$= (e^{2n+1}i\alpha + (e + \frac{1}{2})\alpha + e^{2n+1}i\beta + \frac{1}{2}\alpha, n) (e^{2n+1}i\alpha - (e + \frac{1}{2})\alpha + e^{2n+1}i\beta + \frac{1}{2}\alpha, n)$$

$$= 4e^{2n+1}i\alpha \cos(\phi + e + \frac{1}{2}i\alpha) \cos(\phi - e + \frac{1}{2}i\alpha).$$

Let now (s, t) denote $(t + \frac{1}{2})a + (t + \frac{1}{2})\pi i$, and let

$$H\left(1 + \frac{x}{(s, t)}\right) = H_0 H_1 \left(1 - \frac{x^2}{[(t + \frac{1}{2})a + (t + \frac{1}{2})\pi i]^2}\right) \cdot \left(1 - \frac{x^2}{[(s + \frac{1}{2})a + (t + \frac{1}{2})\pi i]^2}\right),$$

it being understood that the infinite values of t are infinitely greater than the infinite values of s ; then we shall have

$$\rho_1 = \rho_0 H\left(1 + \frac{x}{(s, t)}\right).$$

VIII.

Cauchy's Theory of Doubly-Infinite Factorials.

A product like $H\left(1 + \frac{x}{(s, t)}\right)$ containing a doubly infinite number of factors depending on two variable integers s, t , is not fully defined until we have fixed upon the relations of the infinite values of s and t . Let these be regarded as coordinates of a point in a plane; then we may suppose that the product is formed first with those values of s, t which lie in a certain closed curve surrounding the origin, and that then this curve is allowed to expand without limit, remaining always similar to itself and similarly situated in regard to the origin. The doubly infinite product so obtained will depend in general upon the shape of this curve.

If we suppose the curve to have the origin for a centre, i.e. that every line through the origin is bisected by it, then the value of the product will be determined with the exception of a factor $e^{4\pi^2}$. For suppose two curves to be drawn having the origin for centre, and let H, H' be the products belonging to them. Then

$$\log H - \log H' = \sum \log \left(1 + \frac{x}{(s, t)}\right) - x \sum \frac{1}{(s, t)} + \frac{x^2}{2} \sum \frac{1}{(s, t)^2} \dots$$

the summation extending over those values of s, t which correspond to points lying between the two curves. Now $\sum \frac{1}{(s, t)} = 0$ because of the symmetry; and

let A be the limit of $\sum \frac{1}{(s, t)^2}$, in comparison with which all subsequent terms must vanish, because s, t become infinite when the curves are increased without limit. Therefore

$$\log H - \log H' = 4\pi^2 A,$$

$$\text{or} \quad H = e^{4\pi^2 A} H'.$$

Now we have shown that when $\frac{x}{s} \rightarrow 0$,

$$\rho_1 = \rho_0 H\left(1 + \frac{x}{(s, t)}\right) = \rho_0 H\left(1 + \frac{x}{(s + \frac{1}{2})a + (t + \frac{1}{2})\pi i}\right).$$

But

$$\begin{aligned} \frac{x}{(s+\frac{1}{2})a + (t+\frac{1}{2})\pi i} &= \frac{\frac{x\pi i}{a}}{(s+\frac{1}{2})\pi i + (t+\frac{1}{2})\frac{\pi^2}{a}}, \\ \therefore \theta x &= \theta \cdot \Pi \left(1 + \frac{x}{(s, t)} \right) = \theta \cdot \Pi \frac{\frac{x\pi i}{a}}{(s+\frac{1}{2})\pi i + (t+\frac{1}{2})\frac{\pi^2}{a}} \\ &= \frac{\theta \cdot e^{Ax^2} \cdot \theta \left(\frac{x\pi i}{a}, \frac{\pi^2}{a} \right)}{\theta \left(0, \frac{\pi^2}{a} \right)}, \end{aligned}$$

since, in order that the Π may represent $\theta \left(\frac{x\pi i}{a}, \frac{\pi^2}{a} \right)$, we must have $\frac{t_\infty}{s_\infty} = 0$; the former arrangement regarded the plane as an infinite rectangle, infinitely longer in the direction of t than in that of s ; the present one makes it infinitely longer in the direction of s than in that of t .

We must now determine A so that $e^{Ax^2} \theta \left(\frac{x\pi i}{a}, \frac{\pi^2}{a} \right)$ may be unaltered when x is increased by πi . We have

$$e^{A(x+\pi i)^2} \theta \left(\frac{x\pi i}{a} - \frac{\pi^2}{a}, \frac{\pi^2}{a} \right) = e^{2Ax\pi i - A\pi^2 - \frac{\pi^2}{a} + \frac{2x\pi i}{a}} \cdot e^{Ax^2} \theta \left(\frac{x\pi i}{a}, \frac{\pi^2}{a} \right),$$

whence $A = -\frac{1}{a}$, and the formula may be written

$$\theta \left(0, \frac{\pi^2}{a} \right) \cdot \theta x = \theta \cdot e^{-\frac{x^2}{a}} \theta \left(\frac{x\pi i}{a}, \frac{\pi^2}{a} \right) = \theta \cdot \Sigma e^{-\frac{1}{a}(x+n\pi i)^2}.$$

Observe that the formula may also be written

$$\theta \cdot \theta \left(\frac{x\pi i}{a}, \frac{\pi^2}{a} \right) = e^{\frac{x^2}{a}} \theta \left(0, \frac{\pi^2}{a} \right) \cdot \theta x = \theta \left(0, \frac{\pi^2}{a} \right) \Sigma e^{\frac{1}{a}(x+n\pi i)^2}.$$

Similarly for the other θ , we have

$$\theta' x = \theta' \cdot \Pi \frac{\sin i \cdot (x + \overline{s + \frac{1}{2} \cdot a}) \sin i \cdot (x - \overline{s + \frac{1}{2} \cdot a})}{\sin^2 i \cdot \overline{s + \frac{1}{2} \cdot a}}, \text{ and } \sin x = x \Pi \left(1 - \frac{x^2}{i^2 \pi^2} \right);$$

$$\begin{aligned} \therefore \frac{\sin i \cdot (x + \overline{s + \frac{1}{2} \cdot a})}{\sin i \cdot \overline{s + \frac{1}{2} \cdot a}} &= \frac{x + \overline{s + \frac{1}{2} \cdot a}}{(s + \frac{1}{2})a} \Pi \left(\frac{1 + i \frac{x + \overline{s + \frac{1}{2} \cdot a}}{i\pi}}{1 + i \frac{\overline{s + \frac{1}{2} \cdot a}}{i\pi}} \right) \\ &= \left(1 + \frac{x}{s + \frac{1}{2} \cdot a} \right) \Pi \left(1 + \frac{x}{s + \frac{1}{2} \cdot a + i\pi i} \right), \\ \therefore \theta' x &= \theta' \cdot \Pi \left\{ 1 + \frac{x}{(s, t)} \right\} \quad \left(\frac{s_\infty}{t_\infty} = 0 \right). \end{aligned}$$

$$\text{secondly} \quad \theta_1' v = \theta_1' u \cdot \Pi \left\{ 1 + \frac{x^2}{(v, u)} \right\}, \quad \left(\frac{e u}{t v} : : 0 \right),$$

$$\theta_1' v = \theta_1' u \cdot \Pi \left\{ 1 + \frac{x^2}{(v, u)} \right\} \cdot x, \quad \text{or}$$

and these formulae we may derive the transformations

$$\left(\theta_1 \frac{u^2}{v} \right), (u, v) = \left(\theta_1' u, v' \right) = \theta_1' \theta_1 \left(\frac{u^2}{u}, \frac{v^2}{u} \right) = \theta_1' \Sigma \left(\frac{1}{u} \cdot x(1+u) \frac{1}{2}, u \theta \right),$$

$$\left(\theta_1 \frac{u^2}{v} \right), (\theta_1 u, v) = \theta_1' \theta_1 \left(\frac{u^2}{u}, \frac{v^2}{u} \right) = \theta_1' \Sigma \{ (-) u \cdot x(1+u) \theta \},$$

$$\left(\theta_1 \frac{u^2}{v} \right), (\theta_1 v, v) = \theta_1' \theta_1 \left(\frac{u^2}{u}, \frac{v^2}{u} \right) = \theta_1' \Sigma \{ (-) u \cdot x(1+u) \frac{1}{2}, u \theta \},$$

expressing them in terms of $f(u, h)$, we find $\left\{ f'u \text{ for } f \left(\theta_1 \frac{u^2}{v} \right) \right\}$

$$f' \left(\frac{u^2}{v} \right) = \frac{f'u}{u} = f' \left(\frac{u^2}{u} \right) = \frac{1}{u} \cdot h' \left(\frac{u^2}{u} \right) = \frac{h'v}{uv},$$

hence

$$f'u^2 + f'v^2 = (v^2 + 1) f'u^2, \text{ or } f'u^2 = (1 + f'^2) f'^2 = (1 + f'^2) f'^2,$$

which it follows that f' is the f of the transformed function,

expressed thus, and result in terms of the elliptic functions we have only to consider the transformation changing K into K' and vice versa; so that we

if $\frac{u^2}{v} = \frac{u^2}{v} = \frac{u^2}{v} = \frac{u^2}{v}$, and consequently

$$\theta_1 \left(\frac{u^2}{v} \right) = \theta_1' \left(\frac{u^2}{v} \right) = \theta_1' \theta_1 \left(\frac{u^2}{u}, \frac{v^2}{u} \right) = \theta_1' \theta_1 \left(\frac{1}{u} \cdot x(1+u) \frac{1}{2}, u \theta \right) = \theta_1' \theta_1 \left(\frac{1}{u} \cdot x(1+u) \frac{1}{2}, u \theta \right),$$

in transformation must also be represented as follows. In virtue of the

$$\theta_1 \left(\frac{u^2}{v} \right) = \theta_1' \left(\frac{u^2}{v} \right) = \theta_1' \theta_1 \left(\frac{u^2}{u}, \frac{v^2}{u} \right) = \theta_1' \theta_1 \left(\frac{1}{u} \cdot x(1+u) \frac{1}{2}, u \theta \right),$$

$$\theta_1 \left(\frac{u^2}{v} \right) = \theta_1' \left(\frac{u^2}{v} \right) = \theta_1' \theta_1 \left(\frac{u^2}{u}, \frac{v^2}{u} \right) = \theta_1' \theta_1 \left(\frac{1}{u} \cdot x(1+u) \frac{1}{2}, u \theta \right),$$

$$\theta_1 \left(\frac{u^2}{v} \right) = \theta_1' \left(\frac{u^2}{v} \right) = \theta_1' \theta_1 \left(\frac{u^2}{u}, \frac{v^2}{u} \right) = \theta_1' \theta_1 \left(\frac{1}{u} \cdot x(1+u) \frac{1}{2}, u \theta \right),$$

$$\theta_1 \left(\frac{u^2}{v} \right) = \theta_1' \left(\frac{u^2}{v} \right) = \theta_1' \theta_1 \left(\frac{u^2}{u}, \frac{v^2}{u} \right) = \theta_1' \theta_1 \left(\frac{1}{u} \cdot x(1+u) \frac{1}{2}, u \theta \right),$$

$$\theta_1 \left(\frac{u^2}{v} \right) = \theta_1' \left(\frac{u^2}{v} \right) = \theta_1' \theta_1 \left(\frac{u^2}{u}, \frac{v^2}{u} \right) = \theta_1' \theta_1 \left(\frac{1}{u} \cdot x(1+u) \frac{1}{2}, u \theta \right),$$

$$\theta_1 \left(\frac{u^2}{v} \right) = \theta_1' \left(\frac{u^2}{v} \right) = \theta_1' \theta_1 \left(\frac{u^2}{u}, \frac{v^2}{u} \right) = \theta_1' \theta_1 \left(\frac{1}{u} \cdot x(1+u) \frac{1}{2}, u \theta \right),$$

$$\theta_1 \left(\frac{u^2}{v} \right) = \theta_1' \left(\frac{u^2}{v} \right) = \theta_1' \theta_1 \left(\frac{u^2}{u}, \frac{v^2}{u} \right) = \theta_1' \theta_1 \left(\frac{1}{u} \cdot x(1+u) \frac{1}{2}, u \theta \right),$$

it is evident that θ_1 is the function. Observe that the transformation

$$\int_{-\infty}^{\infty} \frac{dx}{x^2(1+x^2)^{n+1}} = K^n,$$

$$K^n = K^n$$

IX.

Problem of linear transformations.

The transformation just considered amounts to an interchange of the *period* and *quasi-period*; for the function $\theta\left(\frac{x\pi i}{\omega}, \frac{\omega' \pi i}{\omega}\right)$ has the period ω and the quasi-period ω' . If $\alpha, \beta, \gamma, \delta$ are whole numbers, the problem to express the θ -function which has the period $\alpha\omega + \beta\omega'$ and the quasi-period $\gamma\omega + \delta\omega'$ in terms of the θ -function with period πi and quasi-period ω is called the *Transformation Problem*. If we write $\alpha\omega + \beta\omega' = \omega_1$, $\gamma\omega + \delta\omega' = \omega'_1$, the θ -function is $\theta\left(\frac{x\pi i}{\omega_1}, \frac{\omega'_1 \pi i}{\omega_1}\right)$. Now $\alpha\delta - \beta\gamma = 1$, [in which case the transformation is said to be *linear*], we find that $-\omega_1/\delta\omega = \beta\omega'_1/\omega$ and $\gamma\omega'_1/\omega_1$; and on the plane of complex numbers the area of the parallelogram included by ω_1 is the same as that included by ω and ω' . The quotient $\theta\left(\frac{x\pi i}{\omega_1}, \frac{\omega'_1 \pi i}{\omega_1}\right)/\theta\left(0, \frac{\omega' \pi i}{\omega}\right)$ is then equal to the doubly infinite product $\prod\left(1 + \frac{x + \frac{1}{2} + (t-1)\omega'_1/\omega}{(\frac{1}{2} + (t-1)\omega'_1/\omega)}\right)$, $t = 0, \pm 1, \pm 2, \dots$, that is, when the plane is regarded as an infinite parallelogram whose sides are parallel to ω and ω' , but the former side infinitely greater than the latter. Hence we must have

$$\frac{\theta\left(\frac{x\pi i}{\omega_1}, \frac{\omega'_1 \pi i}{\omega_1}\right)}{\theta\left(0, \frac{\omega' \pi i}{\omega}\right)} = \rho_1 A_1 \theta_1^{\nu_1}(x) \quad \text{if } p_1 = 0, \omega_1 = e^{2\pi i \nu_1/\omega'} \quad \text{if } p_1 = 1, q_1 = 1,$$

To determine A_1 , observe that the left-hand side is constant when x is increased by ω_1 which is $-\alpha\omega + \beta\omega'$. Consequently

$$\rho_1 A_1 (x + \alpha\omega + \beta\omega') = \rho_1 A_1 x e^{2\pi i \nu_1 (\alpha + \beta \omega'/\omega)} = \rho_1 A_1 x e^{2\pi i \nu_1 (\alpha + \beta \omega'/\omega)},$$

therefore

$$A_1 (\alpha\omega + \beta\omega') = \rho_1 A_1 x e^{2\pi i \nu_1 (\alpha + \beta \omega'/\omega)},$$

$$A_1 (\alpha\omega + \beta\omega')^2 = -q_1 \alpha\omega + \beta_1 \omega' + 2\nu_1 \omega\omega',$$

or

$$\rho_1 (\alpha\omega + \beta\omega') = -q_1 \alpha\omega + \beta_1 \omega' + 2\nu_1 \omega\omega'.$$

Hence

$$\alpha q_1 - \alpha_1 \beta \pmod{2} \text{ but } q_1 - \beta_1 \delta \not\equiv 1, \quad \nu_1 = \alpha\delta - \alpha_1 \beta \pmod{2}.$$

Now

$$\left(x + \frac{1}{2}\right)\omega + \left(t + \frac{1}{2}\right)\omega' = \left(x + \frac{1}{2}\right)(\alpha\omega + \beta\omega') + \left(t + \frac{1}{2}\right)(\gamma\omega + \delta\omega')$$

$$= (\alpha + \gamma t + \frac{1}{2}\alpha + \frac{1}{2}\gamma)\omega + (\beta + \delta t + \frac{1}{2}\beta + \frac{1}{2}\delta)\omega'.$$

Hence the formula must be

$$\frac{\theta\left(\frac{x\pi i}{\omega_1}, \frac{\omega'_1 \pi i}{\omega_1}\right)}{\theta\left(0, \frac{\omega' \pi i}{\omega}\right)} = e^{2\pi i \nu_1 \left(\frac{\alpha + \gamma}{2} + \frac{\beta + \delta}{2} + \frac{1}{2}\right)} \frac{\theta_1^{\nu_1}(x)}{\theta_1^{\nu_1}(0)} \frac{\theta_1^{\nu_1}(\frac{1}{2})}{\theta_1^{\nu_1}(\frac{1}{2})}.$$

By since if $p, q = 0$ or 1 ,

$$\begin{aligned} \left(a + \frac{p}{\omega}\right)\omega + \left(t + \frac{q}{\omega}\right)\omega' &= \left(a + \frac{p}{\omega}\right)(a\pi i + \beta\omega) + \left(t + \frac{q}{\omega}\right)(\gamma\pi i + \delta\omega) \\ &= (a\gamma + \gamma t + \beta p\alpha + q\gamma)\pi i + (\beta a + \delta t + \beta p\beta + q\delta)\omega, \\ \frac{a' \left(\frac{x\pi i}{\omega} + \frac{\omega'\pi i}{\omega}\right)}{\theta'' \left(\frac{0}{\omega} + \frac{\omega'\pi i}{\omega}\right)} &= \frac{\theta_{q+1}^{p+1}(x + \frac{1}{2}\omega + \frac{1}{2}\omega')}{\theta_{q+1}^{p+1}(\frac{1}{2}\omega + \frac{1}{2}\omega')}, \end{aligned}$$

are odd cases of this. If $\beta\gamma = \alpha\delta = 1$, the four numbers cannot be all odd. If $\beta\gamma$ is odd, α or δ or both may be even; if $\alpha\delta$ is odd, β or γ may be even. Besides the case already considered, the only one of $\alpha, \gamma = \beta, \delta = 1, \beta = 0$ or $\alpha = \omega t, \omega' = \pi i + \omega$; in this we have

$$\begin{aligned} \theta(x, a + \pi i) &= \sum e^{a^2(n^2 + (a + \frac{1}{2})n)} + 2\pi x \dots \theta'(x, a), \\ \theta'(x, a + \pi i) &= \sum e^{a^2(n^2 + (a + \frac{1}{2})n)} + 2\pi x \dots \theta(x, a), \\ \theta_1(x, a + \pi i) &= \sum e^{(n + \frac{1}{2})^2(a + \frac{1}{2})n} + 2(a + \frac{1}{2})x \dots \theta_1(x, a), \\ \theta_1'(x, a + \pi i) &= \sum e^{a^2(n^2 + (a + \frac{1}{2})^2n + (a + \frac{1}{2})n + \frac{1}{4})} + 2(a + \frac{1}{2})x \dots \theta_1'(x, a), \end{aligned}$$

$$\frac{d\theta^2 x}{dx} = k^2 \frac{e^{2\pi x}}{d\theta^2 x} + \frac{e^{2\pi x}}{d\theta^2 x} \frac{d\pi x}{dx} = \frac{d\theta^2 x}{dx} + \frac{1}{d\theta^2 x},$$

the new modulus,

$$\frac{d\theta^2 x}{dx} + \lambda^2 \frac{e^{2\pi x}}{d\theta^2 x} = \frac{1 + \lambda^2 k^2 e^{2\pi x}}{d\theta^2 x} = 1, \text{ or } \lambda^2 = \dots \frac{k^2}{k^2}, \lambda = \frac{ik}{k^2},$$

$$\frac{d\theta^2 x}{dx} = k^2 \frac{e^{2\pi x}}{d\theta^2 x} + \frac{1}{d\theta^2 x} = k^2 \frac{e^{2\pi x}}{d\theta^2 x} + \frac{1}{d\theta^2 x},$$

$$\frac{d\theta}{dx} \left(k^2 x, \frac{ik}{k^2} \right) = k^2 \frac{e^{2\pi x}}{d\theta^2 x} + \frac{d\theta}{dx} \left(k^2 x, \frac{ik}{k^2} \right) = \frac{e^{2\pi x}}{d\theta^2 x} + \frac{d\theta}{dx} \left(k^2 x, \frac{ik}{k^2} \right) = \frac{1}{d\theta^2 x}.$$

Σ.

General Problem of Transformation.

Jacobi's Theorem for product of n θ -functions.

or the product of the n functions

$$\theta(x + u_i, a) = \sum e^{m_i^2 \frac{x^2}{4} + 2m_i x + 2u_i}, \quad (i = 1, 2, \dots, n)$$

we

$$\theta(x + u_i) = \sum e^{m_i^2 \frac{x^2}{4} + 2m_i x + 2u_i},$$

ation being so taken that the numbers m_1, m_2, \dots, m_n take all values from ω . For any particular set of values of the m , let their sum Σm be

Write $x = \frac{1}{2}\pi i$ for x , and remember that n is odd; thus

$$\frac{\theta'(n\tau, n\tau)}{\theta(\tau, n\tau)} = \frac{\Pi\theta'\left(x + \frac{\pi ni}{n}\right)}{\Pi\theta\left(\frac{\pi ni}{n}\right)} \text{ whenever } \frac{\theta'(0, n\tau)}{\theta(0, n\tau)} = \frac{\Pi\theta'\left(\frac{\pi ni}{n}\right)}{\Pi\theta\left(\frac{\pi ni}{n}\right)}.$$

Again, writing successively $x = \frac{1}{2}\pi i$, $x = \frac{1}{2}\pi i + \frac{1}{2}\pi$ for x , we find

$$\frac{\theta_3(n\tau, n\tau)}{\theta_3(0, n\tau)} = \frac{\Pi\theta_3'\left(x + \frac{\pi ni}{n}\right)}{\Pi\theta_3\left(\frac{\pi ni}{n}\right)},$$

$$\frac{\theta_4(n\tau, n\tau)}{\theta_4(0, n\tau)} = \frac{\Pi\theta_4'\left(x + \frac{\pi ni}{n}\right)}{\Pi\theta_4\left(\frac{\pi ni}{n}\right)},$$

whence

$$\frac{\theta_3(n\tau, n\tau)}{\theta_3(0, n\tau)} = \frac{\theta_4^{-1} \prod_{i=1}^{n-1} \theta_4'\left(\frac{\pi ni}{n}\right)}{\Pi\theta\left(\frac{\pi ni}{n}\right)},$$

$$\cos\left(\frac{\pi}{M} + \lambda\right) = \frac{\prod_{i=1}^{n-1} \cos\left(x + \frac{2s_i h}{n}\right)}{\prod_{i=1}^{n-1} \cos\left(K + \frac{2s_i K}{n}\right)},$$

where

$$\frac{1}{M} = \frac{\prod_{i=1}^{n-1} \sin\left(\frac{2s_i h}{n}\right)}{\Pi \sin\left(K + \frac{2s_i K}{n}\right)},$$

$$\cos\left(\frac{\pi}{M} + \lambda\right) = \frac{\Pi \cos\left(x + \frac{2s_i h}{n}\right)}{\Pi \cos\left(K + \frac{2s_i K}{n}\right)},$$

$$\sin\left(\frac{\pi}{M} + \lambda\right) = \frac{\Pi \sin\left(x + \frac{2s_i h}{n}\right)}{\Pi \sin\left(K + \frac{2s_i K}{n}\right)}.$$

XI.

Schröter's Theorem for product of two θ -functions.

We propose to find the value of the product

$$\begin{aligned}\theta(x, pa) \cdot \theta(y, qa) &= \sum e |n^2 pa + 2nx| \cdot \sum e |n^2 qa + 2ny| \\ &= \sum \sum e |m^2 p + n^2 q \cdot a + 2mx + 2ny|.\end{aligned}$$

Let $n-m$ when divided by $p+q$ give s for quotient and μ for remainder, so that $n-m = s(p+q) + \mu$, ($\mu < p+q$).

If then we make

$$m = t - qs,$$

we must have

$$n = t + ps + \mu,$$

and the numbers m, n will take all integer values from $-\infty$ to $+\infty$ if μ takes all positive integer values $< p+q$, and s, t take all integer values from $+\infty$ to $-\infty$. Making this substitution, the exponent of the general term becomes

$$\begin{aligned}& (\mu - qs)^2 pa + (t + ps + \mu)^2 qa + 2(\mu - qs)x + 2(t + ps + \mu)y \\ &= \mu^2 qa + 2\mu y + (t^2 \cdot p + q \cdot a + 2t \cdot x + y + \mu qa) + (s^2 \cdot pq \cdot p + q \cdot a + 2s \cdot py - qx + \mu pqa).\end{aligned}$$

Consequently we have

$$\begin{aligned}\theta(x, pa) \theta(y, qa) &= \sum_{\mu=0}^{\mu=p+q-1} e^{\mu^2 qa + 2\mu y} \theta(x + y + \mu qa, \overline{p+q} \cdot a) \cdot \theta(py - qx + \mu pqa, pq \cdot \overline{p+q} \cdot a) \\ &= \sum_{\mu=0}^{\mu=p+q-1} e^{\mu^2 pa + 2\mu x} \theta(x + y + \mu pa, \overline{p+q} \cdot a) \cdot \theta(qx - py + \mu pqa, pq \cdot \overline{p+q} \cdot a).\end{aligned}$$

For x, y write $y+x, ny-x$, and for $p, q, 1$ and n respectively, in the second formula; then

$$\begin{aligned}\theta(x+y, a) \theta(ny-x, na) &= \sum_{\mu=0}^{\mu=n} e^{\mu^2 a + 2\mu \cdot \overline{x+y}} \theta(\overline{n+1} \cdot y + \mu a, \overline{n+1} a) \theta(\overline{n+1} \cdot x + \mu na, n \cdot \overline{n+1} a).\end{aligned}$$

Assuming now that n is odd, write $x + \frac{1}{2}\pi i$ for x ; then

$$\begin{aligned}\theta'(x+y, a) \theta'(ny-x, na) &= \sum e^{\mu^2 a + 2\mu \cdot \overline{x+y}} (-)^{\mu} \theta(\overline{n+1} \cdot y + \mu a, \overline{n+1} a) \theta(\overline{n+1} x + \mu na, n \cdot \overline{n+1} a).\end{aligned}$$

When we add together these formulae, only those terms remain on the right for which μ is even; we may therefore write 2μ instead of μ , and then

$$\begin{aligned}\theta(x+y, a) \theta(ny-x, na) + \theta'(x+y, a) \theta'(ny-x, na) &= 2 \sum_{\mu=0}^{\mu=\frac{1}{2}(n-1)} e^{\mu^2 a + 4\mu \cdot \overline{x+y}} \theta(\overline{n+1} \cdot y + 2\mu a, \overline{n+1} a) \theta(n+1 \cdot x + 2\mu na, n \cdot \overline{n+1} a).\end{aligned}$$

$\theta_1(u) = 2\theta(2x, 4a)$, $\theta_2(u) = \theta(x = 2y, 4a)$, by direct addition of series,

$$(y, b) + \theta_1(u, a) \theta(y, b) = 2\theta(2x, 4a) \theta(2y, 4b) + 2\theta_1(2x, 4a) \theta_1(2y, 4b),$$

arriving by this formula the left hand of the equation had arrived at, writing $\frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}a$ instead of x, y, a , we find

$$\theta(y = x, ma + \theta_1) + \theta_1(y, a) \theta(y = x, ma) \\ = \sum_{n=0}^{\infty} e^{i\pi n^2(ma + \theta_1)/2} \theta\left(\frac{n+1}{2}y + \frac{n}{2}a, \frac{n+1}{2}a\right) \theta\left(\frac{n+1}{2}x + \frac{1}{2}pma, \frac{n}{2} + \frac{n+1}{2}a\right),$$

and further that $n+1$ is divisible by 1, let $n = 4m+1$, so that

$$\theta(y = x, ma) + \theta_1(y = x, a) \theta_1(y = x, ma) \\ = \sum_{m=0}^{\infty} e^{i\pi(4m+1)^2(ma + \theta_1)/2} \theta(2my + \frac{1}{2}ma, ma) \theta(2mx + \frac{1}{2}pma, ma),$$

writing $x = \frac{1}{2}a$ for x , adding the two formulae, and putting 2μ for μ , the θ terms remain in which μ is even, we get

$$\left. \begin{aligned} & \theta(y, a) \theta(4x = 1, y = x, 4a = 1a) \\ & + \theta(y, a) \theta'(4x = 1, y = x, 4a = 1a) \\ & + \theta(y, a) \theta_1(4x = 1, y = x, 4a = 1a) \\ & + \theta(y, a) \theta_1'(4x = 1, y = x, 4a = 1a) \end{aligned} \right\} \\ = 2 \sum_{\mu=0}^{\infty} e^{i\pi(4\mu+1)^2(4a + \theta_1)/2} \theta(2my + \mu a, ma) \theta(2mx + \mu a, ma),$$

because $4x = 1, a = \frac{1}{2}, y = \frac{1}{2}, \theta = 0$, we find, writing 0 for $\theta(0, 3a)$

$$\theta_1 \theta_1' + \theta \theta' = \theta(0, 3a) \sqrt{2H + 3} \sqrt{2H' + 1},$$

the modular equation in a form given by Jacobi. It is worth observing that more generally

$$\theta_1^2 \theta_1' + \theta \theta' = \theta_1^2 \theta_1'^2 + \theta^2 \theta'^2,$$

XII.

Formulas for theta and integrals of the third kind.

[*Continued*]

Let $\alpha_1 \alpha_2 = B = \frac{1}{\alpha_3^2} (1 - \alpha_3)(1 - \beta_3)(1 - \beta_3') (1 - \alpha_3')$, and if

$$\alpha_1 \rightarrow \int_0^{\alpha_1 \alpha_2 + \beta_3} \frac{dx}{R}, \quad \alpha_2 \rightarrow \int_0^{\alpha_1 \alpha_2 + \beta_3'} \frac{dx}{R}, \\ \alpha_3 \rightarrow \int_0^{\alpha_1 \alpha_2 + \beta_3'} \frac{dx}{R}, \quad \alpha_3' \rightarrow \int_0^{\alpha_1 \alpha_2 + \beta_3} \frac{dx}{R},$$

α_1, α_2 are the roots of a quadratic equation $L + 2Mx + Nx^2 = 0$, whose coeff-

$$e^{2v} = \frac{\theta' \frac{i\pi}{2K} (u_1 - a) \theta' \frac{i\pi}{2K} (u_2 - a)}{\theta' \frac{i\pi}{2K} (u_1 + a) \theta' \frac{i\pi}{2K} (u_2 + a)}.$$

Now since $u = u_1 + u_2$, [we have, transforming the expressions for

$$\wp', \wp'u, \wp'\overline{u_1 + a}, \wp'\overline{u_2 + a}, \wp'_1', \wp'_1'u, \wp'_1'\overline{u_1 + a}, \wp'_1'\overline{u_2 + a},$$

in which the second vanishes, by the formula for the multiplication of four functions, and adding the results]

$$\wp'u, \wp'\overline{u_1 + a}, \wp'\overline{u_2 + a} = \wp'u + a \cdot \wp'a \cdot \wp'u_1 \cdot \wp'u_2 + \wp'_1'\overline{u + a} \cdot \wp'_1'a \cdot \wp'_1'u_1 \cdot \wp'_1'u_2,$$

$$e^{2v} = \frac{\wp'\overline{u - a} \cdot \wp'a \cdot \wp'u_1 \cdot \wp'u_2 - \wp'_1'\overline{u - a} \cdot \wp'_1'a \cdot \wp'_1'u_1 \cdot \wp'_1'u_2}{\wp'u + a \cdot \wp'a \cdot \wp'u_1 \cdot \wp'u_2 + \wp'_1'\overline{u + a} \cdot \wp'_1'a \cdot \wp'_1'u_1 \cdot \wp'_1'u_2},$$

$$\frac{\wp'_1'u_1 \cdot \wp'_1'u_2 \cdot \wp'_1'a}{\wp'u_1 \cdot \wp'u_2 \cdot \wp'a} = \frac{e^{-v} \wp'(u - a) - e^v \wp'(u + a)}{e^{-v} \wp'_1'(u - a) + e^v \wp'_1'(u + a)} = k^2 \sin a \sqrt{x_1 x_2}.*$$

* [Here the MS. ends.]

If we multiply together two series—for example the series for e^x and e^y , we get a result of this kind:

$$\begin{array}{cccccc}
 1 & + & x & + & \frac{x^2}{1!2} & + & \frac{x^3}{1!3} & + & \frac{x^4}{1!4} & + & \dots \\
 + & & + & & + & & + & & + & & \\
 y & + & xy & + & \frac{x^2y}{1!2} & + & \frac{x^3y}{1!3} & + & \frac{x^4y}{1!4} & + & \dots \\
 + & & + & & + & & + & & + & & \\
 \frac{y^2}{1!2} & + & \frac{xy^2}{1!2} & + & \frac{x^2y^2}{1!2 \cdot 1!2} & + & \frac{x^3y^2}{1!3 \cdot 1!2} & + & \frac{x^4y^2}{1!4 \cdot 1!2} & + & \dots \\
 + & & + & & + & & + & & + & & \\
 \frac{y^3}{1!3} & + & \frac{xy^3}{1!3} & + & \frac{x^2y^3}{1!2 \cdot 1!3} & + & \frac{x^3y^3}{1!3 \cdot 1!3} & + & \frac{x^4y^3}{1!4 \cdot 1!3} & + & \dots \\
 + & & + & & + & & + & & + & & \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
 \end{array}$$

This may be called a *doubly infinite series*; the general term is $\frac{x^my^n}{m!n!}$, containing two variable integers m, n , and we obtain the series by giving to each of these all positive integral values, zero included. In summing this series, the terms may be taken in various orders. We may first take all the terms in the first horizontal line, then all those in the second, and so on; this presents the series in the singly infinite form

$$e^x + ye^x + \frac{1}{2}y^2e^x + \dots$$

the sum of which we know to be e^xe^y . Or we may take the left-hand column first, then the second column, and so on; this presents the series in the form

$$e^y + xe^y + \frac{1}{2}x^2e^y + \dots$$

whose sum is again e^ye^x . The former process approaches the quarter of an infinite plane on which the series is spread out as an infinitely long horizontal rectangle whose breadth is increased without limit; the latter as an infinitely long vertical rectangle whose breadth is increased without limit.

We may also reduce the series to a singly infinite one in the following way. Namely, it is equal to

$$\begin{aligned}
 1 + (x+y) + \left(\frac{x^2}{1!2} + xy + \frac{y^2}{1!2} \right) + \left(\frac{x^3}{1!3} + \frac{x^2y}{1!2} + \frac{xy^2}{1!2} + \frac{y^3}{1!3} \right) + \dots \\
 = 1 + x + y + \frac{(x+y)^2}{1!2} + \frac{(x+y)^3}{1!3} + \dots = e^{x+y}
 \end{aligned}$$

since, by the binomial theorem for a positive integer exponent,

$$\frac{(x+y)^n}{1!n} = \frac{x^n}{1!n} + \frac{x^{n-1}y}{1!(n-1)} + \frac{x^{n-2}y^2}{1!(n-2) \cdot 1!2} + \dots + \frac{y^n}{1!n} = \sum \frac{x^ay^b}{1!a \cdot 1!b}, \quad (a+b=n)$$

Now this mode of summing the doubly infinite series takes together the terms on

an oblique line joining corresponding terms of the top row and left-hand column. It approaches the quarter of an infinite plane as a triangle in the shape of half a square whose size is indefinitely increased.

We may sum this doubly infinite series in yet another way which leads to a useful result. Let $x=uv$, $y=\frac{u}{v}$; and let moreover

$$\begin{aligned} fu &= 1 + u + \frac{u^2}{(112)^2} + \frac{u^3}{(113)^3} + \frac{u^4}{(114)^4} + \dots \\ f_k u &= (\partial_u)^k fu = \frac{1}{11k} + \frac{u}{11(k+1)} + \frac{u^2}{112 \cdot 11(k+2)} + \dots \\ &= \sum \frac{u^k}{11k \cdot 11(n+k)}. \end{aligned}$$

Then we shall find for the product $e^{uv} \cdot e^{\frac{u}{v}}$ the doubly infinite series

$$\begin{array}{cccc} 1 & + & uv & + & \frac{u^2 v^2}{112} & + & \frac{u^3 v^3}{113} & + & \dots \\ & + & & + & & + & & + & \\ uv^{-1} & + & u^2 & + & \frac{u^3 v}{112} & + & \frac{u^4 v^2}{113} & + & \dots \\ & + & & + & & + & & + & \\ \frac{u^2 v^{-2}}{112} & + & \frac{u^3 v^{-1}}{112} & + & \frac{u^4}{112 \cdot 112} & + & \frac{u^5 v}{112 \cdot 113} & + & \dots \\ & + & & + & & + & & + & \\ \frac{u^3 v^{-3}}{113} & + & \frac{u^4 v^{-2}}{113} & + & \frac{u^5 v^{-1}}{113 \cdot 112} & + & \frac{u^6}{113 \cdot 113} & + & \dots \\ & + & & + & & + & & + & \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

And if we now sum first all the terms on the middle diagonal going downwards to the right from the term 1, then those in the two lines parallel to this on either side, and so on, we shall obtain

$$e | u(v+v^{-1}) | = f(u^2) + (v+v^{-1}) u f_1(u^2) + (v^2+v^{-2}) u^2 f_2(u^2) + (v^3+v^{-3}) u^3 f_3(u^2) + \dots$$

In this formula write ix for u , and $ie^{i\theta}$ for v , so that

$$u(v+v^{-1}) = -2x \sin \theta, \quad u^k(v^k + v^{-k}) = (-1)^k 2x^k \cos k \left(\theta + \frac{\pi}{2} \right),$$

$$\text{then } e | -2ix \sin \theta | = f(-x^2) - 2ix f_1(-x^2) \cdot \sin \theta - 2x^2 f_2(-x^2) \cos 2\theta - \dots$$

$$= f(-x^2) + 2 \sum_{n=1}^{\infty} (i)^n \cdot x^n f_n(-x^2) \cdot \cos n \left(\theta + \frac{1}{2} \pi \right),$$

so that

$$\cos(2x \sin \theta) = f(-x^2) - 2x^2 f_2(-x^2) \cos 2\theta + 2x^4 f_4(-x^2) \cos 4\theta - \dots$$

$$\sin(2x \sin \theta) = 2x f_1(-x^2) \sin \theta - 2x^3 f_3(-x^2) \sin 3\theta + 2x^5 f_5(-x^2) \sin 5\theta - \dots$$

The function $x^n f_n(-x^2)$ is called Bessel's function of the n^{th} order, and is generally denoted by $J_n(2x)$.

This process of summing the doubly infinite series approaches the infinite area as an infinitely long figure parallel to the middle diagonal, whose breadth is indefinitely increased.

Thus we have considered four ways of approaching the infinite plane, and each of them consists in taking an area of a certain shape and then allowing it to expand indefinitely.

If the two numbers m, n which determine the place of any term in the series are allowed to take negative as well as positive values, the doubly infinite series will cover the whole plane instead of only a quarter of it. The process of summing the series will still consist in taking an area of a certain shape, and allowing it to expand indefinitely while it remains similar to itself and similarly situated in regard to the origin.

The question is, does the sum of the series depend upon the shape of this area? We may show very easily that it does *not* so depend when the terms are all real and positive. For let P be the sum of the series when it is summed in one way; then the sum of no portion of the series, however selected, can exceed P . If then P' be an approximation to P made by taking a large number of steps towards the summation that way, Q' a similar approximation to Q , the sum obtained by another arrangement; then P' cannot exceed Q , nor Q', P . And since $P > P', Q > Q'$ can be made as small as we like by proceeding sufficiently with the summation, it follows that P' cannot exceed Q , nor Q, P , and consequently $P = Q$.

Hence as before the sum is independent of the mode of summation if the series converges when we substitute for each term its modulus.

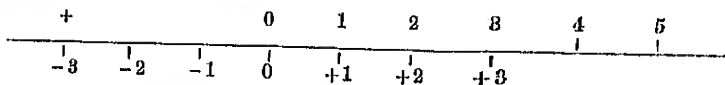
Similar reasoning applies to multiply infinite series, or, as a particular case, to products of any number of singly infinite series.

Now in the case of the product just considered, and in that of the products of θ -functions to be presently treated, it is plain that this condition is satisfied. The transformations of this section, therefore, are all of them valid.

NOTES OF LECTURES ON QUATERNIONS*.

We will define the signs + and - as indicating steps, whereby any magnitude may be increased or diminished, or by means of which we may move from one point of a progression to another.

Let the quantity which has the progressive values be measured along a line; but we shall suppose at first that the numbers stand for amounts of any kind of quantity, not necessarily length.



Then we should have a zero value and successive values marked off, *above* the line, by the numbers 1 2 3 etc. Numbers standing alone mean amounts of the given quantity, while the sign + or - before them means a step is to be taken through successive values of that quantity so as to lead to a different value from that from which we set out.

Thus the equation

$$2 + 3 = 5$$

means that if we start with 2 things represented by 2 places on the line and step through 3 following places we get to the place indicated by 5.

But there is another way of considering this equation. Instead of marking off on our line a scale of numbers merely it may be a scale of steps as by the positive and negative numbers below the line.

And then our equation would be

$$+2 + 3 = +5.$$

Or whatever number we set out from and take two steps to the right and then three again to the right it is the same as taking 5 at once to the right. Here are two interpretations of the equation.

Again consider this form of the same truth

$$5 - 3 = 2.$$

This may mean first—

Starting with a number 5 denoted by its place on the line, and taking three steps backward, to the left, we get to the number 2. Or it may mean, if we consider the scale as one of steps,

* [I am indebted to Miss E. Watson for these *Notes* of lectures delivered at University College, towards the close of 1877.]

A step 5 to the right and then a step 3 to the left leads to the same number as if starting with the same we took a step 2 to the right.

Now let us combine two forward steps,

$$2 + 3 + 4 = 9 = 2 + 7.$$

This means that if we start at the place 2 and take 3 steps to the right and then 4 to the right we get to the place 9; or this is the same as starting with the place 2 and taking a step 7 forward.

But if this is written

$$+2 + 3 + 4 = +9,$$

we should interpret it thus:

Starting with any number and making the steps to the right represented by 2, 3, 4, is the same as making the step 9 to the right.

Further

$$2 = 9 - 4 - 3$$

may mean either that starting at the place 9 and taking 4 steps to the left and then 3 again to the left we get to the place: or it may mean that starting anywhere, making a step 9 to the right and then steps 4 and 3 to the left is the same as making the step 2 to the right.

On the whole then we have two modes of interpreting these operations of addition and subtraction.

The corresponding forms of the last written equation are

$$2 + 3 + 4 = 9,$$

$$+2 + 3 + 4 = +9.$$

The common interpretation would be 2 things and 3 things and 4 things all treated in the same way make 9 things.

I. The first equation consists of a number with two steps performed on it, leading to a number.

II. The second means altogether steps, which lead to a step performed on any number.

Whenever as a result we get a negative expression, it means a step of *decrease* of a quantity or of direction.

We now go on to the symbols of Multiplication.

Again every equation has two distinct interpretations, as well as the common arithmetical one:

$$2 \times 3 = 6,$$

may be read '6 is the product of 2 and 3', in which case numbers are treated in just the same way.

But we shall leave this meaning out of account and read the equation either as

I. Twice three are 6. I perform an operation on three things by the operator 2; or,

II. I take any number, double and triple it. This is equivalent to the operation of sextupling it.

Now let us consider the multiplication of steps instead of things.

In the equation

$$2 \times (+3) = +6$$

the last term on each side is a step, the first is an operator and the equation means by means of doubling I can turn a step 3 to the right into a step 6 to the right.

We notice in passing that in Multiplication the operation to be performed comes first; in addition it comes after the thing or step operated on.

Now what operator is required to turn the step -3 to the left into the step $+6$ to the right?

First we reverse the step by an operator which we will call r , $\{r(-3) = +3\}$; thus it becomes $+3$. Now double it, and the whole operation is written,

$$2r(-3) = +6,$$

so the required operator is $2r$, which means reverse and then multiply by 2.

But we may change the order of the process, viz. double and then reverse and we get the same result

$$r2(-3) = +6.$$

Now we will construct an equation analogous to this but which shall consist entirely of operators. Here we have two steps.

Let $k3$ mean triple without reversing. And let us suppose any step taken, tripled, reversed, doubled and reversed again. The two reversals will clearly destroy each other and give "no reversal" or k , and we shall have our step hex-tupled without reversal. This may be written as an equation like the last. And in the same way we have two others in which the direction is reversed:

$$\begin{array}{ll} \text{I.} & r2(-3) = +6, \\ \text{II.} & r2(r3) = k6, \end{array} \quad \left| \quad \begin{array}{l} r2(+3) = -6, \\ r2(k3) = r6; \end{array} \right.$$

and we are led to assign a new meaning to the symbols $+$, $-$; we may use them instead of k and r respectively.

Thus their meaning is extended from that of indicators of steps to operators on steps. Unless this extended meaning is borne in mind, many equations would be unmeaning, e.g. the familiar one,

$$-2(-3) = +6.$$

For there is no meaning in Multiplying one step by another.

We may assign two reasonable interpretations. Either both $-$ signs mean reverse, or the second is a step and the first means reverse.

In every equation of multiplication the last factor on each side may mean a step and then all the rest must be operations, or they may all be operations. It will be necessary to examine in the particular cases whether both meanings are allowable or whether only one may be given.

Let us now take lengths to stand for quantities either commensurable or incommensurable. If they are incommensurable (that is if they are among the values of a continuous quantity) the only way to represent them is on a certain scale.

Quantities of length are generally added by placing them end to end, treating them simultaneously in exactly the same way.

But we shall always suppose one length given and the other numbers with their signs to mean operations to be performed on it. Addition shall be represented by a step to the right along the line of the given length and subtraction a step to the left along the same line.

With the meaning of a quantity so far developed the product of two quantities represented by lines will be given by the rectangle on these lines. Then any product of higher degree than it will be unmeaning because it will be of higher dimensions in space than 3.

We learn however that before Descartes, this linear way of representing quantities was the only one used for the solution of equations. Vieta, in his treatise on Algebra imagined space of 0 dimensions in order to explain his equations.

The different orders began with Linear, Planum, Plano-planum, and went down to, solido solido solidum.

The equation

$$x^3 + 6x^2 + 6x + 6$$

was interpreted as the sum of a number of solidi, viz

$$\text{cube of } x + 6 \text{ linear } x^2 + 6 \text{ planum } x + 6 \text{ solidum,}$$

and the equation was actually solved by cutting up a cube.

Just as in the first treatise on Algebra introduced into Europe from Arabia, by Cardan*, the equation $x^2 + 2x - 15$

was solved by a construction in a plane (see Fig. 69) which gives the value

$$x = -3.$$

Descartes that gave another meaning to the product of two quantities. He involved it here by letting numbers stand for the *values* of quantities. The length of any line would then be the operation which is necessary to convert the unit of length into that line.

And the product of two quantities becomes the ratio compounded of the ratios which they bear to the unit.

With such a change of meaning we may write x^2 without supposing a figure of n dimensions. It will now simply be the n th power of the ratio of the line x to the unit. This is clearly equivalent to our second way of looking at multiplication. But we may also use the first way as well. x^2 may then mean, not the rectangle on xb , but either

I. The line which bears the same ratio to b which x bears to 1,

II. Where a and b are both operations, the ratio compounded of the ratios of a and b to the unit.

Just as before, the only choice we have is with the last number on either side, which may be either a quantity or an operation.

* *De Arte Invenendi*, 1545, p. 66. *Opera*, ed. J. Prætorius, Lugduni Batavorum, 1722, 212 f.

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cube of $x + a$ linear $x^2 + b$ planum $x + c$ solidum,

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$$\text{the equation } x^2 + 2x = 15$$

was solved by a construction in a plano [see Fig. 60] which gives the value

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And the product of two quantities becomes the *ratio* compounded of the ratios which they bear to the unit.

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I. The line which bears the same ratio to b which a does to 1.

II. Where a and b are both operations; the *ratio* compounded of the ratios of a and b to the unit.

Just as before, the only choice we have is with the last number on either side, which may be either a *quantity* or an *operation*.

* [de Arithmetica, lib. x. cap. v.; cf. also Chasles, *Aperçu historique*, pp. 482, 541.]

Or if instead of a scale of quantities we have a scale of steps,

$$(a)(-b) = -c$$

may mean

I. Take a step b to the left and increase it in the ratio a to 1 and you get a step c to the left, or

II. Take any step, multiply it by the ratio b , reverse it, multiply it by the ratio a ,—then the result is the step multiplied by c and reversed.

As yet we have considered only scalar quantities; which are either steps of addition or subtraction, i. e. steps of position on a straight line, or operations performed on those steps. These last may be equally considered as ratios of steps. We extended the meaning of quantity from simple number to these two: steps and operations. We were led to 'steps' by the appearance of negative quantity. But there is still another unexplained symbol, namely the square root of a negative quantity. Descartes' method takes no account of such a quantity.

We again define the symbol $+$ by the equation

$$OP = OM + MP,$$

so that $+$ now means a step in a plane [Fig. 61].

This equation holds as a definition whenever a step can be made in two instalments.

Whatever may be the angles made between the component steps we shall write [Fig. 62]

$$ac = ab + bc.$$

If OM and MP are to stand for the ratios of these lines to a unit line, we must have a unit in the direction of OM and another in that of MP . Then if x and y are ratios,

$$OP = x \text{ times } OI + y \text{ times } OI'.$$

x and y are scalars, ratios of steps to the unit steps. x is positive whenever P is on the right of O and y is positive whenever P is above OX .

Thus two scalar numbers x and y must be assigned in order to determine OP .

We will lay down a rule for adding two steps. Place the beginning of the second at the end of the first, then the line joining the beginning of the first to the end of the second will be their sum.

In giving this rule we make an assumption, viz. that the step PR is the same as OQ . Assuming this,

$$OR = OP + OQ.$$

But we have also

$$OR = ON + NR.$$

And ON is the sum of the horizontal parts of OP and OQ and NR of their vertical parts. We may express this by the equations—

if $OP = x \text{ times } OI + y \text{ times } OI'$; and $OQ = x' \text{ times } OI + y' \text{ times } OI'$;

$$OP + OQ = \overline{x + x'} \text{ times } OI + \overline{y + y'} \text{ times } OI'.$$

So that if (xy) are the pair of numbers required to describe OP and $(x'y')$ the pair required to describe OQ we must add corresponding numbers from those two pairs, if we want to get the pair required to determine $OP + OQ$, or

$$(x, y) + (x', y') = (x + x', y + y'),$$

the reason of this from the figure is that OQL is the same triangle as PRS only in a different position.

We have considered

1. Quantities, 2. Steps of these quantities, 3. Ratios of steps of quantities.

number a = length on a line, = length on a plane, = length in space to be considered hereafter.

We have seen that with the extended meanings given to $+$ and $-$,

$$ab + bc = ac,$$

and extending this result to vectors in space

$$ab + bc + cd + de + ef = ac + ef,$$

On adopting a notation used by Hamilton,

$$a + ab = b,$$

$$b - a = ab,$$

$$(\text{point} + \text{line}) = \text{point},$$

$$(\text{point} - \text{point}) = \text{line};$$

or in Hamilton's terms,

$$(\text{vector}) + (\text{vector} \times \text{vector}) = \text{vector}, \quad (\text{vector}) - (\text{vector}) = \text{vector}.$$

He calls the operation symbolized by the first equation "ordinal synthesis." It is a putting together of a line and a point, and the result is a point. The second operation he calls "ordinal analysis." And he enunciates this theorem: If we start with the result of a synthesis and perform on it the corresponding analysis we shall get the instrument of synthesis.

Using this notation, the equation for addition of vectors in space becomes

$$b - a + c - b + d - c + e - d + f - e = f - a,$$

which is an identity in this form.

It is convenient to represent steps by points in this way when we have to do with steps originating at the same point.

We have seen (Fig. 3b),

$$ab + bc + cd + de + ef = af,$$

that is in this sense, Head point forms

$$a + b + c + d + e + f = af,$$

and hence

$$a + b = af,$$

or may then be said, the sum of the two points a, b .

Similarly, if we have been previously shown to any point of ab in terms of oa, ob , see below, (Fig. 4.3),

$$ab = oa + ob = oa + \frac{ob}{a} \cdot ab,$$

where $l : m$ is the ratio in which f divides ab . That is

$$af : mb = l : m \quad (ab = a + b),$$

and

$$(l + m)af = (l + m)mb : m, \text{ or } m : a = l : m + b,$$

and hence

$$af = \frac{l}{l + m} \cdot a + \frac{m}{l + m} \cdot b.$$

Expressing this in terms of the points

$$(l + m)(f - a) = l(a - a) + m(b - a),$$

$$\text{i.e. } (l + m)f = l + mb : m, \text{ or } m(f - a) = l(b - a). \quad (A)$$

From this we are led to consider points in a plane not having masses. The expression for f shows that it is the centre of gravity (or centre of mass) of the masses l and m placed at b and a respectively. The point f divides ab in the inverse ratio of the masses.

It may also be expressed in terms of slope, transposing in the last equation

$$l(f - a) + m(b - f) = 0,$$

that is

$$l, af + m, bf = 0.$$

We may make use of this result to prove the vector form of equation (A).

We have of course

$$af = a + f \text{ and } bf = b + f.$$

Hence

$$(l + m)af = l, a + m, b + (l + m)f = 0, \text{ as just found,}$$

or as before

$$(l + m)af = l, a + m, b.$$

If we want to extend the rule for finding the mean of two points to a greater number of points, the sought point is called their *mid-centre* and is defined by this equation

If m is mid-centre and a, b, \dots etc. the points,

$$ma + mb + mc + \dots + mf = 0,$$

that is

$$a + b + c + \dots + f = m, \text{ or}$$

or

$$a + b + c + \dots + f = na,$$

where n = number of points.

Then

$$m = \frac{a + b + c + \dots + f}{n}.$$

For example, the mid-centre of four points a, b, c, d is by this rule

$$m = \frac{a + b + c + d}{4}.$$

Clearly it lies at the point H in the figure—[Fig. 65].

In the case of an odd number of points, $2n + 1$, we should first find the mid-centre H of $2n$ of them and then joining it to the last divide this line in the ratio $1 : 2n$.

We can now find the resultant of n steps. It is n times the step from the mid-centre of the beginnings to the mid-centre of the ends.

This is obvious in the case of two steps.

We have then, [Fig. 60],

$$ad + cd = 2H_2,$$

that is

$$(b + d) - (c + a) = 2H - 2H,$$

which follows from the definition of the mean of two points.

If there are four steps and p is the mid-centre of the a 's and q of the b 's, we have [Fig. 61]

$$\begin{aligned} & a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 \\ &= a_1(b_3 + b_4) + b_4(a_3 + a_4) - (a_1 + a_3 + a_2 + a_4) \\ &= -4p - 4q = -4(p + q), \end{aligned}$$

$$4q = -p,$$

and this is the resultant.

This construction is sometimes more convenient than the tangent arrangement. For the mid-centre lies always within the area covered by the steps. The construction is always convenient provided the steps are given.

If we want to add together certain multiples a_1, a_2, a_3 of steps we attribute masses h_1, a_2, a_3 to the points a, b, c etc. to the points b . Then we should have,

$$h_1a + h_2b + h_3c + \dots + h_n = (h_1 + ah_2 + ah_3 + \dots + ah_n)(q - p),$$

or may be easily seen, by joining each of the points b, c, d, a, e etc. to the mid-centre of the set.

We will now apply the new interpretation of the signs a, b, \dots to the description of motion.

Let p be a point moving uniformly along a line. Then if r is the space traversed in t , q being any one end, we get

We must find an expression for the steps we have to get from a to the position of p at any time t . The points p and a are at $\{2H, 0\}$ in (1).

Then we have,

$$ap = a + t \cdot ab,$$

$$p = a + t \cdot b$$

This is the equation of uniform motion of a point on a line.

We may also find the equation of uniform circular motion.

Let the origin a be at 1 for our [Fig. 62], then the arc described in t is at . If this is represented by the angle, the circular measure of the angle ap is t . We have

$$ap = a + at \cdot ab; \quad ap - a = at \cdot ab.$$

Hence

$$ap = \cos \alpha \sin \theta \sin \phi \sin \psi \sin \theta \sin \phi \sin \psi,$$

or

$$p = a \sin \alpha \sin \theta \sin \phi \sin \psi.$$

This is the step from the centre to the position of p at any time t .

Suppose now this uniform plane circular motion projected on an inclined plane. The curve becomes an ellipse, the pairs of axes being such that which from the centre are always proportional to the corresponding axes of the circle (Fig. 70).

$$\frac{a^2 \sin^2 \theta}{b^2 \sin^2 \phi} = \frac{\sin^2 \psi}{\cos^2 \psi},$$

$$\frac{a^2 p^2}{b^2 h^2} = \frac{\sin^2 \psi}{\cos^2 \psi}.$$

Hence we have the same form of expression for h as we have for an ellipse h for uniform motion on a circle, viz.

$$h = a \sin \alpha \sin \theta \sin \phi \sin \psi.$$

But here a and b are not at right angles to each other.

The rate of change of ap is the same as the velocity of the point p , as we see at once.

If $ap \rightarrow 0$,

$$h \rightarrow 0 \text{ and}$$

$$\sin \psi \rightarrow \cos \psi \text{ as } ap \text{ approaches 0.}$$

If we extend this to n steps up the spiral, the rate of change of ap in this each step,

i.e. if

$$p = a \sin \theta \sin \phi \sin \psi, \quad p' = p_1 \sin \theta_1 \sin \phi_1 \sin \psi_1 \quad (11) \quad (12)$$

We have found, putting $a = a_1$ (length of,

$$p = a \sin \theta \sin \phi \sin \psi,$$

and we can at once obtain from this an expression for the velocity of p at any instant.

Since the angle described in $\Delta t = \alpha_1$

$$\text{the arc} = a_1 \sin \theta_1 \sin \phi_1 \sin \psi_1$$

this then is the numerical value of the velocity. The direction is in the tangent line at p (Fig. 72).

Hence if we draw a radius of possibility h to p , we have, since $ap \rightarrow 0$,

$$p = a \sin \theta \sin \phi.$$

But ap is what ap becomes when at is increased by $\frac{a}{x}$.

$$a_1 p_1 = a \sin \theta \left(\sin \theta + \frac{a}{x} \right) + a_1 \sin \phi \left(\sin \phi + \frac{a}{x} \right)$$

$$= a \sin \theta \sin \phi + a_1 \sin \theta \sin \phi$$

The rule then for finding the vector-velocity from the position or vector-radius is—multiply this last vector by n and increase its angle by $\frac{\pi}{2}$.

We have gone on the supposition that α and β are at right angles, and equal to each other; but the result is true independently of this, since

$$\partial_t \cos nt, \partial_t \sin nt$$

always equal $-n \sin nt, n \cos nt$ respectively.

If then we take the elliptic projection of the circular motion, we may still conclude that velocity of $p = n \cdot \overline{oq}$ if \overline{oq} is parallel to the tangent at p [Fig. 73]. For the velocity at p is always parallel to the tangent at p .

Now consider the velocity of the point when it arrives at q . For this point

$$\rho = \alpha \cos \left(nt + \frac{\pi}{2} \right) + \beta \sin \left(nt + \frac{\pi}{2} \right).$$

Hence, multiplying by n and increasing angle by $\frac{\pi}{2}$,

$$\begin{aligned} (\text{for } q) \quad \dot{\rho} &= n\alpha \cos (nt + \pi) + n\beta \sin (nt + \pi) \\ &= -n\alpha \cos nt - n\beta \sin nt \\ &= -n \cdot \overline{op} = n \cdot \overline{po}. \end{aligned}$$

But this velocity must be in the direction qs of the tangent at q . Hence we may conclude, if we draw oq parallel to the tangent at p , then op will be parallel to the tangent at q .

The characteristic property of conjugate diameters is proved of the projections of diameters of a circle at right angles to each other.

We have seen that

$$\begin{aligned} op &= \overline{oa} \cdot \cos nt + \overline{ob} \cdot \sin nt, \quad \text{and also } op = om + np, \\ oq &= -\overline{oa} \cdot \sin nt + \overline{ob} \cdot \cos nt, \quad \text{and} \quad oq = on + nq. \end{aligned}$$

Hence, we have

$$\begin{aligned} \frac{om}{oa} &= \cos nt, & \frac{np}{ob} &= \sin nt, \\ \frac{nq}{ob} &= \cos nt, & \frac{on}{oa} &= -\sin nt, \end{aligned}$$

and from these equations

$$\frac{om}{oa} = \frac{nq}{ob}, \quad \frac{on}{oa} = -\frac{mp}{ob},$$

a result which is thus interpreted :

We have two pairs of semi-conjugate diameters, oa and ob ; and op and oq . The projection of op on oa is to oa as the projection of oq on ob to ob . And the projection of oq on oa is to oa as the reversed projection of op on ob to ob .

The first of these equalities is the same as

$$\frac{om}{oa} = \frac{or}{ob}.$$

Since \dot{p} is a vector the expression for it may be treated in the same way as that for ρ . If we take n an improper fraction so that $og' = n \cdot og$, the extremity of the vector \dot{p} will trace out an ellipse similar and similarly situated to the first. Generally, the curve which describes the way in which the path of a moving point is gone over is called the hodograph (of that path). In the case of uniform rectilinear motion $\rho = a + \beta t$, $\dot{p} = \beta$, and the hodograph reduces to a point.

For uniform circular motion it is a circle with dimensions n times those of the first circle; and for harmonic motion in an ellipse, n times that ellipse.

We will now consider the velocity of q as it moves round its ellipse.

We have

$$\ddot{oq} = \dot{p} = na \cos (nt + \frac{1}{2}\pi) + n\beta \sin (nt + \frac{1}{2}\pi).$$

The process for finding its velocity is the same as in the case of op . Hence

$$\begin{aligned} \dot{p} &= n^2 a \cos (nt + \pi) + n^2 \beta \sin (nt + \pi) \\ &= -n^2 \cdot \overline{op} \\ &= n^2 \cdot \overline{po}. \end{aligned}$$

Thus we learn that in harmonic motion in an ellipse the acceleration of the moving point is directed towards the centre of the ellipse and is proportional to the distance from the centre.

We will now consider a new kind of motion of which the equation is

$$\rho = ae^{nt} + \beta e^{-nt}.$$

These exponential functions are quantities which are equally multiplied in equal times.

Consider quantities, s , defined by this property.

We may see at once that their rate of change at any instant is proportional to the value of the quantity at that instant.

We have by def. $s_2 = ks$, and moving the interval $t_2 - t$, along the axis [Fig. 74], we still have $s_2 = ks_2$. Hence we infer the rates of change of the two quantities at the beginning and end of the interval are to each other as $1 : k$, or

$$\dot{s}_2 = k\dot{s}_1.$$

Hence we have the proportion

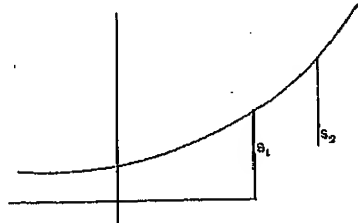
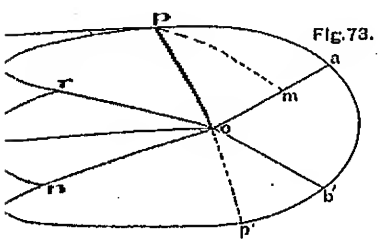
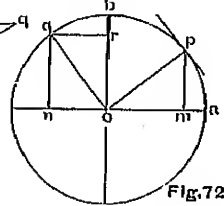
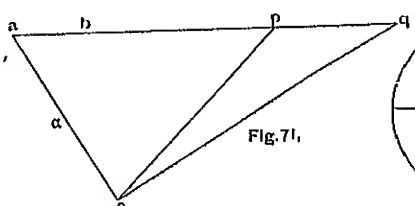
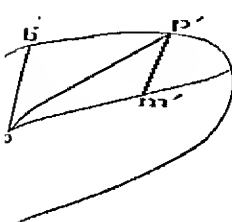
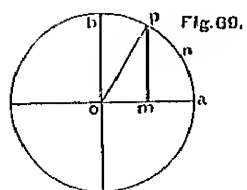
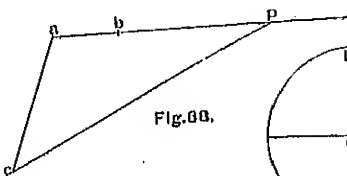
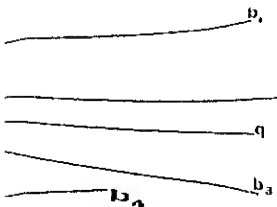
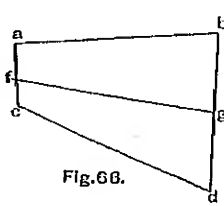
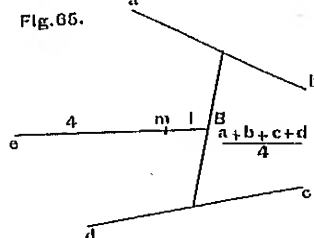
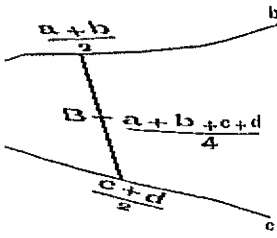
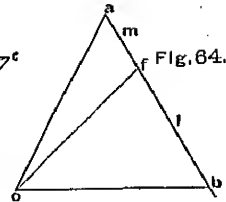
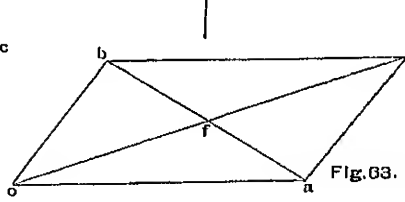
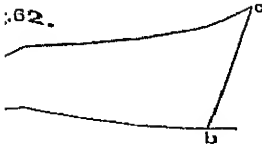
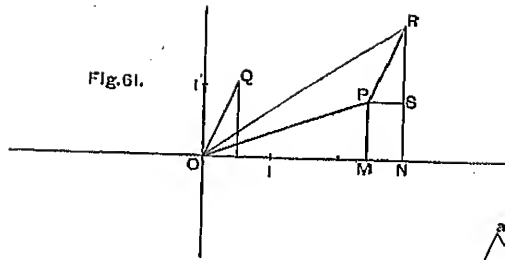
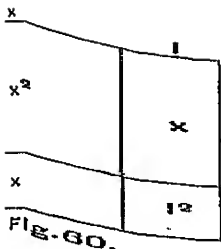
$$\dot{s}_2 : \dot{s}_1 :: s_2 : s_1,$$

and therefore

$$\dot{s} = ps.$$

Calling p the logarithmic rate, the last equation declares that s is a quantity which increases at uniform logarithmic rate.

If we change our unit of time from $1''$ to n'' , s becomes ns , and therefore p becomes np .



Having given that the quantity varies at logarithmic rate p , we are required to find P the number by which it is multiplied in $1''$. In n seconds s is multiplied by P^n ; and n must be a commensurable number, whole or fractional.

If it is a fraction, e.g. $\frac{1}{2}$, we must give one value only to \sqrt{P} , because at half a second after the time at which we begin, s will have a perfectly definite value.

If n is incommensurable, it may be defined by this physical property.—Let P^n mean the multiplier which s gets in this incommensurable time. Let us take a quantity $=1$, which increases at logarithmic rate 1. Then at the end of $1''$ it will have reached a certain definite value. Call it e . Then e is the result of making unity grow at logarithmic rate 1 for $1''$.

Then, I say, the result of making unity grow at logarithmic rate 1 for $t''=e^t$.

For since in $1''$ it is multiplied by e , in t seconds it will be multiplied by e^t .

Now change the unit of time to n seconds. We see then that the result of making unity grow at logarithmic rate 1 for nt'' of old units $=e^{nt}$; therefore in the new unit e^{nt} results from making 1 grow at logarithmic rate n for t'' .

The rate of change of e^{nt} is then ne^{nt} ($\dot{s}=ps$).

If the logarithmic rate is negative the quantity decreases. Rate of change of $e^{-nt} = -ne^{-nt}$.

We wrote the equation of the curve $\rho = ae^{nt} + \beta e^{-nt}$, where $a=oa$, $\beta=ob$.

But $\rho = om + mp$. [Fig. 75].

Hence $\frac{om}{oa} = e^{nt}$, $\frac{mp}{ob} = e^{-nt}$.

Hence $om \cdot mp = oa \cdot ob = \text{const.}$

This is the relation between the vector coordinates of p , and hence its locus is a hyperbola. We see that the axes of a and β are asymptotes to the curve. For by taking nt large enough, that is the length of $om=e^{nt}$ very large, we can make e^{-nt} or the length of mp as small as we like. And by taking nt small enough we can make e^{-nt} or mp as large and e^{nt} or om as small as we like.

For the flux of ρ we have

$$\dot{\rho} = nae^{nt} - n\beta e^{-nt},$$

a similar curve to that for ρ ; but it is turned round the α axis so that the β coordinates are negative. It is shown by the dotted line in the figure.

We see that the tangent at p is parallel to oq ,

$$oq = om + mq = om - mp.$$

And the velocity which is measured in this direction is n times oq .

The figure is drawn for $n=\frac{1}{2}$.

The locus of q is the conjugate hyperbola, and op and oq are semi-conjugate diameters. In the ellipse both of them meet the curve, but in the hyperbola one meets the conjugate branch.

We have,

$$\rho = ae^{nt} + \beta e^{-nt} = (a + \beta) \frac{1}{2} (e^{nt} + e^{-nt}) + (a - \beta) \frac{1}{2} (e^{nt} - e^{-nt}).$$

We will call

$$\begin{aligned} \frac{1}{2} (e^{nt} + e^{-nt}), & \text{ the hyperbolic cosino of } nt, \text{ } hc . nt, \\ \text{and } \frac{1}{2} (e^{nt} - e^{-nt}), & \text{ ,, ,, sine of } nt, \text{ } hs . nt. \end{aligned}$$

If we also write

$$\alpha + \beta = \gamma, \quad \alpha - \beta = \delta,$$

the last formula becomes

$$\rho = \gamma hc . nt + \delta hs . nt.$$

We thus have an expression for ρ in terms of two semi-conjugate diameters, analogous to the one we found for the ellipse.

γ and δ are clearly the values of op and oq for $t=0$.

The corresponding expression for the flux is

$$\dot{\rho} = n (\gamma hs . nt + \delta hc . nt),$$

since each of the functions $hs . nt$, $hc . nt$ has for its flux n times the other. Or we may see that this follows from the first expression found for $\dot{\rho}$.

If we take any two semi-conjugate diameters, OC and OD , and also two others OP and OQ [Fig. 76], we have,

$$\begin{aligned} \rho &= OP = OM + MP \\ &= \gamma hc . nt + \delta hs . nt. \end{aligned}$$

Hence,

$$\frac{OM}{OC} = hc . nt; \quad \frac{MP}{OD} = hs . nt,$$

and

$$\begin{aligned} \frac{\dot{\rho}}{n} &= OQ = ON + NQ \\ &= \gamma hs . nt + \delta hc . nt. \end{aligned}$$

Hence,

$$\frac{ON}{OC} = hs . nt; \quad \frac{NQ}{OD} = hc . nt.$$

From these we find

$$\frac{OM}{OC} = \frac{NQ}{OD} \quad \text{and} \quad \frac{ON}{OC} = \frac{MP}{OD},$$

for any two pairs of conjugate diameters.

These equations declare that,

The projection of OP on OC is to OC as the projection of OQ on OD is to OD ; and the projection of OQ on OC is to OC , as the projection of OP on OD is to OD .

From the equation for \dot{p} , we see that the hodograph is a curve of n times the dimensions of the conjugate hyperbola, and is similarly situated to that hyperbola.

For the acceleration, we find,

$$\begin{aligned}\ddot{p} &= n^2 (\alpha e^{nt} + \beta e^{-nt}) \\ &= n^2 p,\end{aligned}$$

by repeating the operation by which \dot{p} was found.

Thus the acceleration of a point P moving on a hyperbola is in the direction of OP , but away from the centre.

In the ellipse the acceleration is directed towards the centre.

Velocities and accelerations resemble steps in having magnitude and direction. All three belong to the class of "vectors," which are characterised by these two properties, and also by a third, not pointed out by Hamilton, namely, that they have no definite position.

Hence we must consider vectors as steps, not of points, but of a rigid body. And similarly, velocities and accelerations must be regarded as velocities and accelerations, of translation, of a rigid body.

Thus the uniform circular motion which we have discussed, is to be considered as the revolution of a rigid body, round a centre, so that all its particles describe circles of the same size.

Any enclosed area may be represented by a line drawn perpendicular to its plane, and of length proportional to its magnitude, so that there are as many linear units in the line, as there are square units in the area.

Now the area is not completely defined until the direction in which we go round it is known. This direction may be either clock-wise or the reverse, and the corresponding vectors would be drawn in opposite directions, upwards in the one case, downwards in the other.

The rule for drawing the representative vector is this:—It must be drawn so that when we look back along it the area appears to be gone round counter clock-wise.

For the area shewn in the figure [Fig. 77] it must therefore be drawn upwards.

If we have a figure of 8 [Fig. 78] bounding two areas which are gone round in opposite directions, looked at from one side, the true value of the whole area is the difference of the two parts. It is represented by the difference of the lengths of the vectors whose lengths correspond to their magnitudes; for these are drawn in opposite directions. (We will call the rotation opposite to that of the clock-hands, positive rotation.) This method for the addition of areas was given by Möbius.

We may take a more complex plane curve [Fig. 79]. It is easy to see that the two areas in the centre are to be counted as 0. The lower one for example is seen to be formed by the overlapping parts of a positive and a negative and the superposed parts destroy one another.

The addition of areas in space corresponds to the addition of vectors which are not in the same straight line.

If we want to represent an area on a curved surface, for example, a cone, we break it up into small areas, nearly plane, and draw from each of these its representative vector in the manner before described. These lines will not be parallel, and must be added together according to one of the rules for finding the resultant of vectors in space. Their resultant represents the area. It represents not the actual size of the area, but in a certain physical sense, the area of the contour. Hence every closed surface may be considered as a quantity having a certain magnitude and direction. This point of view was first taken by Hayward*.

If A is any plane area, its projection on a plane making an angle θ with its plane is $A \cos \theta = A'$.

We know that for any plane area there is a set of planes on which the projection is a maximum, viz. all parallel planes, and another set on which the projection is a minimum, viz. planes perpendicular to the first.

All this holds good for areas, not plane, which are represented in the way we have described.

There is a certain set of planes on which the projection is a maximum. Call it A . On all other planes making an angle θ with the maximum planes, the projection is $A \cos \theta$. And there is a plane on which the projection is zero.

Take as an example of a non-plane area, two triangles and a parallelogram in different planes. [Fig. 80.]

Let ab , bc , cd be the vectors representing the areas 1, 2, 3, then ad represents the whole area.

If now we project the areas 1, 2, 3 on a plane D , it is clear that the new representative vectors are the projections of the lines 1, 2, 3 on a line perpendicular to the plane D .

For the angle between two representative vectors = the angle between their planes. This being so, the projection of the compound area (1, 2, 3) on any plane is represented by the projection of ad , on a normal D to that plane.

Hence all planes perpendicular to ad are maximum planes for the compound area, and its projection on them is A , the projection on any plane making an angle θ with them is $A \cos \theta$. And the projection on planes parallel to ad is a minimum. If then the maximum planes are known, and the maximum projection, we find the line representing the compound area by this rule:—Draw a line perpendicular to the maximum planes, of length A .

Consider first two triangles in a plane, having a common side and vertex; we can prove that their sum is the triangle having the same side, with its base equal to the (vector-) sum of their bases.

If the two triangles are [Fig. 81] OAB , OAC , it is to be shown that

$$OAD = OAB + OAC.$$

* [Proc. of L. Mathematical Society, Vol. IV. pp. 230—31, 417.]

If we complete the parallelogram with sides AC , AB , the triangle

$$CAD = DAB,$$

for their areas are equal and the motion round both is positive.

Taking account of the way in which we go round their contours,

$$OAD = OAC + OCD + CAD$$

$$= OAC + OCD + DAB.$$

Now,

$$OAB = OCD + DAB,$$

(for, drawing perpendiculars to a line at right angles to AB , the areas are,

$$OAB = \frac{1}{2}ln \cdot AB,$$

$$DAB = \frac{1}{2}mn \cdot AB,$$

$$OCD = \frac{1}{2}lm \cdot AB.$$

Hence

$$OAB = OCD + DAB.)$$

substituting, this gives

$$OAD = OAB + OAC.$$

If we agree to call the common side OA the height of the triangles, we may write the last result thus:—The sum of two triangles with the same height, is a triangle having the same height and with its base equal to the sum of their bases.

We may now extend this to triangles in space, which are represented by vectors.

Take two triangles standing up from the plane of the paper, and first suppose their common "height" perpendicular to the plane.

We have to shew that the line representing OAD [Fig. 82] is the sum of the lines representing OAC and OAB .

This is at once seen by looking at the space-figure. We may represent OAC by a vector perpendicular to AC , numerically equal in length to AC . Let this be AC' . Then OAB and OAD will be represented in the same way by AB' , AD' . These representative vectors must be all drawn in the plane of AB and AC , that is, the plane of the paper. For each of the triangles is perpendicular to this plane. The new vectors are the sides and diagonals of the same parallelogram turned through a right angle, and we see that $AC' + AB' = AD'$.

This proves the proposition.

We may easily pass to the general case where the height OA is not supposed perpendicular to the plane of AB and AC .

If we draw through O a plane perpendicular to OA it cuts the plane of each triangle in a line perpendicular to OA : Take OAD [Fig. 83] for example; then

$$\text{area } 2OAD = OP \times AD = OA \cdot \frac{OP}{OA} \cdot AD$$

$$= OA \cdot AD \cdot \cos \theta = OA \cdot OF.$$

Similarly, if OG , OIH are the projections on the plane perpendicular to OA of the bases AC , AB , their areas will be $OA \cdot OG$, $OA \cdot OH$, respectively. Now the lines OG , OIH , OF are the sides and diagonals of a parallelogram; hence by the addition of vectors, $OF = OG + OH$. And since we may represent the triangles by lines drawn from O in the plane perpendicular to all their planes, at right angles to the plane of each, and equal in length to OG , OIH , OF , these representative vectors form the same parallelogram turned through a right angle. If they are OF' , OG' , OIH' we still have

$$OF' = OG' + OIH',$$

and hence for the corresponding areas

$$OAD = OAB + OAC.$$

In the ancient geometry the product of two lines was the rectangle contained by them. We will now extend this representation of a product, and say that the product of two lines inclined at any angle is the area of the parallelogram contained by them. Thus the product of two vectors is the vector perpendicular to their plane and proportional in length to the area they enclose. Thus we make the definition $OA \cdot OB = 2 \cdot \text{triangle } OAB$.

Remembering the convention about signs of areas, we see that OAB is positive.

By interchange of letters, the formula gives,

$$OB \cdot OA = 2 \cdot \text{triangle } OBA.$$

Now OBA is $-OAB$.

Hence we learn that the product of two vectors is altered in sign when the order of multiplication is reversed.

We may now interpret the proposition about the sum of two triangles in space, by this formula.

Their areas will be

$$2AOD = AO \cdot AD,$$

$$2AOB = AO \cdot AB,$$

$$2AOC = AO \cdot AC,$$

.. since

$$AOD = AOB + AOC,$$

$$AO \cdot AD \{= AO(AB + AC)\} = AO \cdot AB + AO \cdot AC.$$

Hence we conclude that vector multiplication is distributive.

Using shorter symbols the theorem may be written

$$(1) \quad V\alpha(\beta + \gamma) = V\alpha\beta + V\alpha\gamma.$$

We have seen that $OA \cdot OB = -OB \cdot OA$, or

$$(2) \quad V\alpha\beta = -V\beta\alpha.$$

This shows that vector multiplication is not commutative.

If we change the order in each term of equation (1), we change the sign of each term and get,

$$(3) \quad V(\beta + \gamma)\alpha = V\beta\alpha + V\gamma\alpha.$$

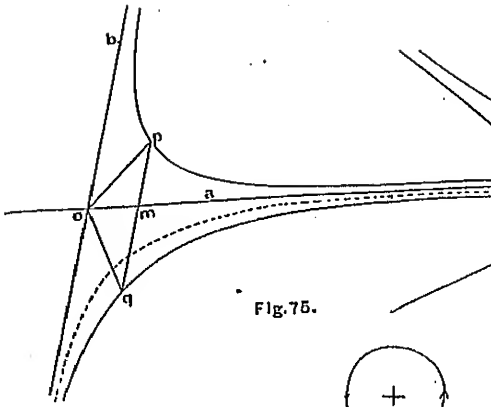


Fig. 75.

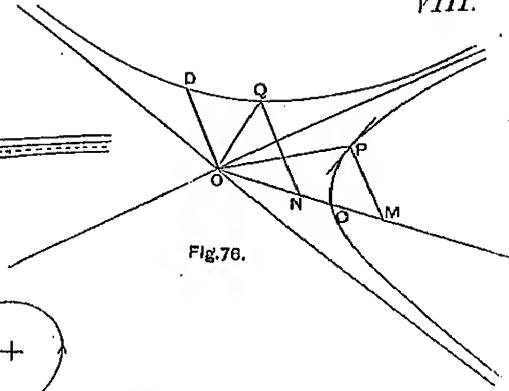


Fig. 76.

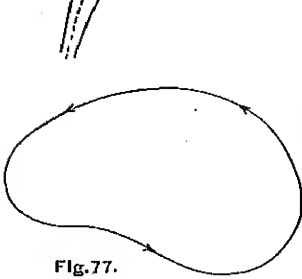


Fig. 77.

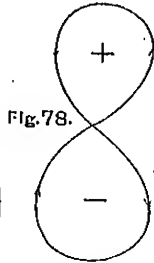


Fig. 78.

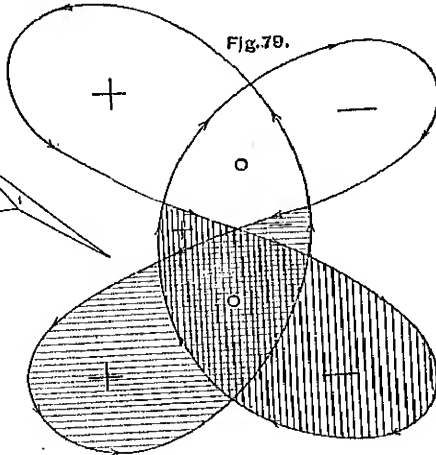


Fig. 79.

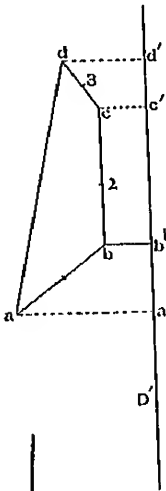


Fig. 80.

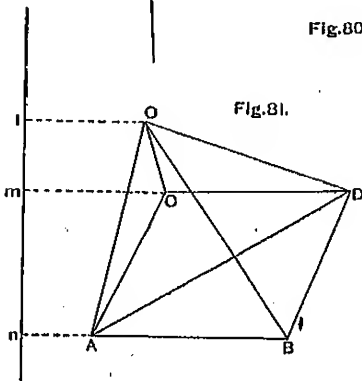


Fig. 81.

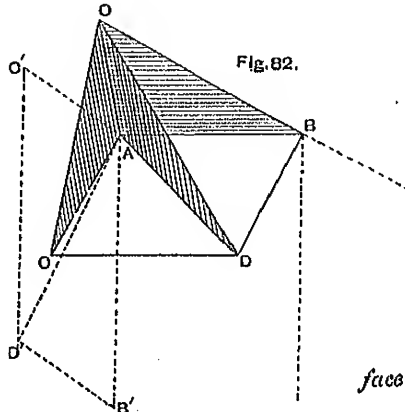


Fig. 82.

This however is not the same equation as (1). The signs of its terms are opposite to those of the terms in (1).

Grassmann called the vector-product of two lines their "outer product," because it has no existence unless one is outside the other.

The following proof of equation (2) was given by Sylvester. It shows on what geometrical fact the rule depends.

If OA and OB coincide they will enclose no parallelogram, and we shall have,

$$(4) \quad Va\alpha = 0.$$

This being true, try it on $V(a + \beta)(a + \beta)$.

This product is broken up into four parts,

$$0 = Va(a + \beta) + V\beta(a + \beta) = Va\alpha + Va\beta + V\beta\alpha + V\beta\beta.$$

The first and last terms vanish, and hence

$$0 = Va\beta + V\beta\alpha.$$

If we had assumed (2), equation (4) would follow at once from it, for we must have

$$Va^2 = -V(a^2).$$

This kind of multiplication has been called "polar multiplication." We see then that any kind of multiplication which is distributive and where the square of any quantity = 0, must be polar.

We may now use the theorem about the geometrical representation of products by areas, to investigate the rotation of rigid bodies.

Let the angular velocity of a rigid body about the axis ab be ω . Any particle p of the body describes a circle round ab . At this moment p is moving perpendicular to the plane of the paper and its velocity (are described in 1") is, $\omega \cdot mp$.

If we measure off on ab a length ab proportional to ω , the velocity of p is twice the area of the triangle abp . [Fig. 84.]

It is necessary however to take into account the direction of motion of p , and to give a different sign to the triangle which represents the velocity.

Suppose ab measured so that, looking back from b to a , the motion of p appears to be counter-clockwise; p must be moving at this moment back from the paper. We make this convention because the sign of abp is negative, and therefore its representative vector must be measured downwards, or in the negative direction.

We have found, then,

$$\dot{p} = \omega \cdot mp = 2 \cdot abp = Va\bar{b} \cdot \bar{a}\bar{p} = Vap.$$

Similarly for every point in the space occupied by the body there is a certain vector representing the velocity of a particle at that point, and which is expressed in terms of its position-vector p . The aggregate of these vectors forms a velocity system.

We might have also an acceleration system; that is a definite acceleration belonging to every point of space. Both of these are vector-systems.

Now if we have to compound two vector-systems we shall compound the two vectors belonging to the systems respectively at every point of space. As an application of this we will compound the velocity systems corresponding to two rotations about a fixed axis. These rotations may be called "spins." They are completely defined when we know the rotation axis and the angular velocity.

The problem then is to find the resultant of two spins with angular velocities ω_1 , ω_2 , and with axes passing through a fixed point a .

We have $\dot{\rho}_1 = V\omega_1\rho$, $\dot{\rho}_2 = V\omega_2\rho$
for any point of space which has the position-vector ρ .

Since the velocities are represented by the areas abp , acp we have for the resultant velocity ($a\dot{\rho} = a\dot{b}p + a\dot{c}p$) [Fig. 85]

$$\begin{aligned}\dot{\rho} &= \dot{\rho}_1 + \dot{\rho}_2 = V\omega_1\rho + V\omega_2\rho \\ &= V(\omega_1 + \omega_2)\rho.\end{aligned}$$

Hence a velocity system compounded of two spins is a velocity system with a spin which has the sum of the component angular velocities.

This way of writing the theorem of distributive vector multiplication is only a short-hand for the geometrical proof of the same proposition.

This expression for the resultant does not mean that the body has first a little turn about one axis and then about the other. The spin of the body is made up of two spins but it does not get those spins actually. We may easily extend this result to find the resultant of any number of angular velocity systems with axes passing through a fixed point.

We shall be led to consider another kind of product of two vectors:—the scalar product.

In the ancient geometry the product of these lines was represented by the rectangular parallelepiped which they contained.

We will extend this to the parallelepiped contained by any three lines meeting in a point, and we may then make the definition

$$OA, OB, OC :: 6 \times \text{tetrahedron } OABC.$$

Now if we draw OM perpendicular to OAB and proportional in length to $2OAB$ [Fig. 86], we have

$$OA, OB, OC :: OM, OC.$$

$$\text{Now } 6 \cdot OABC = OM \cdot ON :: OM, OC \cos \phi.$$

Then, if the lengths of OA , OB , OC , OM are a , b , c , d we have

$$OA, OB, OC :: a, b, c \cos \phi.$$

We must now take account of signs.

The volume of $OABC$ shall be reckoned positive when the three points A , B , C , looked at from O are gone round counter-clockwise. The volume in the figure then is negative. It has been seen that the product of two vectors OA and OB is a directed quantity for it is an area, considered as to size and aspect, and may therefore be represented by a vector. Its magnitude is $ab \sin \theta$

and it is perpendicular to the plane of OA and OB . It is the vector-product. But now the product of three vectors has also been reduced to the product of two, viz. OM and OC . And since this is a volume, it can only have quantity not direction, and it is called the scalar product of two vectors. In the scalar product we consider one of the vectors OM as the area, the other as a vector. While in the vector-product both are considered as vectors.

This distinction between two kinds of vector was first made by Maxwell. One kind of vector corresponds to a force, the other to a flow.

The product of two flows or of two forces will be a vector-product. The product of a force and a flow will be a scalar-product.

The point of view we have taken in considering an area as a product of two vectors of the same kind and a volume as a product of three vectors of the same kind, is that of Grassmann, not of Hamilton.

Since the volume represented by OC , OM is to be negative we must draw OM so that looking back along it the rotation of OA to OB is in the negative direction. It is drawn right then in the figure. If the parallelogram were rectangular OM would fall on OC , and supposing it of the same length we should have

$$Sc^2 = \text{square length of } a.$$

We may see at once that a scalar-product is not altered when we change the order of the factors. For interchanging OC and OM , the sign of ϕ is altered but not that of $\cos \phi$.

$$\text{Therefore } Sc\phi = S\phi c.$$

We may define the scalar-product of two vectors geometrically as a solid, one of the vectors being an area; or physically as the product of three vectors of the same kind. Similarly the vector-product may be defined geometrically as an area considered in respect and size or physically as the product of two vectors of the same kind.

When the scalar-product of two vectors is one vector multiplied by the projection of the other upon its direction, we can at once deduce the distributive law for scalar multiplication, or

$$Sa(\beta + \gamma) = Sa\beta + Sa\gamma,$$

since the projection of $\beta + \gamma$ on a is the same as the sum of the projections on it of β and γ . [Fig. 107.]

Illustration of a scalar product in physics. The rate of doing work by a body moving with a velocity v and acted on by a force $\omega = S\omega x$.

Another illustration. The kinetic energy of a rotating body $= S \cdot \text{spin} \times \text{moment of momentum}$. Each of these last quantities is a vector.

Again, if a plate is bent out of its plane or deformed in its plane,
potential energy $= S \cdot \text{strain} \times \text{stress}$.

Recalling the double interpretation of the equality 2 x 3 = 6; viz. 1st, 2 and 3 multiplied together give 6; 2nd, 3 (operated on) by 2 given 6; we will

make a corresponding second interpretation of the product of two vectors plus, and ask what operation must be performed in order to turn *only* into *another*.

To change *OA* into *OB* I must multiply it first by a number to change length to that of *OB* and then turn it through an angle so that it may lie on *The whole of the operation required in order to do these two things is called q .*

$$q, OA \rightarrow OB, \quad \{\text{Fig. 50a}\}$$

Drawing *ab* parallel to *OB* we see that

$$q, Oa \rightarrow Ob,$$

If we want to apply the same operator *q* to any other vector *oc*, we draw another triangle *OB'D'*, and if this is similar and of the same sign as *OB*,
 $O'D' = OB' \cdot (OB/OA) = (OB/OA) \cdot Oa,$

This proposition would not hold for a triangle *OB'c'* similar to *OB* in different sign.

We make use of the operator *q* to prove any operation about the middle have,

If *obL, ocL, odL* [Fig. 50] are three similar triangles *obL* and *ocL* are about *o*; and if the parallelogram *obL, ocL* are complete, we can check if the triangle *obL* is similar to each of the three triangles which are put together at *o*.

Let *q* be the operator which takes *OA* into *OB*, $q, OA \rightarrow OB$.

$$obL = q, oaL$$

$$ocL = q, ocL$$

$$odL = q, odL$$

$$obL = q, oaL \rightarrow q, obL$$

Adding we have

$$obL + q, obL = obL + q, obL$$

Therefore the triangle *obL* is similar to triangle *obL, q, obL*.

We may suppose all the parallelograms to be complete (and hence) all corners.

We may remove all the parts except the triangles *obL* and the parallelogram *obL, q, obL*, or all except *obL, obL* and *q, obL, q, obL*.

Then, in each case, *obL* must still be of the same shape.

The curve described by *obL* in the above case will be the curve *obL*. We can be sure that a curve which can be described with one *obL, q, obL* can also be described equally well with two other three-fold rotations.

The operator *q* which we have seen to be the product of three rotations is then the sum of two parts.

If [Fig. 90]

$$q \cdot OA = OB, \text{ and } OB = OM + MB,$$

$$\begin{aligned} q &= \frac{OM}{OA} + \frac{MB}{OA} \\ &= \frac{OM}{OA} + \frac{MB}{OA'} \cdot \frac{OA'}{OA} \\ &= a + b \cdot \frac{OA'}{OA}, \end{aligned}$$

where a and b are numbers, with positive or negative signs.

We want a symbol for the operator which turns a step through a right angle in the same plane. If i stands for this operator

$$q = a + bi;$$

i may be regarded as a handle perpendicular to the plane of OA and OB which turns OA through a right angle.

It is at once seen to be a property of i that if it is used twice on a vector it reverses that vector; or

$$i^2 = -1.$$

We can now use the operator i to establish some important results. First however we must find an expression for the rate of change of a product of two numbers, pq . Take their values p_2q_2 at the time t_2 , and p_1q_1 at the time t_1 , then for the mean rate of change of the product during the interval we have,

$$\frac{p_1q_1 - p_2q_2}{t_1 - t_2} = p_1 \cdot \frac{q_1 - q_2}{t_1 - t_2} + \frac{p_1 - p_2}{t_1 - t_2} \cdot q_2.$$

Making the beginning and end of the interval coincide, we get the actual rate of change of the product; that is

$$\partial_t pq = p\dot{q} + \dot{p}q \dots\dots\dots(1).$$

In getting out this result all we have assumed is that multiplication is distributive. Hence the result is true when p or q or both of them are vectors. Letting q only equal a vector ρ , we have

$$\partial_t (p\rho) = p\dot{\rho} + \dot{p}\rho \dots\dots\dots(2).$$

If both are vectors the result holds good for either the scalar or the vector-product; hence

$$\partial_t V(\alpha\beta) = V\alpha\dot{\beta} + V\dot{\alpha}\beta \dots\dots\dots(3),$$

$$\partial_t S(\alpha\beta) = S\alpha\dot{\beta} + S\dot{\alpha}\beta \dots\dots\dots(4).$$

We may apply equation (2) for one quantity a scalar and the other a vector to the problem of resolving the acceleration of a moving point along the tangent and the normal to its path.

The velocity of p , \dot{p} , is in the direction of the tangent at p , and its magnitude equals $\dot{s} = v$, where s is the length of the curve up to p , measured from a fixed point a . [Fig. 91.]

Hence $\frac{\dot{p}}{p}$ is a vector of unit length parallel to pt . Let oe represent it; and denote it by p' .

Then we have

$$\dot{p} = \dot{p}' + vp',$$

Hence by equation (2) the acceleration of p is

$$\ddot{p} = \dot{p}' + v\dot{p}'.$$

The first part is the acceleration parallel to the tangent. Its magnitude is \dot{v} .

The second term be multiplied by the velocity of e . Since e is always at the end of a unit vector its path is a circle, of radius 1. Hence the velocity of e is always in a direction perpendicular to oe , and to pt , that is along the normal. Its magnitude is the angular velocity of oe , that is of the tangent pt .

The magnitude then of the normal component of the acceleration is the velocity of the point multiplied by the rate of turning round of the tangent.

But we know that,

$$\frac{\dot{p}'}{v} = \frac{\text{rate of turning round of tangent}}{\text{velocity of } p} \\ = \frac{\text{rate of turning round of tangent per unit length of curve}}{\text{curvature}}.$$

Hence the normal acceleration is $v^2 \times$ curvature, curvature being measured by a line drawn along the line normal to the curve.

Suppose a point or system of points rigidly connected, moving in a plane which also moves over a fixed plane. Each of these is to be a velocity-system consistent with rigidity. The characteristic of such a system is that the distance of any two points remains constant. This is the same as saying that the velocity of any point relatively to any other is perpendicular to the line joining them. For since ab is of constant length, and we suppose a fixed for a moment, b clearly moves perpendicular to ab .

If we compound with this system another velocity system consistent with rigidity, for the same points a and b , the velocity of b relatively to a is a line perpendicular to ab , suppose it standing up perpendicular to the plane of the paper. Then the resultant of the two velocities of b is again in a direction perpendicular to ab . Hence the resultant of two velocity systems each consistent with rigidity is a velocity system consistent with rigidity. If we have any velocity-system of this nature in which some point a is moving with a certain velocity and combine with it a motion of translation of the whole system, equal and opposite in direction to the velocity of a , its velocity is destroyed and there results a spin about a as centre. Conversely, any velocity-system consistent with rigidity is made up of a translation and a spin round some point.

Now let us find an expression for the acceleration of any point P moving in a moving plane. At any moment the motion of P is a spin about some point a which is called the instantaneous centre.

We have then

$$\dot{P} = I, aP, \omega,$$

where ω is the angular velocity about a .

In general the instantaneous centre will move about; and then we find from equation (2) for the acceleration of P

$$\begin{aligned}\ddot{P} &= I, aP, \dot{\omega} + I, aP, \dot{\omega} \\ &= I(\dot{p} + \dot{a})\omega + I, aP, \dot{\omega} \\ &= aP, \omega^2 + I, aP, \dot{\omega} + I, \dot{a}, \omega \dots\dots\dots (1),\end{aligned}$$

where we have used the expression for a vector symbolically in terms of points, viz.,

$$aP = p - a, \quad \therefore a\dot{P} = \dot{p} - \dot{a},$$

and have then substituted for \dot{P} its value, I, aP, ω .

It is important to notice that \dot{a} does not mean the velocity of the point a which is a point fixed in the plane, but the velocity of the instantaneous centre.

The first term in the expression for \ddot{P} is the normal acceleration of P , supposing it to move uniformly round a . The second term is the tangential acceleration.

The third term is the acceleration of P at right angles to the velocity of the instantaneous centre.

We will apply this result to examine the motion of the instantaneous centre. Let the point P coincide with the instantaneous centre; calling it O , we have

$$\dot{O} = I, \dot{a}, \omega,$$

where $aP = aO$ vanishes.

Here \ddot{O} is the acceleration of the point O , fixed in the moving plane, \dot{a} is the velocity of the instantaneous centre in the fixed plane.

The problem is the same as that of determining the relative motion of the two planes. If we suppose them fixed alternately, all angular velocities of the first with regard to the second will be the opposite of the corresponding velocities of the second with regard to the first.

Hence if O_1 is a point in the other plane (supposed fixed before), we have, when a is changed to a_1 ,

$$\ddot{O}_1 = I, \dot{a}_1, \omega,$$

where \dot{a}_1 is now the velocity of the instantaneous centre in the plane which was moving before and is now considered as fixed.

But the acceleration of O in one plane with regard to a point O_1 considered as fixed in the other is the same as the acceleration of O_1 with regard to O considered as fixed, only in the reversed direction; that is

$$\begin{aligned}\ddot{O}_1 &= -\ddot{O} \\ \therefore I, \dot{a}_1, \omega &= I, \dot{a}, \omega,\end{aligned}$$

that is

$$\dot{a}_1 = \dot{a}.$$

The velocity of the instantaneous centre in the fixed plane is the same as the velocity of the instantaneous centre in the moving plane. Hence these velocities are always in the same direction we see that the curves traced out by the instantaneous centre in the fixed plane and the moving plane respectively must touch. Further since the velocities of a and a_1 are equal in magnitude, we have $\frac{1}{\lambda_1} = \frac{1}{\lambda_2}$, where λ_1, λ_2 are the lengths of the two curves, but of instantaneous centres, measured from any fixed point. Hence the curves must roll on each other without sliding, and the point P will lie on a roulette.

We shall be able to find the curvature of this roulette.

Let us be the normal to the two loci of instantaneous centres. [Fig. 93.]

The normal acceleration of P along NP is the first term of equation (4); the resolved parts of the other two terms. But the second part has no component along NP . Hence calling the magnitude of the velocity of a_1 v , we have

$$\text{normal acceleration of } P = \frac{v^2}{NP^2} + \frac{v^2}{r^2} \cos \theta \cos \phi,$$

Also

$$\text{velocity of } P = v \sin \theta,$$

And

$$\text{curvature} = \frac{\text{normal acceleration}}{(\text{velocity})^2}$$

Hence

$$\text{curvature} = \frac{1}{NP^2} + \frac{1}{r^2} \cos \theta \cos \phi$$

Now $\frac{d\theta}{dt}$ rate of turning round of axis line with the moving plane per unit length of the curve of instantaneous centres

Hence we have

$$\text{curvature of roulette} = \frac{1}{NP^2} + \frac{d\theta}{dt} \frac{1}{r} \cos \theta \cos \phi$$

There are two circles, but respectively of points which have no normal, and no tangential acceleration. Their intersection is a point having no velocity at all.

We now go on to apply the question to a train of epicycloidal teeth.

We define r^{θ} as the result of making θ vary at logarithmic rate p for t seconds; and the rate of change of this quantity is got by multiplying itself by p .

Let us now mention what meaning we can give to r^{θ} . This is a step in the plane, the rate of change of which is got by multiplying itself by p , that is by turning it through a right angle.

Suppose NP was originally of length 1, and θ 0.

Then since the rate of change of NP is always perpendicular to its direction, P moves on a circle.

We can show in a special case that r^{θ} is a step in the plane making an angle of circular measure θ with the fixed axis line oa .

Suppose $t = 1$. Then we have

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{2 \cdot 3 \cdot 4} + \dots$$

We can construct the step e^t by operating on $OA = 1$ by this series.

The series converges rapidly. The first five terms take us up to E . [Fig. 93.] Some point very near this gives the end of the step e^t . Now the angle subtended by the arc $AE = t$; E is therefore the point on the circle whose arc measured from A is the radius. Since the radius was taken = 1, the arc $AE = 1 = t$. Hence for this case ($t = 1$) we have verified the meaning given to e^t .

From the figure we have, if now AE be any arc t ,

$$OE = OM + ME,$$

or

$$e^{it} = \cos t + i \sin t.$$

Hence from the series for e^{it} we can get the series for the sine and cosine in powers of the angle.

Any step p of greater length than the radius, be thus expressed

$$p = re^{i\theta},$$

where r is the length of the step; θ the angle it makes with OA .

From equation (2) we can find the radial and transversal velocities and accelerations

$$\begin{aligned} \dot{p} &= i\dot{r}e^{it} + ir\dot{e}^{it}, \\ v &= (\dot{r} - r\dot{\theta}^2)e^{it} + i(2r\dot{\theta} + r\dot{e}^{it}). \\ \text{(radial)} & \quad \text{(transversal)} \end{aligned}$$

We will return to the general consideration of rectangular versors.

In one plane if the lengths of oa , ob [Fig. 94] are x , y we have

$$ob = \left(\frac{oa}{\cos t} + \frac{ob}{\sin t} i \right) \cos t \\ (= 1) \cos t \text{ for any versor,}$$

A rectangular versor is the operator which turns a vector through a right angle in its plane and stretches it. We may then represent it by a handle perpendicular to the plane in which the turning takes place and of length numerically equal to the ratio in which the length of the vector is to be altered. If the handle is of length 1 the vector is not stretched. We will now consider what is meant by the sum of two rectangular versors.

If we have

$$A_1 \pm B_1 = C_1,$$

where A , B and C are rectangular versors, how can we represent C ?

Let A and B lie in the same plane. Then we must take the vector operated on *is* perpendicular to each of these versors and therefore perpendicular to their plane.

Suppose it is of length 1.

Operating on it by the two versors successively we see that A turns it down into the plane of the paper and increases its length to A , B also turns it down into the same plane and increases its length to B , Ap is of course perpendicular to A , Bp perpendicular to B . The result of both operations is then $Ap+Bp$ [Fig. 95]; if now we ask what versor would be required in order to bring p into this position and to give it this length, the answer clearly is—a versor perpendicular to $Ap+Bp$ and equal to it in length; that is C , the diagonal of the parallelogram formed by A , B .

Hence $A+B=C$,

and the sum of two rectangular versors is formed in the same way as the sum of two vectors.

To find the sum of three rectangular versors not in the same plane,

$$A+B+C,$$

we must add $A+B$ by the preceding rule and then add the resulting rectangular versor to C by the same rule. The way in which we take the pairs is indifferent. We see that it is necessary to take the operations in instalments if we are to give any interpretation to successive steps. For if A , B , C are not all in the same plane, there is no vector on which they can operate simultaneously.

Of course the preceding result is independent of the length of p being unity. We may multiply each term of the equation by any number and the result will still hold good.

We now go on to consider the product of two rectangular versors. We assume them each of length unity. Let them be A and B inclined at an angle θ . [Fig. 96.]

The vector to be operated on must be placed so that after A has operated on it B will be able to operate on it. It must therefore be in the plane of AB perpendicular to A . After both operations it will be in the position p' perpendicular to B and in the same plane. The effect of the product is then a versor not rectangular, which turns p through the supplemental angle of θ .

Since BA turns oa into ob , we have,

$$BA = \frac{ob}{oa} = \frac{om}{oa} + \frac{mb}{oa} = (-\cos \theta + I \sin \theta),$$

where I is a unit vector perpendicular to the plane of AB . If these versors have any lengths x , y , ob will of course be xy times the length of oa , and we shall have

$$BA = (-\cos \theta + I \sin \theta) xy.$$

Now we have seen that $-\cos \theta \cdot xy$ is the scalar-product of B , A , regarded as vectors. We saw also that $I \sin \theta \cdot xy$ was their vector-product. We shall now say that these are respectively the scalar and vector parts of the product and write

$$BA = S.BA + V.BA.$$

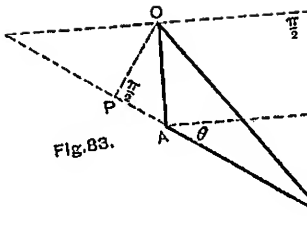


Fig. 83.

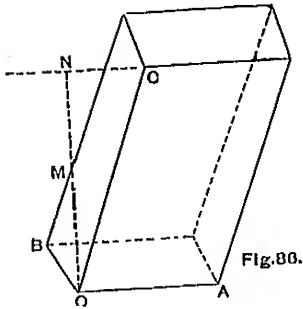


Fig. 86.

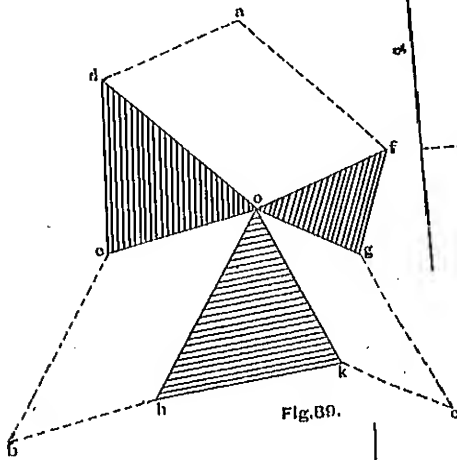


Fig. 89.

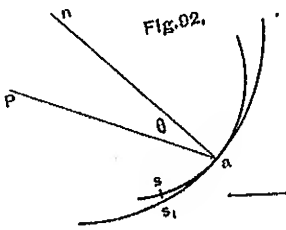


Fig. 92.

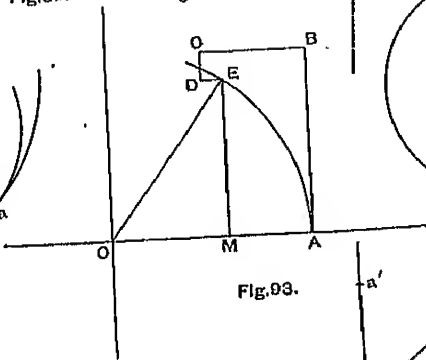


Fig. 93.

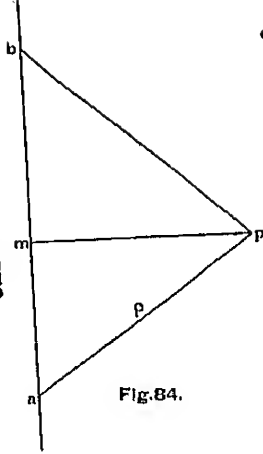


Fig. 84.

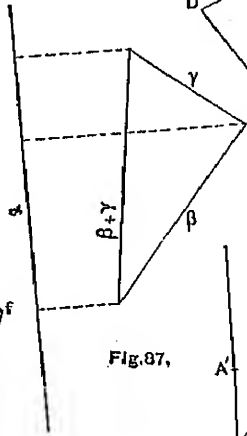


Fig. 87.

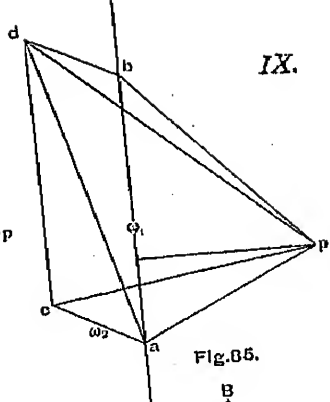


Fig. 85.

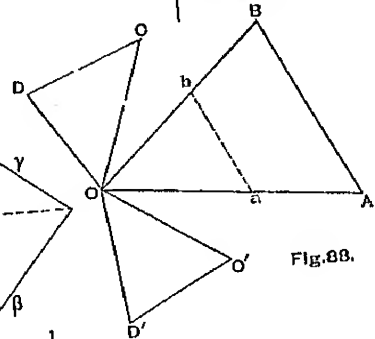


Fig. 88.

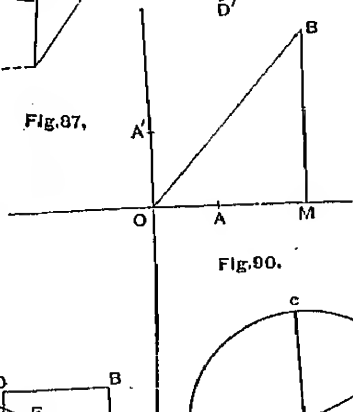


Fig. 90.

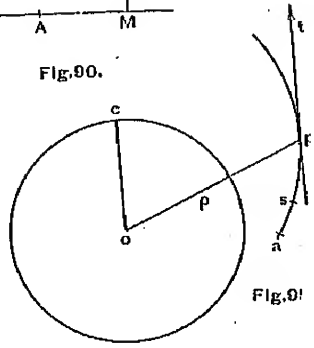
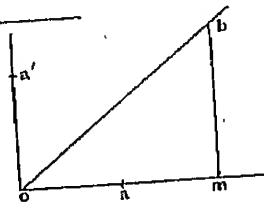


Fig. 91.



In order that $I \sin \theta . xy$ may represent the vector part of the product we need only agree to measure I so that looking back along it B is turned into A by a positive rotation.

In the case where the two versors are at right angles to each other the scalar part of the product vanishes and we can at once trace the analogy between the double interpretation allowable here and that which we gave to equations of multiplication in arithmetic.

We said that when two factors are multiplied together, either they may both be operators or the first may be an operator and the second a step. Similarly, when quantities can be measured in any direction in a plane, an equation of multiplication for two directions at right angles to each other may be written in two ways

$$BA = V . BA, \quad Ba = V . Ba.$$

In the first equation both factors are versors; in the second, the first only is a versor, the second is a vector.

We saw that in arithmetic the rule for finding the product was the same for both interpretations, e.g.

$$(-3)(+2) = -6.$$

Rule. Multiply the numbers together and remember that like signs give +, unlike signs -. The interpretation is either—

The product of the operations of first tripling and reversing, and next doubling is equivalent to the operation of sextupling and reversing; or,

The effect of the operation of tripling and reversing on the step +2 is to turn it into the step -6.

Similarly if B and A , or B and a are at right angles to each other, we have the same rule for finding the product in each case, viz. measure off the product of their lengths along a vector at right angles to their plane, and drawn so that looking back along it, we go from B to A , counter-clockwise.

The second factor may be either a vector or a versor, but the first *must* be a rectangular versor; and in both cases the directions must be at right angles to each other.

If however B and A are not at right angles the double interpretation is no longer possible. We must not regard A as a vector a . For a versor can operate only on vectors perpendicular to it. In this case both B and A must be rectangular versors. Then also the scalar part of the product would not vanish. We should have

$$BA = V . BA + S . BA.$$

The rule found for the addition of rectangular versors enables us to represent any one in terms of three units at right angles to each other.

If i, j, k , are unit lines at right angles to each other, then any vector rectangular versor may be represented by

$$xi + yj + zk,$$

where x, y, z are components of its length.

And

$$ij = bc \quad ca = bc^2a = -ba = ab = k, \text{ etc.}$$

Thus the secondary rules may be considered as bound up in the primary rules, if we agree to let i, j, k stand for these binary products. We may suppose a, b, c are points or stars at an infinite distance in three directions at right angles to each other, then the step ab will clearly be a rectangular versor.

Since any versor not rectangular is the sum of a scalar and a vector part and any vector may be expressed in terms of i, j, k , we are led to consider quantities of the form

$$\omega + xi + yj + zk,$$

which may also be written

$$m(\cos \theta + I \sin \theta),$$

where

$$m^2 = \omega^2 + x^2 + y^2 + z^2.$$

This is called a *quaternion* since it involves four numbers. It is the operation of turning one vector into another in the plane perpendicular to I , that is to the rectangular versor $xi + yj + zk$. This is its most general form. We may consider however that we have already studied quaternions in which the versors were in the same plane and in the same direction.

If the quaternion q is represented by

$$q = \omega + xi + yj + zk = \omega + \rho,$$

its conjugate is

$$Kq = \omega - xi - yj - zk = \omega - \rho.$$

Hamilton calls the part ρ the vector part of the quaternion. Strictly speaking it can only be a rectangular versor. For a quaternion, and therefore each of its parts, is an operation on a vector. It is convenient however to regard a rectangular versor as a vector. They are subject to the same laws.

The stretching power of Kq and also its plane are the same as those of q , but the angle through which it turns a vector is in the opposite direction to the angle through which q turns it.

$$\text{If} \quad q = \frac{OB}{OA} = \frac{OM}{OA} + \frac{MB}{OA} \quad [\text{Fig. 97}]$$

we shall have (by definition)

$$Kq = \frac{OM}{OA} - \frac{MB}{OA} = \frac{OM}{OA} + \frac{MB'}{OA} = \frac{OB'}{OA}.$$

The effect of q, Kq , or of $Kq \cdot q$ is merely to lengthen the line OA . For the turning successively through equal angles in opposite directions brings OA back to the same direction.

And in fact, the expressions for q and Kq in terms of ω and ρ give $q \cdot Kq$ a scalar quantity

$$q \cdot Kq = (\omega + \rho)(\omega - \rho) = \omega^2 - \rho^2 = \omega^2 + x^2 + y^2 + z^2.$$

The fundamental property of the conjugate of a quaternion is that it is another quaternion such that being multiplied by the first it gives a scalar.

We will now examine the meaning of $Kq.r.q$, where r is any rectangular vector.

If the lines here drawn [Fig. 99] represent great circles on a sphere, we see that AC is $Kq.r.q$. That is the result of putting r between a quaternion and its conjugate in to slide r along the great circle of q through twice the length of the arc q ; or to make r rotate round the pole of q through twice the angle of q .

In supposing the arcs drawn on a sphere, that is assuming the versors do not stretch, we have really made no restriction. Whatever the stretching powers are, the stretched vectors from the centre come out at the same points of the sphere.

A quaternion is an operator which turns any one vector into another. We will now consider an operator which turns any two vectors α and β into two others γ and δ . Hamilton called this a linear and vector function. We will denote it by ϕ . Then

$$\begin{aligned}\phi\alpha &= \gamma, \\ \phi\beta &= \delta, \\ \phi(l\alpha + m\beta) &= l\gamma + m\delta.\end{aligned}$$

The last equation expresses what is meant by the function being linear.

The effect of a quaternion, operating on all the vectors in a plane, for example a picture, would be to turn the whole picture through a certain angle and to increase its size, so that the result is a similar picture.

The effect of the function ϕ would be that while all parallel lines remain parallel to one another, their distances as well as their directions are altered. They are all altered in the same proportion. We may express this by saying "All parallel vectors are multiplied by the same quaternion, but the quaternions are different for different vectors."

A circle would become an ellipse. For the equation of a circle may be put into the form

$$\rho = \alpha \cos \theta + \beta \sin \theta \dots\dots\dots (1),$$

if α and β are equal and at right angles.

Operating on this with ϕ we get

$$\begin{aligned}\phi\rho &= \phi\alpha \cos \theta + \phi\beta \sin \theta, \\ \text{or} \quad \phi\rho &= \gamma \cdot \cos \theta + \delta \cdot \sin \theta \dots\dots\dots (2),\end{aligned}$$

where γ and δ are in general not equal and not perpendicular. But this is the equation of an ellipse.

Putting θ proportional to the time, the first equation is that of uniform motion in a circle; the second that of harmonic motion in an ellipse.

If the lines which correspond to the axes of the circle are at right angles to each other, they give the axes of the ellipse; for parallel lines remain parallel.

The tangents at the ends of one axis must be parallel to the other axis.

Then all we have to do is to construct an ellipse on these lines as axes and the property in question being true for the axes is at once proved for all the diameters.

We have secured that

$$\phi(ou) = i(ob),$$

$$\phi(ob) = i(ou).$$

Then we have the function of any other diameter [Fig. 100],

$$\phi(op) = \phi(om) + \phi(mp)$$

$$= \frac{om}{ou} \cdot \phi(ou) + \frac{mp}{ob} \cdot \phi(ob)$$

$$= i \left(\frac{om}{ou} \cdot ob + \frac{mp}{ob} \cdot ou \right)$$

$$= i \cdot t \cdot (mq + on) \quad \{\text{see (1)} \text{ p. 510}\} = i \cdot oq.$$

Hence pure strain of a body is this:—Every line (diameter of the strain ellipse) is changed into one equal in length to the conjugate diameter and perpendicular to it.

An exactly similar proof applies to the hyperbola where we start with the supposition

$$\phi(ou) = i \cdot ob \text{ and } \phi(ob) = i \cdot ou,$$

that is we suppose *ou* is reversed in direction. Then

$$\phi(op) = i \cdot oq.$$

We may include both cases in the following theorem:

If $\phi(ou) = \pm i \cdot ob$ and $\phi(ob) = i \cdot ou$,

then

$$\phi(op) = i \cdot oq.$$

The upper sign applies to the hyperbola; the lower to the ellipse.

There is an important difference between the two strains. The elliptic strain is the only one which occurs in nature. Lines are not reversed in strain. Hyperbolic strain is however the same as elliptic strain, followed by turning the body round through an angle of continuation. Hyperbolic strain is an ideal case if we are merely representing a strain by the function ϕ .

If however the linear function of a vector represents the displacement of a point at the end of that vector, both the ellipse and the hyperbola can be used. The displacement of a point may be in the positive or negative direction along the vector to that point.

We shall find that if the displacements of points round a fixed point *o* are all outwards from, or inwards towards that point we must use an ellipse. If some are outwards and some inwards we must use a hyperbola. Of course all we have to do is to examine whether the two principal displacements along *ou*, *ob* are outwards or inwards.

Taking, as before, two rectangular vectors *ou* and *ob* such that

$$\phi(ou) = i \cdot ob, \quad \phi(ob) = i \cdot ou,$$

then if e and f are the elongations along the two axes we have

$$\psi(a) = e \cos \alpha, \quad \psi(b) = f \cos \alpha.$$

Hence writing a, b for a', b' , respectively,

$$e \cos \alpha = b, \quad f \cos \alpha = a.$$

Hence

$$e = \frac{b}{\cos \alpha}, \\ f = \frac{a}{\cos \alpha}.$$

The squares of the axes are inversely proportional to the elongations,

If both elongations are of the same sign, the x -axis and xy -axis are of the same sign; this is the case in the ellipse. If the elongations are of different sign we have the square law, and this is the case of a hyperbola.

If we add to the stretching of pure strain, a rotation round some axis, we get the whole effect of a homogeneous strain, that is a strain in which all parallel lines remain parallel though their directions are taken in any order. This time a circle will be an ellipse, the axes of which are the axes of the strained body. Hence a sphere is the body in which all lines are elliptical.

The whole effect of the strain being to pull the z -axis out in three directions at right angles to each other and to turn it away from the xy -plane, the axes of the ellipsoid are in them three directions; they are the shortest and the longest lines and lie at right angles to them.

Any three diameters of the sphere at right angles to each other become three conjugate diameters of the ellipsoid. The parallel planes are no longer parallel and we know that the tangent plane at the end of a diameter of the sphere is parallel to the planes containing the other two diameters at right angles to it. This property must also belong to the corresponding diameters of the ellipsoid. Hence the three diameters perpendicular to each other in the sphere are changed into three diameters of the ellipsoid such that the tangent plane at the end of any one is parallel to the plane containing the other two. And this is the definition of conjugate diameters.

The strain ellipsoid in space gives the three axes of the sphere and any vector in a similar way to that in which it is given by the ellipse in a plane.

Next, namely, is the original position of a vector, its new position is represented by the conjugate axis, which is perpendicular to this axis and proportional to it in magnitude. Then a vector xy to the original position of the vector yz .

If however we choose to represent the elongation of one shell and a surface bearing the same relation to the ellipsoid that the hyperbola bears to the ellipse.

These are got by making the hyperbola and the conjugate hyperbola revolve round the major axis and then expressing both its new direction perpendicular to this axis. Thus we get hyperbolic shells and lines which are perpendicular.

The ellipsoid was got in the same way by making the ellipse revolve round the major axis and then expressing it in a direction perpendicular to this.

These surfaces are then got by homogeneous strain from surfaces of revolution; and the plane conjugate to a diameter remains parallel to the tangent plane at the end of it. If the strained body is shortened or lengthened along all the axes we must use the ellipsoid in order to get the function ϕ which represents displacement. If it is shortened along one or two axes and lengthened along the other we must use the hyperboloid. The displacement of any diameter op is represented by the conjugate area opq which in the figure [Fig. 101] is an ellipse. Some conjugate sections however will cut the hyperboloid in hyperbolae. What meanings can we give to the conjugate area in this case? We must take the area of the ellipse on the same axes in order to keep the same rule for all possible diameters.

This is what is meant by a linear and vector function of a vector in space. Every vector is turned into its linear function which is a vector represented by the area conjugate to the original one. This is pure strain. To get homogeneous strain generally which includes rotation we must first get the linear vector function and then turn it round.

In the preceding investigation all the geometry we require is given by the properties of the homogeneous strain itself.

If we take an elastic rod of square section and twist and bend it there is a certain strain at every point, which may be represented by a vector. For instance, if, before the twisting and bending, a body is moving along the rod with uniform velocity then the strain at any point is represented in magnitude and direction by the velocity of this body. The rod tends to untwist and unbend itself. The action at any point which balances this tendency is the stress. There is a bending and also a twisting couple which compounded give another couple and this is represented by a vector perpendicular to its plane. Hence at every point the stress and the strain can be represented by vectors. It is found that the stress is a linear vector function of the strain. It is moreover a pure function. There is no turning round.

As another example we may take an elastic plate bent in two different ways, one kind of bending would at any point change the plane surface into a surface bent like a sphere. The other kind tends to bend the plane like a saddle. These two strains have two corresponding stresses. The plate tends to flatten itself again from both kinds of bending.

Hence again the stress is a pure function of the strain.

Another instance is furnished by a solid body in which one point is fixed. Rotating it, there is one axis about which the moment of momentum is a maximum. If ω is the vector representing the spin and the momentum is represented by p , then

$$p = \phi(\omega), \text{ momentum a pure function of spin.}$$

This pure function depends on an ellipsoid called the momentum ellipsoid.

We can only give a short sketch of a part of the theory in which little was done by Hamilton, but which has been worked out by Tait; the part that is which treats of the operator

$$\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2.$$

Suppose a scalar quantity having various values all over a plane. The height of the plane for instance is such a quantity. At any point in a sloping plane or on the side of a hill there is one direction in which the height increases most rapidly. If we draw a line from any point in the direction of this greatest increase and proportional to the rate of increase, this is called the slope of the height.

If z is the height, slope of $z = i\partial_x z + j\partial_y z$.

We must then set off lines representing the rate of increase in two directions at right angles to each other and take their resultant. This will give the magnitude and direction of the greatest increase of height, that is the slope of z . Suppose for example that going northwards [Fig. 102] we rise one foot in 30, and going eastwards, one in fifty, the resultant of the lines $\frac{1}{30}$ and $\frac{1}{50}$ is the slope.

Now take a scalar quantity u having different values at different points of space. The slope of u is then

$$i\partial_x u + j\partial_y u + k\partial_z u.$$

It is the rate of increase in the direction in which u increases fastest and is found in this way: measure off, in three directions at right angles to each other, lengths which represent rates of increase and find their resultant.

Physical examples of slopes.

If u = potential energy, ∇u = force.

If u = velocity potential, ∇u = velocity.

In these cases ∇ operates on a scalar. Tait investigated the effect of ∇ on a vector.

Let a vector σ have the constituents p, q, r . Then

$$\begin{aligned}\nabla \sigma &= (i\partial_x + j\partial_y + k\partial_z)(ip + jq + kr) \\ &= -(\partial_x p + \partial_y q + \partial_z r) + i(\partial_y r - \partial_z q) + j(\partial_z p - \partial_x r) + k(\partial_x q - \partial_y p).\end{aligned}$$

The quantity in the first part is a scalar; the remaining part is a vector.

The two parts have definite relations to the vector σ . This vector may mean the velocity of a fluid at any point.

Then $-S \cdot \nabla \sigma$ represents the expansion, that is, the rate of change of unit volume. In general there is a certain spin at every point of a fluid. $V \cdot \nabla \sigma$ represents twice this spin.

If σ has a different meaning the interpretations given to the two parts of its slope are of course different. Suppose it means the vector called by Faraday the "electrotonic state" or magnetic induction.

Then a law of electricity is that $S \cdot \nabla \sigma$ equals 0. But $-S \cdot \nabla \sigma$ is the expansion in a moving fluid. Maxwell therefore interprets the law thus:—there

X.

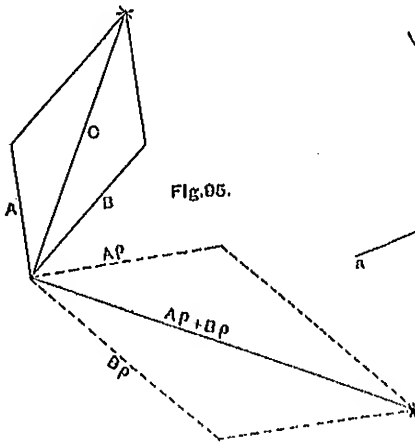


Fig. 05.

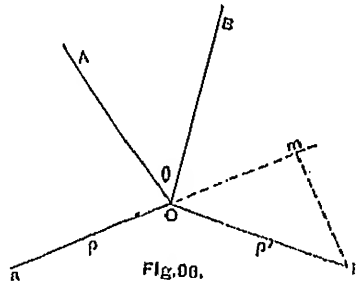


Fig. 06.

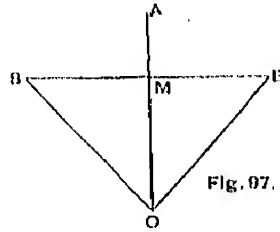


Fig. 07.

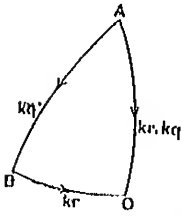


Fig. 08.

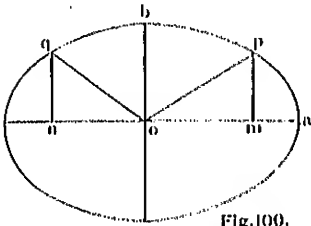
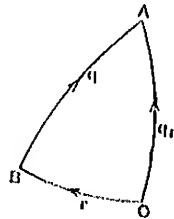


Fig. 100.

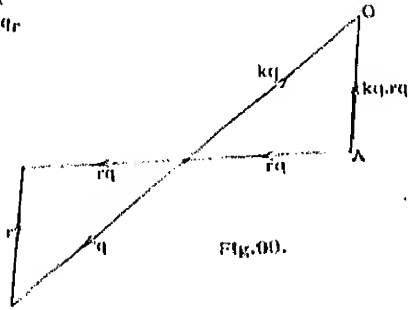


Fig. 101.

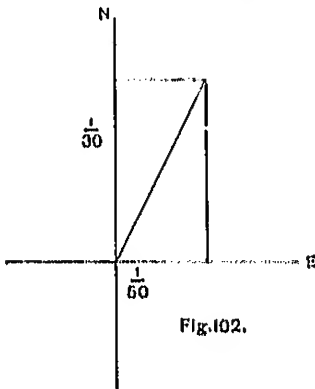


Fig. 102.

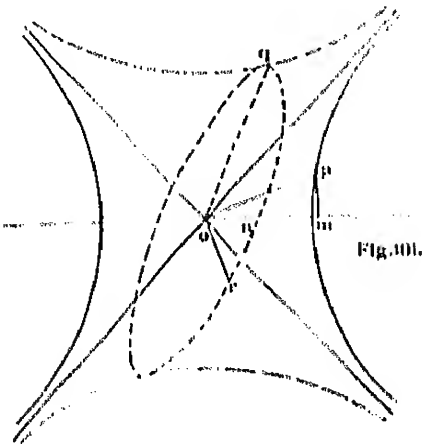


Fig. 103.

is no expansion of the ether in electrical motions. But the analogy is not exact, for the electrostatic state σ represents a momentum, not a velocity. $V \cdot \nabla \sigma$ here represents twice the magnetic induction.

The slope of the slope, $\nabla^2 \sigma$ has no meaning for a fluid-motion, but for σ the electrostatic state it means electromotive force.

The equation

$$\nabla \sigma = S, \quad \nabla \sigma \cdot V = \nabla \sigma$$

is interpreted for σ any vector,

slope of a vector = convergence + curl,

$\nabla^2 \sigma$ = the slope of the slope = the concentration.

Laplace's operator

$$= (\partial_x^2 + \partial_y^2 + \partial_z^2) = \nabla^2$$

is the square of

$$i\partial_x + j\partial_y + k\partial_z = \nabla.$$

Hamilton laid great stress on his having thus discovered the square root of this operator, which is of importance in many physical investigations. Its properties come out in a remarkable way from the properties of ∇ .

SYLLABUS OF LECTURES ON MOTION*.

Division of the Subject.

The science which teaches how to describe motion accurately, and how to compound different motions together, without considering the circumstances under which motions take place, is called *Kinematic* (*κίνημα*, motion).

The simplest kind of motion is that in which a body without changing its size or shape moves so that all straight lines in the body remain parallel to their original positions; this motion is called a *Translation*. As all parts of the body move alike, we may confine our attention to any one of them, however small; for this reason that part of Kinematic which treats of translations is often called the Kinematic of Particles.

A body which does not change its size or shape during the time considered is called a *rigid* body. That part of Kinematic which treats of motions in which there is no change of size or shape is called the Kinematic of Rigid Bodies.

A change of size or shape, considered without reference to change of position, is called a *strain*. The Kinematic of Strains teaches how to describe strains accurately, and how to compound them together. Bodies which change their size or shape are called *elastic*; and the corresponding branch of Kinematic is called the Kinematic of Elastic Bodies.

The science which teaches under what circumstances particular motions take place is called by one or other of two different names according to the view that is taken of it. If it is regarded as mainly based upon the Law of Force, and if its results are expressed in terms of force, it is called *Dynamics* (*δυναμὴς*, force); but if it is regarded as mainly based upon the Law of Energy, and if its results are expressed in terms of energy, it is called *Energetics* (*ἐνεργεῖα*). In either case it is divided into two parts; *Static*, which treats of those circumstances under which *rest* or *null motion* is possible, and *Kinetic*, which treats of circumstances under which actual motion takes place. Properly speaking, Static is a particular case of Kinetic which it is convenient to consider separately.

We may also make divisions between the Static and Kinetic of particles, rigid bodies, and elastic bodies; but the Static of particles and of rigid bodies

* [This syllabus appears to me to have been drawn up for lectures at University College Michaelmas Term, 1878.]

is generally treated as one subject, while the Kinematic and Dynamic or Energetics of elastic bodies are grouped together as the science of *Elasticity*.

These divisions may be represented by the following scheme :—

Science of Motion.

| | | | | |
|---|-----|---------------------|--------|--|
| Kinematic, Dynamic, or Energetics, viz. | } { | Static, Kinetic, | } of { | Particles (Translations), Rigid Bodies (Twists), Elastic Bodies (Strains). |
|---|-----|---------------------|--------|--|

Translations.

Def. If two bodies *A* and *B* are in motion, the motion of *B* is said to be compounded of the motion of *B* relative to *A*, and the motion of *A*.

Prop. Translations represented by the sides of a parallelogram compounded together into a translation represented by the diagonal.

Def. A *vector* is a quantity having magnitude and direction. A translation is a particular kind of vector, and the composition of translations is equivalent to their addition as vectors; it satisfies the law

$$a + b = b + a.$$

Def. Uniform rectilinear motion is that in which equal spaces are traversed in equal times.

Its equation is

$$p = at + b.$$

Prop. Two uniform rectilinear motions compound into a uniform rectilinear motion.

Harmonic Motion.

Def. Uniform motion in a circle is that in which equal arcs are traversed in equal times.

Def. If a point *P* move uniformly in a circle, and a perpendicular *PM* be always drawn from it to a fixed diameter *AA'* of the circle, the foot *M* of the perpendicular will oscillate to and fro in the diameter; this motion of the point *M* is called a *Simple Harmonic Motion*.

Its equation is

$$p = a \cos (nt + c).$$

Def. The radius of the circle is called the *amplitude* of the s. h. m.

Def. The time which *P* takes to go once round the circle is called the *period* of the s. h. m.

Def. The circular measure of the arc described by *P* from the era of reckoning till it came to the positive end of the diameter *AA'* is called the *epoch* of the s. h. m.

Def. The portion of the whole period which has elapsed since the point *M* last passed through its middle position in the positive direction is called the *phase* of the s. h. m.

The resultant of any number of simple harmonic motions whose periods are commensurable in a periodic motion, its period being the least common multiple of their periods.

Fourier's Theorem. Every rectilinear periodic motion of period P may be resolved into a series of simple harmonic motions whose periods are $P, \frac{1}{2}P, \frac{1}{3}P$, etc.

Let $\phi(t)$ be the distance of the moving point from a fixed point on the line at a time t , then the periodicity of the motion is expressed by the fact that $\phi(t+P) = \phi(t)$, whatever t is. And the theorem asserts that in this case the quantities a, b can always be found so as to make true the following equation, where

$$\phi = \frac{2\pi t}{P} \phi(t) = \frac{1}{2} b_0 + b_1 \cos \theta + b_2 \cos 2\theta + \dots \\ + a_1 \sin \theta + a_2 \sin 2\theta + \dots$$

The amplitudes and epochs of the several harmonic components may be represented as follows. Let a vertical cylinder revolve about its axis, while a pencil moves up and down one of its generating lines, so as to trace out a curve on the cylinder. If the motion of the pencil is periodic, and has a period equal to that of the cylinder or any exact multiple of it, this curve will return into itself and be a finite curve on the cylinder. Now let the pencil have the given periodic motion which it is required to resolve into harmonic constituents. When the cylinder revolves once in the period P_1 let the curve described be called C_1 ; when it revolves twice in that period let the curve be called C_2 ; when it revolves n times, let this curve be called C_n . And let a circle be drawn on the cylinder whose height is the mean height of the curve C_1 ; this will be called the mean circle.

If a plane be drawn through the axis of the cylinder, any curve traced on the cylinder may be orthogonally projected on that plane. It is necessary now to define the area between this projection and the line in which the plane is cut by the plane of the mean circle. Let AB be this line [Fig. 104], and let $PMQNP$ be the projection, where PMQ is projected from the near half of the cylinder, and QNP from the further half. Then for the near half, the area APM which is below AB must be considered negative, and the area MQH which is above it, positive. For the further half, QNH must be considered negative, and NPA positive. Thus the area is

$$= APM + MRQ - NRQ + APN \\ = APM + MNQ - MPNQ.$$

The same rule is to be applied when the curve cuts itself or the line AB any number of times. Now it is found that for every closed curve traced on a cylinder, there is one plane through the axis such that the area of the projection on it is zero; and that for the plane at right angles to it the area is the greatest possible; while for an intermediate plane the area varies as the sine of the angle which it makes with the zero plane. It is thus possible to draw an ellipse upon the cylinder, the area of whose projection upon any plane whatever through the axis shall be the same as that of a given closed curve. Let the ellipse E_1 have

the same projected area as the curve C_{10} , C_{20} , hold that of the curve C_{10} , C_{20} , etc., with that of the curve C_{10} and so on. If, while the cylinder revolves once on its axis during the period P , the point P is able to follow the ellipse E_{10} always remaining in the same vertical line, it will have a $c = w$ with the period P . If while the cylinder revolves m times during the period P , the point P is able to follow the ellipse E_{m0} , it will have a $c = m \cdot w$ with the period $\frac{1}{m} P$. These motions are the harmonic components of the plane periodic motions; and that motion may be reproduced by compounding them all together*.

Definition of Velocity

Prop. If a rectilinear motion be such that the space traversed every second is proportional to the square of the time elapsed, the compounded rectilinear motion, the resultant will be such as to be a curvilinear motion.

Its equation is

$$y = w \cdot x^2 / 2g.$$

Velocity

Def. If a body is in uniform rectilinear motion, and traverses a certain space in every second, the body is said to have a *velocity* or *rate of motion* of so many units per second, or simply a *velocity*.

Def. If a body undergoes a translation from *a* to *b* in a path of *t* seconds in any manner by any path from *a* to *b* in *t* seconds, the body is said to have a *mean velocity* $\frac{AB}{t}$ in that interval of *t* seconds.

The two quantities here denoted by *mean velocity* and *velocity*, then, are *vectors*. A velocity may be expressed in terms of distance in the same unit, inches per second; in feet or miles per second; in inches per foot, or, when *t* is given, as, *per second* or *per secondly*; it is said to be *given in absolute measure*. In uniform rectilinear motion the mean velocity is the same in any interval whatever, and has just the inclination or velocity of each instant, but the latter is a property which the body possesses at any epoch or point of time, while the former is a fact relating to its motion during an interval.

Def. If any rectilinear motion of a body be compounded with a uniform motion of unit velocity at right angles to it, the curve traced out by the point is called the *curve of positions* for that rectilinear motion.

Lemma. PT is the tangent at a point P of a curve (Fig. 110). Any angle being proposed, it is always possible to find a point Q on the circle so near to P that the chord of arc Q is bisected in R at right angles with the tangent PT at an angle less than the proposed angle.

Let C be the centre of the circle; make CP less than the proposed angle, and draw CM perpendicular to CP . Then CP is the angle which the chord Q makes with PT , and it is always less than CP , therefore less than the proposed angle. *Q.E.D.*

* [IX, Dynamics, p. 21, where it is said a period of P motion is traversed with the g which the time g represents.]

DEF. R, P are points on any curve, Q moves from R along the curve towards P ; if when any angle is proposed, it is always possible to take Q so near to P that the chord of every arc pq included in PQ shall make with a certain line TP an angle less than the proposed angle; then the curve is said to have TP for a tangent at the point P [Fig. 106].

If S is a point on the other side of P and if Q moves from S towards P , there may be another line PT' such that an arc PQ may always be taken in which no chord shall be inclined to PT' so much as by a proposed angle. In this case we may speak of TP as the tangent up to P and of PT' as the tangent on from P . When TPT' is a straight line, the curve is said to be *elementally straight* or to have the property of *elemental straightness* at the point P ; for the more it is magnified, the more will a portion containing P of given length in the magnified figure approach to the straight line TPT' in shape and position. For this, three conditions are necessary; there must be a tangent up to P , a tangent on from P , and these tangents must be in one straight line.

PROP. If the curve of positions of a rectilinear motion has a tangent at a point P , then it is possible to choose an interval ending at the instant corresponding to the point P so that the mean velocity in that interval (and in all intervals included in it) shall differ less than by a given amount from a certain quantity.

Let QP [Fig 107] be a portion of the curve of positions, PT the tangent at P ; QN, PM parallel to the (vertical) rectilinear motion considered, and perpendicular to the (horizontal) uniform motion with which it is compounded; QR perpendicular to PM . Since the uniform motion has unit velocity, the number of units of length in NM is equal to the number of seconds in which the body has performed the vertical motion RP , and the mean velocity in the interval NM is therefore $\frac{RP}{NM}$. Now take AB a horizontal line equal to the unit of length, and draw AC, AD parallel to PT, PQ , meeting the vertical line through B in C, D . Then BD represents the mean velocity in the interval NM . Similarly if pq be any arc included in PQ (pm, qn perpendicular to NM), and if we draw Ad parallel to the chord pq , Bd will represent the mean velocity in the interval nm . Now it is possible by hypothesis to choose Q so near to P that the angle QPT , which is equal to CAD , shall be less than any proposed angle; and that the angle which any chord pq makes with PT , which angle is equal to CAd , shall be less than the proposed angle. Therefore it is possible so to choose N that for every interval included in NM the length Cd shall be less than a proposed amount; or so that the mean velocity shall differ from the velocity represented by BC by a quantity less than the proposed amount. *q. e. d.* The quantity Bc or $\frac{MP}{TM}$ is then called the *instantaneous velocity* of the rectilinear motion at the instant M .

DEF. Let Q, P be successive positions of a moving point, and let BD represent the mean velocity during an interval included in the passage from Q to P ; then if it is always possible to find Q so near to P that for all intervals between Q and P the distance DC from D to a fixed point C shall be less than a proposed

length, the point at the instant of arriving at P is said to have an *instantaneous velocity* PC in magnitude and direction [Fig. 109].

PROV. If a moving point has a velocity, the curve described has a tangent in the same direction; and if a length equal to the arc AP be measured off on a straight line as Q moved, this rectilinear motion will have a velocity whose magnitude is equal to that of Q .

PROV. If each of two motions has a velocity at a certain instant of time, the motion compounded of them has a velocity which is compounded of the two velocities by the rule for addition of vectors.

Let AB and AC be the given velocities compounded to the parallelogram $ABDC$ [Fig. 109]. Let also AD , AE be the mean velocities during an interval which ends at the given instant; if the parallelogram $ABDC$ be completed, we know that AD is the mean velocity of the resultant motion. Now the interval may be so chosen that for it and all shorter ones included in it AD and AE are each less than half of any proposed length, and therefore DE , which is their vector sum, less than the proposed length. Consequently AD is the velocity of the resultant motion at the given instant, *q. e. d.*

It is to be noticed that in accordance with our definition a motion may have one velocity *up to* a certain instant and another velocity *from* that instant, *q. d.*, as we may say, an *initial* and a *degenerate* velocity. Such motions are for mathematical convenience supposed to take place in the theory of vibrations; but it is believed that they do not occur in nature, and that the constant initial-perturb velocities are always identical. If a point has an initial and a degenerate velocity at a given instant and if they are identical, its motion is said to be *elementally uniform*; for if a small portion of the path containing the position of the point at that instant be magnified to a definite length, and the time of traversing different parts of it be preserved in their proportions, then the smaller the portion taken, the nearer will the path approach to a straight line and the motion to uniform motion along it.

PROV. The velocity in the n is u_n

$$p = a \cos \alpha \cos \beta \cos \gamma$$

in

$$A = a \cos \alpha \cos \beta \cos \gamma$$

(when the position-vector of a point is called p , its velocity is denoted by p .)

The n , u , m , has a velocity, because its curve of position has a tangent, being produced by unrolling an ellipse from a cylinder. Its motion is *elementary* motion being compounded of two simple harmonic motions, its velocity is compounded by the law of addition of vectors. Thus the velocity of P is compounded of the velocities of M and N . But these velocities are respectively perpendicular to the lines CP , CM , and MP , the vector CP being equal to $CM + MP$ [Fig. 110]. The velocities are therefore proportional to the lengths of these lines, and as the velocity of P is u , CP along the tangent, the velocities of M and N are $u \cdot MP$ and $u \cdot CM$ along MP and CM respectively. But a length $u \cdot MP = u \cdot AC$ in PCM along MP is equal to $u \cdot AC$ in PCM along AC .

Fig. 104.

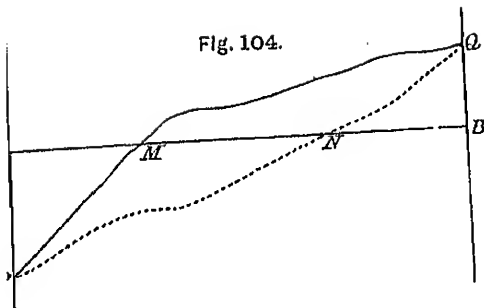


Fig. 105.

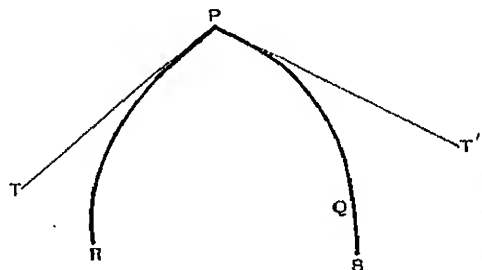
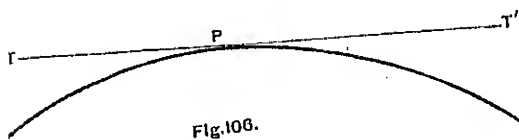
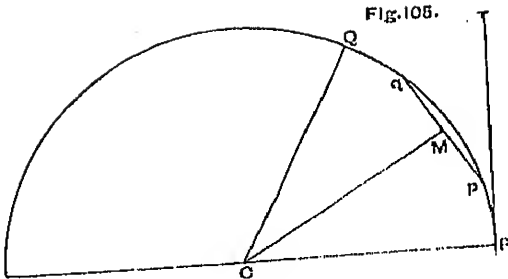
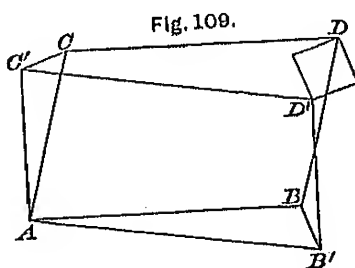
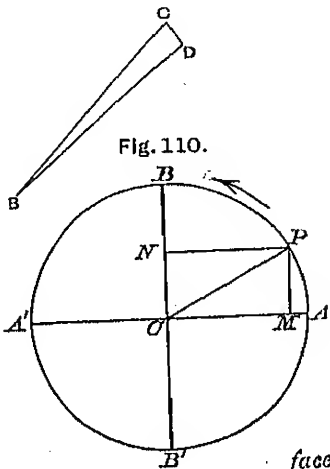
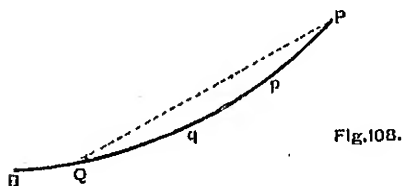
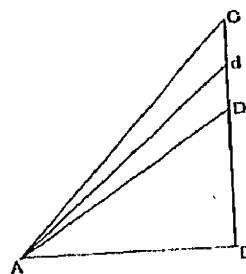
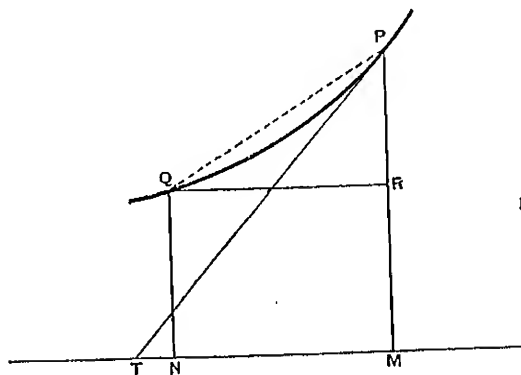


Fig. 107.



PROV. The velocity in the elliptic harmonic motion,

$$\begin{aligned} \rho &= a \cos (nt + \epsilon) + \beta \sin (nt + \epsilon), \\ \dot{\rho} &= -na \sin (nt + \epsilon) + n\beta \cos (nt + \epsilon) \\ &= n \left\{ a \cos \left(nt + \epsilon + \frac{\pi}{2} \right) + \beta \sin \left(nt + \epsilon + \frac{\pi}{2} \right) \right\}, \end{aligned}$$

and is therefore proportional to the conjugate diameter.

PROV. The velocity in the parabolic motion

$$\begin{aligned} \rho &= at + \frac{1}{2} \gamma t^2 \\ \dot{\rho} &= a + \gamma t. \end{aligned}$$

Let t_1, t_2, t_3, t be successive values of t , these quantities being therefore in ascending order of magnitude; $\rho_1, \rho_2, \rho_3, \rho$ the corresponding values of ρ . Then the mean velocity in the interval from t_2 to t_3 is

$$\frac{\rho_3 - \rho_2}{t_3 - t_2} = a + \gamma (t_2 + t_3).$$

Since t_2 and t_3 are intermediate between t_1 and t , this vector differs from $a + \frac{1}{2}\gamma t$ less than $\frac{1}{2}\gamma t_1$ does; that is, less than $\frac{1}{2}\gamma(t - t_1)$. Now it is possible to choose t_1 that this shall be shorter than any proposed length $\gamma\epsilon$; that is, it be possible to choose an interval ending at t , so that the mean velocity for every interval included in it differs from $a + \frac{1}{2}\gamma t$ by less than a proposed amount. The same thing may be shown for intervals beginning at t . Therefore the motion is elementarily uniform and has $a + \frac{1}{2}\gamma t$ for its velocity. Q. E. D.

PROV. In the motion whose equation is

$$\begin{aligned} \rho &= at^n \\ \dot{\rho} &= nat^{n-1}, \end{aligned}$$

(n a positive integer), the velocity in the interval from t_2 to t_3 is

$$\frac{\rho_3 - \rho_2}{t_3 - t_2} = a \frac{t_3^n - t_2^n}{t_3 - t_2} = a (t_2^{n-1} + t_2^{n-2}t_3 + \dots + t_2^{n-1}).$$

Since t_2 and t_3 are intermediate between t_1 and t , this quantity differs from nat^{n-1} less than nat_1^{n-1} does; that is, less than $na(t^{n-1} - t_1^{n-1})$, which by proper choice of t_1 can be made less than an assigned quantity. Whence as before.

without revisiting any of its former positions. According to this a plane and space itself are infinitely extended in every direction.

5. *Perpendicular and Parallel.* From the first two hypotheses it may be proved that straight lines can only intersect in two points. These straight lines which meet therefore divide space into four regions, each of them is called an angle. If these are all of the same shape the lines are called perpendicular. By adding now the third hypothesis we may show that two lines meet in only one point. Two lines making equal angles with the same line can then not meet at all. They are called parallel. It appears probable from experience that only one line can be drawn through a fixed point parallel to a given line. We may assume either that lines meet only in one point, and that this probable experience is accurately true, or we may assume the third hypothesis; either assumption involves the other.

6. *Quantity and Measuring.* A discrete assemblage of things, as of chairs, say, is estimated by counting the number of them. It is assumed that the number is the same in whatever order they are counted. From this assumption flow the theorems of addition and multiplication $a + b = b + a$, $ab = ba$, $a(b + c) = ab + ac$. We cannot however count the number of points in a piece of line. Yet we suppose it to have a certain magnitude; we speak of another piece as greater, or less, and greater or less in a certain degree. The degree in which one piece is greater than another is called the *ratio* of the two pieces. The ratio of $A + B$ to C is called the sum of the ratios of A to C and B to C . The ratio of A to C is called the product of the ratio of A to B and B to C . It is assumed that two things may each be broken up into any number of parts and the parts rearranged without altering their ratio. From this assumption flow as with numbers the theorems of addition and multiplication $a + b = b + a$, $ab = ba$, $a(b + c) = ab + ac$.

7. *Calculus of Ratios.* Every quantity is therefore measured by the ratio which it bears to some fixed quantity, called the unit. But between any two ratios is an infinite number of ratios; it is therefore impossible to tabulate all ratios, or to give them names. A ratio then can only be described approximately, as being very near to the ratio of two numbers, that is, of two quantities which have a common measure. On the assumption that two ratios which are always greater or less than the same numerical fraction are equal, it may be shown, as in Euclid, that similar triangles have proportional sides.

8. *Analysis of Position on a Straight line.* On a straight line take a fixed point o , and a fixed length oa in a given direction from o ; then the position of a point p on the line is known if we know the ratio of op to oa and the side of a on which p is, and *vice versa*. Let p denote the ratio, then $op = p \cdot oa$; p may be regarded as a direction to perform the following operation; change the value of oa in the ratio of 1 to p . The equation asserts that by performing this upon oa we obtain the value of op (op and oa may be regarded as quantities of motion). Subtraction will mean motion in the contrary direction. Hence $-$ may be regarded as an abbreviation for $\times -1$ or *reversed addition*.

9. *Vectors. Ratios of Vectors.* The quantity of motion which carries a point from the position a to the position b is called the *vector* ab . The vector ab is said to be equal to the vector cd when b is on the same side of a that d is of c ,

these vectors. The angle ψ is called the *argument* of the same complex number. Thus if r, ϕ are the modulus and argument of z ,

we have $z = x + iy = r(\cos \phi + i \sin \phi)$,

$$x^2 + y^2 = r^2, \quad \tan \phi = \frac{y}{x}.$$

13. *Addition.* To add together two complex numbers, we must add separately the parts which are not multiplied by i and the parts which are so multiplied. Thus $(a + ib) + (c + id)$, and

$$ab = (a + ib)(c + id) = ac + ibc + ida + id^2 = ac + id(a + b) + id^2.$$

Hence, on substituting for these vectors their ratios to the fixed vector OI , we obtain the rule enumerated. From the fact that the opposite sides of parallelograms are equivalent vectors flows the theorem

$$ac + id(a + b) = bc + id(b + a), \text{ or } a + b = b + a.$$

14. *Multiplication.* To compound the ratios of OI to Oa , and of OI to Ob is to find a vector which bears the same relation to Ob that Oa does to OI . Let Oc be such a vector, then length $Oc = \frac{Oa \cdot Ob}{OI}$, and angle $\angle Oca = \angle Oa + \angle Ob$.

Hence to multiply two complex numbers, multiply their moduli and add their arguments. This rule shows that $z^n = az^n$. By altering the triangle Oab in a certain ratio ($\cos z$) and turning it through a certain angle ($\arg z$) we may show that $z(\cos z + i \sin z) = z + iz^n$, $\left(a = \frac{Oa}{OI}, b = \frac{Ob}{OI} \right)$.

15. *Expansion of $F(x + y)$.* By means of the three laws of addition and multiplication which we have now proved true for complex numbers we may Newton show that the Binomial Theorem with a positive integral exponent is true for these numbers; that is, that $(x + y)^n = \sum_{r=0}^n \binom{n}{r} a^r b^{n-r}$, where a, b take all values consistent with the equation $a + b = n$. Now if $F(x) = ax^n + bx^{n-1} + \dots + kx + l$, by applying this theorem to each term we may show that

$$F(x + y) = F(x) + yF'(x) + \frac{y^2}{2!} F''(x) + \dots \quad \text{Taylor,}$$

where $F'(x), F''(x), \dots$ are rational integral functions of x ; we shall return to consider the method of deriving these from $F(x)$.

16. *Transformation $z = F(x)$. Similarity of smallest parts.* Let the complex number z be as before the ratio of the vector Ox to the vector OI in a certain plane, and let z be the ratio of the vector Ox to the vector $O'I'$ in another plane. Then by the equation $z = F(x)$ (a rational integral function) z is determined as soon as x is known. Let x receive a small change and become $x' = x + y$, and let z consequently become $z' = z + v$. Then by (15) $z = y \cdot F'(x) +$ terms containing y^2 . Suppose y so small that it may be neglected in comparison with 1, and therefore y^2 may be neglected in comparison with y . In that case u is got from y by increasing it in the ratio $\cos F'(x) : 1$ and turning it through the angle $\arg F'(x)$.

Gauss.

Thus the direction of zz' makes a constant angle with the direction of xx' , and we take three points $xx'x''$ very near to each other, the triangle formed by their corresponding points $zz'z''$ will be similar to $xx'x''$. So if any picture be drawn in the first plane, while x describes the lines of this picture z will describe the lines of a distorted copy; but the two pictures will be *similar in their smaller parts*, and any two lines in the one will cut at the same angle as the corresponding lines in the other.

Argand.

17. *The Equation $F(x)=0$ has n roots.* By properly choosing x I may that we can make z come to the origin. For if not, there is some position of z which is the nearest to the origin that it can possibly have. Consider this position, and the corresponding position of x ; and let λ be the corresponding value of $F'(x)$. If x move in any direction, z moves in a direction making an angle $\arg. \lambda$ with this; if therefore x move in a direction making an angle $\arg. \lambda$ with the line joining z to the origin, z will move straight towards the origin, contrary to the supposition that it could not get any nearer. Hence there is some value of x for which $F(x)=0$. Let α be this value, then we know that $F(x)=(x-\alpha)F_1(x)$. But now as before $F_1(x)$ must have a root, say β ; then $F(x)=(x-\alpha)(x-\beta)F_2(x)$, and so on. Thus finally $F(x)$ has as many roots as dimensions.

Cauchy.

18. *Number of Roots in a given Closed Contour.* The argument of $F'(x)$ is then the sum of the arguments of $x-\alpha, x-\beta, \dots$. Now if x describe any closed contour containing no roots, these arguments may increase or decrease but will ultimately resume their original value. But if α is within the contour the argument of $x-\alpha$ will increase by 2π . Hence the number of roots of $F(x)$ within the contour is the number of (2π) 's added to its argument, when

[a page is missing.]

22. *Position on a Plane. Cartesian formulae.* In the method of Descartes the position of a point is defined by its distances from two fixed straight lines X and Y called the *axes*; the distance from each axis is measured parallel to the other, as pn, pm in Fig. [111]. The signed numbers $\frac{np}{Ol}, \frac{mp}{Ol}$ are denoted by x and y , and called the *coordinates* of the point. The position of a straight line is defined by its distance p from the origin, measured perpendicular to the line, and by the angles α, β which this perpendicular makes with Y and X respectively. These are connected by the equation $\alpha + \beta = \omega$ (the angle between X and Y) which may also be written

$$\cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta \cos \omega = \sin^2 \omega.$$

23. *Expressions for the simplest geometric magnitudes.*

(1) Distance of points $(x_1 y_1) (x_2 y_2)$

$$\delta^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2) \cos \omega.$$

(2) Area of triangle $(x_1 y_1) (x_2 y_2) (x_3 y_3)$

$$\frac{2\Delta}{\sin \omega} = x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1 = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Perpendicular distance of point (x, y) from line (p, α, β)

$$= p - x \cos \alpha - y \cos \beta.$$

Angle between the lines (p, α, β) (p', α', β') ; $\theta = \alpha - \alpha'$ or $\beta' - \beta$, or

$$\sin^2 \theta = \cos \alpha \cos \alpha' + \cos \beta \cos \beta' - (\cos \alpha \cos \beta' + \cos \alpha' \cos \beta) \cos \omega$$

$$\sin^2 \theta = \cos \alpha \cos \beta' + \cos \alpha' \cos \beta.$$

Equations. If we know that $x \cos \alpha + y \cos \beta - p = 0$, we know that the point (x, y) is on the line (p, α, β) , thus all the values of x, y which satisfy this represent all the points on that line. The equation itself may then be present all the points on the line; or, (less accurately) to represent the general, the condition that (x, y) may be on a known curve is called the *eqn of the curve*.

$$\left(\text{In the following } \omega \text{ is taken} = \frac{\pi}{2} \right).$$

1st is on a

circle Its distance from a fixed point (the centre) is constant,

any distance of (x, y) from (a, b) is equal to r ,

$$\therefore (x - a)^2 + (y - b)^2 = r^2.$$

2nd Its distance from a fixed point (focus) \pm dist. from fixed line

any distance of (x, y) from (a, b) \pm dist. from $\left(a, x_1 = \frac{\pi}{2}\right)$,

$$\therefore (x - a)^2 + y^2 = (a + x)^2, \quad \text{or } y^2 = 4ax.$$

3rd Its distance from a fixed point $\pm c$ times dist. from fixed line,

Ellipse any dist. of (x, y) from $(ae, 0)$ $\pm c$ times dist. from $\left(\frac{a}{e}, 0, \frac{\pi}{2}\right)$,

$$\therefore (x - ae)^2 + y^2 = c^2 \left(\frac{a}{e} - x\right)^2, \quad \text{or } \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

Hyperbola c is less than 1, and we write $a^2(1 - e^2) = b^2$,

Hyperbola c is greater than 1, $a^2(1 - e^2) = -b^2$.

Reduction of Equation of the First Order. The equation $lx + my + n = 0$ (reduced to the form $x \cos \alpha + y \cos \beta - p = 0$) if we multiply it by a quantity such that $R^2(l^2 + m^2 - 2lm \cos \omega) = \sin^2 \omega$. From the value of R thus indicated the formula,

$$\begin{aligned} l \sin \omega &= R \sin \omega \\ x + m' - 2lm \cos \omega &= R \cos \beta = \frac{m \sin \omega}{\sqrt{(l^2 + m^2 - 2lm \cos \omega)}} \cdot R = \frac{m \sin \omega}{\sqrt{(l^2 + m^2 - 2lm \cos \omega)}} \\ \text{substitution} \end{aligned}$$

Perpendicular from $x'y'$ on $lx + my + n = 0$ is $\frac{lx' + my' + n}{\sqrt{(l^2 + m^2 - 2lm \cos \omega)}} \sin \omega$;

cosine lines $lx + my + n = 0$, $l'x' + m'y' + n' = 0$,

$$\cos \theta = \frac{R^2 + mm' - (lm' + l'm) \cos \omega}{\sqrt{(l^2 + m^2 - 2lm \cos \omega)} \sqrt{(l'^2 + m'^2 - 2l'm' \cos \omega)}}$$

$$\sin \theta = \frac{(lm' - l'm) \sin \omega}{\sqrt{(l^2 + m^2 - 2lm \cos \omega)} \sqrt{(l'^2 + m'^2 - 2l'm' \cos \omega)}}$$

26. *Grassmann Notation for points and lines.* A point is denoted by a single small letter, and a line by a single large letter. The symbol aB or Ba is taken to mean the result of substituting the coordinates of the point a in the equation of the line B . The symbol ab stands for the line joining the points a, b , and AB for the point of intersection of the lines A, B . Accordingly abc stands for the determinant formed with the coordinates of the points a, b, c , and ABC for the determinant formed with the coefficients of the lines A, B, C . $abc=0$ means that the three points are in a line, and $ABC=0$ means that the three lines meet in a point. Let a mean the point (x_1y_1) , b the point (x_2y_2) and c the point (x_3y_3) ; also let A be the line $lx+my+n=0$, B the line $l'x+m'y+n'=0$, and C the line $l''x+m''y+n''=0$. Then $aB=Ba=l'x_1+m'y_1+n'$.

$$abc = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \quad ABC = \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} \quad \begin{array}{l} \text{Equation of} \\ ab \text{ is} \end{array} \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Coordinates of AB are given by

$$\frac{x}{m'n' - m''n} = \frac{y}{n'l' - n'l} = \frac{z}{ln' - l'n}.$$

[I have not met with any further *Notes*.]

ANALYSIS OF LOBATSCHESKY.

Propositions common to the two theories.*

1. A straight line fits itself in all its positions: i.e. if we turn the surface containing it about two points of the line, the line does not move,
2. Two straight lines cannot cut in two points.
3. A straight line may be produced indefinitely.
4. Two lines perpendicular to the same line and in the same plane cannot meet each other.
5. One straight line must cut another if it has points on both sides of it.
6. Angles (plane or dihedral) vertically opposite are equal.
7. Straight lines making equal angles with a third straight line cannot meet.
8. In a triangle equal angles are opposed by equal sides, and conversely.
9. Greater angle opposed to greater side. In right-angled triangle, hypotenuse is greater than either of the other sides, and the two angles next it are acute.
10. Equalities of triangles.
11. A line perpendicular to two other lines not in the same plane with it is perpendicular to every other line in their plane.
12. The intersection of a sphere and a plane is a circle.
13. A line perpendicular to the intersection of two perpendicular planes, lying in one of them, is perpendicular to the other.
14. In a spherical triangle equal sides are opposed by equal angles, and conversely.
15. Two spherical triangles are equal when they have equal sides containing equal angles, or equal angles adjacent to equal sides.

* (i.e. to Euclid's and Lobatschewsky's.)

THE POLAR THEORY OF CUBICS.

1. Consider a net of conics B_2 (viz., $B_2^{(1)}, B_2^{(2)}, \dots$). All the conics are harmonic of all these form a tangential net. The locus of 1 polars in respect of the entire net B_2 meet in a point is a curve of degree H_3 . So also the envelope of lines X whose poles in respect of net b_2 are in a line is a curve of the third class h_3 . These are defined by the equations

$$(xB_2, xB_2', xB_2'') = x^3 H_3,$$

$$(Xb_2, Xb_2', Xb_2'') = X^3 h_3.$$

Let y be the point of intersection of the polars of x , and X which the poles of X lie. Then y is clearly a point on H_3 , and X to h_3 . x, y are called corresponding points of H_3 , and X, Y are tangents of h_3 .

2. The points x, y and the lines X, Y satisfy the equations

$$xyB_2 = 0, \quad XYb_2 = 0,$$

where B_2, b_2 are any conics of their respective nets. These show that the pair xy belongs to the net B_2 and that the line-pair XY belongs to the net b_2 . for each net includes all conics that are harmonic of all conics. Again, since the whole net B_2 is harmonic of the points xy , and every line of the net can be drawn through any two points, there must be a line containing the line xy as half of it; and this line must be tangent to h_3 . So likewise the point XY must be on H_3 . We have the theorems:—

If a conic b_2 breaks up into two points x, y , these are points on H_3 , and the line xy is a tangent to h_3 .

If a conic B_2 breaks up into two lines XY , these are tangents to h_3 , and the point XY is on H_3 .

B. Let a be a point subtended in involution by b_2, b_2', b_2'' , be so chosen that

$$a(\lambda b_2 + \mu b_2' + \nu b_2'') = 0,$$

which means that a is one of the two points into which b_2, b_2' break up; or that a is a point on H_3 . Thus all points on H_3 are subtended in involution by the net b_2 .

In a similar way we should prove that *all tangents to h_3 are cut in involution by the net H_2* . Or we may define H_2 as the locus of points subtended in involution by the h_2 , and h_3 as the envelope of lines cut in involution by the H_2 .

4. Let x be the intersection of XY , and y its corresponding point. Let also $x'y'$ be a pair of corresponding points very near to xy . Then $x'y'$ is a point-pair belonging to the net h_2 , and XY is a line-pair belonging to the net H_2 . But every conic of the net h_2 is harmonic of every conic of the net H_2 . Therefore xx', X, xy', Y is a harmonic pencil. Now xx' is the tangent to h_3 at x , and xy' is the same as xy . Hence

The three tangents to h_3 from a point x on H_3 , together with the tangent to H_3 at x , form a harmonic pencil.

The three points in which H_3 cuts a tangent X to h_3 , together with the point of contact of X , form a harmonic range.

6. There are three cubics C_A, C_A', C_A'' which have H_3 for their hessian, and three curves of the 3rd class c_A, c_A', c_A'' which have h_3 for their hessian. These correspond in pairs, so that we have

$$C_A'c_A = 0, C_A''c_A' = 0, C_A''c_A'' = 0.$$

I attend only to the pair C_A, c_A .

We have now also

$$C_A h_A = 2AS, c_A H_A = 2At, H_A h_A = Tst,$$

where S, T are the fundamental invariants of C_A , and s, t of c_A .

The conics H_2 are the first polars of C_A , and the h_2 are the first polars of c_A . In fact, x being any point and X any line,

$$x(C_A'X - H_2), H_2c_A = 0, H_A h_A = x,$$

$$X(c_A' - h_2), h_2c_A' = 0, h_A H_A = X.$$

Suppose H_A and h_A to break up; then we have the theorem:

The mixed polar in regard to h_3 of two of its corresponding tangents is a point on H_3 .

The mixed polar in regard to H_3 of two of its corresponding points is a tangent to h_3 . (CAYLEY.)

We may here take the mixed polar in either case with regard to any cyclographic table.

The equations $H_2 h_A = x$, &c., are virtually given by Salmon, *Conics*, 5th ed., p. 310.

6. The condition that xc_A' shall touch X is of the second order in x , X and the coefficients of C_A . Denote it by $(xc_A', X)^2$, then if $X = yz$, we have

$$(xc_A', X)^2 = xy^2c_A', xz^2c_A' + xgy^2c_A'^2.$$

Regard X as fixed; then x describes a conic, the (second) polar envelope of X , which

Cayley,

Reiner,
Cremona,

Cayley.

- (a) is the locus of points whose first polars touch X ,
- (b) is the envelope of second polars of points on X ,
- (c) is the locus of poles of X in regard to first polars of its points,
- (d) breaks up when X touches Δ_1 .

Salmon, &c. Let ax_a^2 denote the tangential form of the first polar of a ; then $ax_a^2H_2 = x^2t_a^2N_1$ is the first polar in regard to the cubic of a point on the cubic harmonic of the first polar of that point in regard to the cubic.

The condition that X shall pass through an intersection of at_a^2 , xt_b^2 (that is, through a double conjugate of x) is, of course,

$$x^2t_a^2t_b^2t_c^2 = 0,$$

X being fixed, this is a quintic curve whose equation may also be written

$$F_2X_2 + XH_2H_2 = 0,$$

where F_2 is the second polar envelope of X in above. This equation indicates the properties of the curve.

7. Operating with the mixed concomitant $x^2t_a^2t_b^2$ on H_2 we obtain a new concomitant g_2 of the 6th order in the coefficients of x^3 , which is then seen to be the envelope of a line whose polar envelope is harmonic of its first polar in respect of H_2 .

8. Two point cubics 123 and 123' are said to be harmonic of one another when

$$11', 22', 33' + 12', 23', 31' + 13', 21', 32' + 13', 24', 34' \\ + 12', 21', 33' + 11', 22', 32' = 0,$$

or when

$$211', 22', 33' = 0.$$

If 123 is harmonic of 123' and 1 coincides with 1', it must have the same harmonics in regard to 23, 32.

If three of the six points coincide, a fourth coincides with them.

Every point cubic is harmonic of itself.

The envelope of a line cut harmonically by two cubic curves is of the third class. This envelope is the same for any two cubics through the nine points of intersection.

Two syzygetic cubics cut every line harmonically. Two sets of cubics will thus correspond a net of curves of the third class, one for every pencil belonging to the net.

The condition that X may be cut harmonically by t_a^2 and $xt_a^2xt_b^2$ is the vanishing of the mixed concomitant $xt_a^2xt_b^2xt_c^2$.

If t_b is the harmonic envelope of the pencil Xt_a^2 on H_2 , the tangents to t_b from a point of intersection are the three flexes at that point.

ON PFAPFIANS,

I.

Consider $2n$ alternate multi $a_1 a_2 \dots a_{2n}$, and form a linear function of their binary products, viz.

$$\sum p_{hk} a_h a_k \quad [h, k=1, 2, \dots, 2n.$$

Since to every term in this $p_{hk} a_h a_k$ there corresponds a term $p_{kh} a_k a_h$, the complete coefficient of $a_h a_k$ is $p_{hk} + p_{kh}$, and the whole expression involves only these differences; hence it is convenient to make $p_{kh} = -p_{hk}$, and then

$$\sum p_{hk} a_h a_k = 2 \sum' p_{hk} a_h a_k,$$

where the \sum' signifies that the summation is to be extended to those values only for which $k > h$.

The n^{th} power of $\sum' p_{hk} a_h a_k$ is $(a_1 a_2 \dots a_n)$ multiplied by a numerical constant and a function of the p which is called a *Pfaffian of the n^{th} order*; viz. it is

$$\frac{1}{N} \sum' p_{12} p_{34} \dots p_{2n-1, 2n},$$

where the suffixes are to be permuted in all possible ways, and signs prefixed according to the rule for determinants. Each term occurs $N = 11n$ times and accordingly the result is divided by this number. The equation may then be written

$$(\sum' p_{hk} a_h a_k)^n = H_n \cdot P_n \cdot (a)^n.$$

The Pfaffian of the first order is p_{12} , or generally any constant. For the second order we have

$$\begin{aligned} & (p_{12} a_1 a_2 + p_{23} a_2 a_3 + p_{13} a_1 a_3 + p_{24} a_2 a_4 + p_{34} a_3 a_4)^2 \\ & = 2 (p_{12} p_{34} + p_{13} p_{24} + p_{14} p_{23}) a_1 a_2 a_3 a_4, \end{aligned}$$

so that

$$P_2 = p_{12} p_{34} + p_{13} p_{24} + p_{14} p_{23}.$$

It is sometimes convenient to denote this by P_{1234} , and so for higher orders.

series that
the P 's are
rational
functions of n
variables

This being so, we have for any n unit $a_1 a_2 \dots a_n$

$$(\sum P_{ik} a_k a_i)^2 = 2 \sum P_{iklm} a_k a_i a_l a_m,$$

$$(\sum P_{ik} a_k a_i)^3 = 6 \sum P_{iklmno} a_k a_i a_l a_m a_n a_o,$$

and generally

$$(\sum P_{ik} a_k a_i)^{n+1} = H_{n+1} \sum P_{iklm \dots} a_k a_i a_l a_m \dots \quad (2n+1),$$

We may also form products such as $(\sum P_{ik} a_k a_i)(\sum Q_{jk} a_k a_j) \dots$ these give rise to what may be called mixed Pfaffians, the relation of which to the preceding may be easily recognised.

II

Let there be two sets of n variables $a_1 a_2 \dots a_n$ and $\beta_1 \beta_2 \dots \beta_n$. If we form the product of the n linear functions

$$a_1 + \sum P_{1k} a_k a_n + a_2 + \sum P_{2k} a_k a_n + \dots + \sum P_{nk} a_k a_n,$$

it is known that $H(a) = (p) - H(a) = H(a)$ if we now form the linear function $\sum (q_k P_{ik} a_k + \beta_k)$, we shall find

$$(\sum P_{ik} a_k a_i)^n = \sum (q_k P_{ik} a_k + \beta_k)^n = \sum (q_k P_{ik} a_k + \beta_k)^{n-1} H(a) = H(a) H(\beta),$$

that is, the n th power of a linear binary function of two sets of n units is the product of the units into a numerical multiple of the determinant formed by the coefficients.

Now suppose that $P_{ik} = -P_{ji}$, $P_{kk} = 0$, then

$$\sum P_{ik} a_k a_i = \sum P_{ik} a_i a_k = \sum P_{ik} \beta_k a_i \quad (i = 1 \text{ under } \sum),$$

or we have a linear function of the binary determinants formed with the a and the β . Under these circumstances the determinant p is also symmetrically q , or the n th power of a binary function of the binary determinants of the a, β is $H(a) H(a) H(\beta)$ by a skew symmetrical determinant.

These binary determinants are symmetrical to each other and therefore $a_k a_i = a_i a_k = a_k a_i + \beta_k a_i - \beta_i a_k$, $\beta_k \beta_i = \beta_i \beta_k$. We may easily prove the following theorems, viz.

$$(\beta_i, \beta_j)^2 = -2 a_i a_j \beta_k \beta_k,$$

$$(\beta_i, \beta_j)(\beta_k, \beta_l) = -a_k a_l \beta_i \beta_j + a_j \beta_k \beta_l,$$

Hence with $n=3$ we have

$$\begin{aligned} & \frac{1}{6} (p_{11}(1, 2) + p_{11}(1, 3) + p_{11}(2, 3))^2 = 2 p_{12} a_1 a_2 + p_{13} a_1 a_3 + p_{23} a_2 a_3 \\ & \quad + (\beta_2 a_1 \beta_3 + \beta_3 a_1 \beta_2 + \beta_3 a_2 \beta_1). \end{aligned}$$

that with $n = 4$,

$$= \frac{1}{3} \{ \sum P_{hk} (h, k) \}^3 = (\sum P_{hk} a_h a_k) \cdot (\sum P_{hk} \beta_h \beta_k) + 2 \sum P_{hklm} (a_h a_k \beta_l \beta_m + \dots 6 \text{ terms}).$$

Here the coefficient of $2P_{hklm}$ is worthy of attention. Let us write

$$\left. \begin{array}{c} a \ a \ \beta \ \beta \\ h \ k \ l \ m \end{array} \right\} (h, k, l, m) \\ = a_h a_k \beta_l \beta_m + a_l a_m \beta_h \beta_k + a_h a_m \beta_k \beta_l + a_k a_l \beta_h \beta_m + a_h a_l \beta_m \beta_k + a_m a_k \beta_h \beta_l,$$

then

$$\begin{aligned} (h, l, l, m) &= 6 a_h a_k a_l a_m \beta_h \beta_l \beta_l \beta_m \\ (h, l, l, m)(h, l, l, n) &= a_h a_k a_l \beta_h \beta_k \beta_l (\alpha_m \beta_n + \alpha_n \beta_m) \quad \text{[say } \{m, n\}] \\ (h, l, l, m)(h, l, n, n) &= a_h a_k \beta_l \beta_l (l m n) \\ (h, l, l, m)(h, n, n, n) &= a_k \beta_l \beta_l \{l m n, m n\} \end{aligned}$$

where $\{l m n, m n\} = \sum (\alpha_m \alpha_n \beta_l \beta_l \beta_l \beta_n + \alpha_n \alpha_m \beta_l \beta_l \beta_l \beta_n)$, the $l m n$ being permuted among themselves and the $m n$ among themselves, according to the rule of signs.

In the same way we find

$$\begin{aligned} \frac{1}{3} \left(\sum P_{hk} \cdot \left\{ \begin{array}{c} a \ \beta \\ h \ k \end{array} \right\} \right)^3 &= (\sum P_{hk} a_h a_k)^2 \cdot (\sum P_{hk} \beta_h \beta_k)^2 - 2 (\sum P_{hk} a_h a_k) (\sum P_{hk} \beta_h \beta_k) \\ &\quad \cdot (\sum P_{lmkn} \cdot \left\{ \begin{array}{c} a \ \beta \ \beta \\ l \ m \ l' \ m' \end{array} \right\} + 4 \sum P_{lklmkn} \cdot \left\{ \begin{array}{c} a \ a \ a \ \beta \ \beta \ \beta \\ h \ k \ l \ m \ h' \ k' \ l' \ m' \end{array} \right\}) \\ \frac{1}{6} \left(\sum P_{hk} \cdot \left\{ \begin{array}{c} a \ \beta \\ h \ k \end{array} \right\} \right)^4 &= (\sum P_{hk} a_h a_k)^4 \cdot (\sum P_{hk} \beta_h \beta_k)^2 - 3 (\sum P_{hk} a_h a_k)^3 (\sum P_{hk} \beta_h \beta_k)^2 \\ &\quad + 3 \sum P_{lmkn} \cdot \left\{ \begin{array}{c} a \ a \ \beta \ \beta \\ l \ m \ l' \ m' \end{array} \right\} + 6 (\sum P_{hk} a_h a_k) (\sum P_{hk} \beta_h \beta_k) \cdot \sum P_{ab} (a^2 \beta^2) + 37 \sum P_{ab} (a^2 \beta^2). \end{aligned}$$

To write the general formula more conveniently let us abbreviate thus ;

$$p_{ab} = \sum P_{hk} a_h a_k, \quad p_{\alpha\beta} = \sum P_{hk} \cdot \left\{ \begin{array}{c} a \ \beta \\ h \ k \end{array} \right\}; \text{ then}$$

$$(p_{\alpha\beta})^2 = A_1 p_{\alpha\beta}^2 p_{\beta\alpha}^2 = A_1 p_{\alpha\beta}^{\alpha-1} p_{\beta\alpha}^{\beta-1}, \quad \sum P_{ab} (a^2 \beta^2) = A_2 p_{\alpha\beta}^{\alpha-2} p_{\beta\alpha}^{\beta-2} \sum P_{ab} (a^2 \beta^2) + \text{etc.}$$

where the A are numerical multipliers.

ANALYSIS OF CREMONA TRANSFORMATIONS

Quartic transformation [1]

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ t \end{pmatrix} \quad \text{Jacobian is three times } (a_1 a_2 a_3 a_4) \quad \text{Jacobian is three times } (a_1 a_2 a_3 a_4)$$

[11.1] Cubic transformation [11.1] = [11.1] mod 2

a_1, a_2, a_3, a_4

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ t \end{pmatrix} \quad \text{Jacobian is three times } (a_1 a_2 a_3 a_4) \quad \text{Jacobian is three times } (a_1 a_2 a_3 a_4)$$

[11.1] 1. Cubic transformation [11.1] = [11.1] mod 2

a_1, a_2, a_3, a_4

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ t \end{pmatrix} \quad \text{Jacobian is three times } (a_1 a_2 a_3 a_4) \quad \text{Jacobian is three times } (a_1 a_2 a_3 a_4)$$

[11.1] 2. Cubic transformation [11.1] = [11.1] mod 2

a_1, a_2, a_3, a_4

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ t \end{pmatrix} \quad \text{Jacobian is three times } (a_1 a_2 a_3 a_4) \quad \text{Jacobian is three times } (a_1 a_2 a_3 a_4)$$

Quintic Transformations.

[Cr.] 5. 1. [8001]=[3](001 $\bar{\chi}$ 002 $\bar{\chi}$ 003).

| | a_1 | a | b | c | d | e | f | g | h |
|----|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

 $\mu(a_1 a_2 \beta \gamma \delta \epsilon \zeta \eta \theta)$ gives $\overline{5\mu - 4a_1 - \Sigma a} \begin{cases} 4\mu - 3a_1 - \Sigma a \\ \mu - a_1 - a, \dots \end{cases}$
Jacobian is quartic $a_1^3 abcdefgh$ and the lines $a_1^4 a, \dots$ In De Jonquières' transformation of the $n+1$ th order $[2n, \dots 1] \mu(a_n, a\beta\gamma \dots)$ gives $\overline{(n+1)\mu - na_n - \Sigma a(n\mu - n - 1a_n - \Sigma a, \mu - a_n - a, \dots)},$ and the Jacobian is $n! a_1^{n-1} a\beta\gamma \dots$ and the $2n$ lines $a_1 a, a_1 \beta, \dots$

[8001](4, 0111111)=[3].

[Cr.] 5. 2. [3310]=[3](001 $\bar{\chi}$ 001).

| | a_2 | a_3 | b_3 | a_1 | b | c |
|----|-------|-------|-------|-------|-----|-----|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 1 | 1 | 1 | 1 | 1 | 1 |

 $\mu(a_1 a_2 \beta_3 \gamma_3 a\beta\gamma)$ gives $\overline{5\mu - 3a_3 - 2\Sigma a_2 - \Sigma a} \begin{cases} 3\mu - 2a_3 - \Sigma a_2 - \Sigma a \\ 2\mu - a_3 - \Sigma a_2 - a, \dots \\ \mu - a_3 - a_2, \dots \end{cases}$
Jacobian is cubic $a_1^2 a_2 b_3 c_2 abc$ 3 conics $a_1 a_2 b_3 c_2 a, \dots$ 3 lines $a_3 a_1, \dots$

[3310](3, 222, 011)=[3].

[Cr.] 5. 3. [0600]=[3](000 $\bar{\chi}$ 111).

| | a_1 | b_2 | c_2 | d_2 | e_2 | f_2 |
|----|-------|-------|-------|-------|-------|-------|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 1 | 1 | 1 | 1 | 1 | 1 |

 $\mu(a_1 \beta_2 \gamma_2 \delta_2 \epsilon_2 f_2)$ gives $\overline{5\mu - 2\Sigma a_2} (2\mu - \Sigma a_2 + a_2, \dots).$
Jacobian is six conics $b_2 c_2 d_2 e_2 f_2$, &c.

[0600](02222)=[0600]

[0600](002222)=[01040000]=Cr. 9. 4.

Sextic Transformations.

[Cr.] 0. 1. [10, 0001]=[3](001 $\bar{\chi}$ 002 $\bar{\chi}$ 003 $\bar{\chi}$ 004).

See 5. 1.

[Cr.] 0. 2. [14200]=[3](000 $\bar{\chi}$ 011).

| | a_3 | b_3 | a_2 | b_2 | c_2 | d_2 | a |
|----|-------|-------|-------|-------|-------|-------|-----|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

 $\mu(a_1 \beta_3 a_2 \beta_2 \gamma_2 \delta_2 a)$ gives $\overline{6\mu - 3\Sigma a_3 - 2\Sigma a_2 - a}$
 $\begin{cases} 3\mu - a_3 - 2\beta_3 - \Sigma a_2 - a, \dots \\ 2\mu - a_3 - \beta_3 - \Sigma a_2 + a_2, \dots \\ \mu - a_3 - \beta_3 \end{cases}$

Jacobian is 2 cubics

 $a_1^2 b_3 a_2 b_2 c_2 d_2 a; a_1 b_3^2 a_2 b_2 c_2 d_2 a$ 4 conics $a_1 b_3 b_2 c_2 d_2$, etc.1 line $a_3 b_3$.

[14200](33, 2222, 0)=[3].

ANALYSIS OF CREMONA'S TRANSFORMATIONS.

Octavia Transformations.

[Cr.] 8. 2. $[32\bar{3}0100] = [3](001\bar{x}001\bar{x}002).$

| | a_4 | a_3 | b_3 | c_3 | a_2 | b_2 | a | b | o |
|----------|-------|-------|-------|-------|-------|-------|-----|-----|-----|
| δ | 0 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 |
| δ | 2 | 1 | 1 | 1 | 1 | 1 | 1 | | |
| δ | 2 | 1 | | | | | | 1 | |
| δ | 2 | | 1 | 1 | 1 | 1 | | | 1 |
| δ | 1 | 1 | 1 | 1 | | 1 | | | |
| δ | 1 | 1 | 1 | 1 | 1 | | | | |
| δ | 1 | 1 | | | | | | | |
| δ | 1 | | | | | | | | |
| δ | 1 | | | 1 | | | | | |
| δ | 1 | | | | | | | | |

[Cr.] 8. 8. $[1822000]=[B](001\bar{x}001\bar{x}011)$.

| | a_1 | b_1 | a_2 | b_2 | a_3 | b_3 | a_4 | b_4 |
|----|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 13 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 14 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 15 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 17 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 18 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 19 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 20 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 21 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 22 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 23 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 24 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 25 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 26 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 27 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 28 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 29 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 30 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 31 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 32 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 33 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 34 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 35 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 36 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 37 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 38 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 39 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 40 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 41 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 42 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 43 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 44 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 45 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 46 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 47 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 48 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 49 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 50 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 51 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 52 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 53 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 54 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 55 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 56 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | | | | | | | | |

[Cr.] 8. 4. $[0070000]=[3](001\bar{1}001\bar{1}11\bar{1}222)=[7, 8](222).$

| | a_3 | b_3 | c_3 | d_3 | e_3 | f_3 | g_3 |
|---|-------|-------|-------|-------|-------|-------|-------|
| 0 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 1 | 2 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 2 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 2 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 2 | 1 |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 | 2 |

[Cr.] 8, 5. $[1000010] = [8](000\chi002\chi004) = 6.4\chi004).$

[Cr.] 6. $[6018000] = [8](001\bar{1}002\bar{1}000) = 4.1\bar{1}000$.

| | a_1 | a_2 | b_1 | c_1 | d_1 | e_1 | f_1 | a | b | c |
|----|-------|-------|-------|-------|-------|-------|-------|-----|-----|-----|
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 13 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 14 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 15 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 17 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 18 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 19 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 20 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 21 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 22 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 23 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 24 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 25 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 26 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 27 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 28 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 29 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 30 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 31 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 32 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 33 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 34 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 35 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 36 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 37 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 38 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 39 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 40 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 41 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 42 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 43 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 44 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 45 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 46 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 47 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

BITANGENT CIRCLES OF A CONIC.

1. If two conics have double contact, any point on the chord of contact has the same polar in respect of them.

For the polar of O , a point on AB , must pass through O where the tangents at A and B intersect, and also through P the harmonic of O in respect of A, B . [Fig. 112.]

Con. If one of the conics is a circle, and the point O at infinity, OP is a diameter of both conics which bisects chords at right angles to itself; i.e. an axis. Hence

If a circle have double contact with a conic, its centre must lie on one of the axes and the chord of contact must be perpendicular to that axis.

2. Through three points to draw a conic having double contact with a given conic.

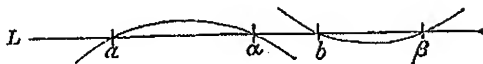
Let O, P, Q be the points; let OP meet the given conic in AB , OQ in CD . Let X, Y be the points which divide harmonically both AB and OP , X', Y' those which divide harmonically both CD and OQ . Then $XY, XX', YY', Y'X'$ are the four positions of the chord of contact. For suppose X to be a point on the chord, then by the last proposition it must have the same polar in respect of both conics; and therefore the axes conjugate in respect of AB and OP . [Fig. 113.]

We see thus that the conics, having double contact with the given one, which pass through OP divide themselves into two systems; viz., those whose chords of contact pass through X and those whose chords of contact pass through Y . In the particular case in which O, P are the points at infinity on a circle, X and Y become the points at infinity on the axis of the circle, and so

Circles having double contact with a given conic divide themselves into two systems, and according as the chord of contact is parallel to one axis or the other; and two circles of each system can be drawn through a given point.

3. The locus of a point whose powers in respect of two given conics are in a fixed ratio is a conic passing through their points of intersection.

Let L be an arbitrary line which cuts the two conics in aa and $b\beta$ respectively.



It is required to find a point x on L such that

$$xa \cdot x\alpha = m \cdot xb \cdot x\beta.$$

Let two points p, q be so related that

$$\frac{pa}{pb} = m \frac{qb}{qa},$$

then these two points uniquely determine each other, and therefore there are two positions at which they coincide, or two positions of the point x ; that is, two points of the locus are on the arbitrary line L .

Therefore the locus is of the second order, and it passes through the intersections of the conics because two quantities which are in a fixed ratio must vanish simultaneously.

It follows conversely that if three conics pass through the same four points, the ratio of the powers of any point on one in respect of the other two is constant. For let U, V, W be the conics, and consider a point m on W . The powers of m in respect of U, V are in a certain ratio. Now the locus of a point whose powers in respect of U, V are in that ratio is a conic through the intersections of U, V and of course also through m . Thus it has five points in common with W and must therefore coincide with it.

In particular, let W be made up of the two chords L, M of U and V . Then the power of any point on U in respect of V is in a constant ratio to the product of its distances from the lines L, M . [Fig. 114.]

If we suppose L, M to coincide, then U and V will have double contact; and we learn that in this case the power of any point on U in respect of V is in a constant ratio to the square of its distance from L . [Fig. 115.]

Now the power of any point in respect of a circle is the squared tangent from the point to the circle: hence we see that

The tangent from any point of a conic to a bitangent circle is in a constant ratio to the perpendicular on the chord of contact.

4. If we suppose the point on the curve to go to infinity in the direction CP , the ratio becomes ultimately $\frac{GM}{GP}$, and is therefore the same for all circles of this system. For circles of the other system it is $\frac{PM}{CP}$. These two ratios may be called ϵ and ϵ' . [Fig. 116.]

5. The sum or difference of the tangents drawn from any point of a conic to two bitangent circles of the same system is constant.

Let A, B be the circles, L, L' their chords of contact. Then

$$PM = \epsilon \cdot PA,$$

and

$$PN = e \cdot PB;$$

$$\therefore MN = e(PA \pm PH)$$

according as P is between L and L' or outside of them. [Fig. 117.]

(The radical axis of the circles is clearly midway between their chords of contact.

Since the four points of contact are symmetrically situated in respect of the axis, a circle will pass through them having its centre on the axis. This is called the circle of contact.)

6. The product of the tangents drawn from any point of a conic to two bitangent circles of the same system is equal to the square of the tangent drawn from the same point to their circle of contact.

For the conic, the circle of contact, and the pair of chords of contact, make three conics through the same four points: and therefore the power of any point on the conic in respect of the circle of contact bears a constant ratio to the product of its distances from the chords of contact, and therefore a constant ratio to the product of the tangents drawn to the two circles. By considering the points at infinity on the conic we see that this ratio must be one of equality.

OF POWER-COORDINATES IN GENERAL.

The equation of a circle contains *three* disposable constants; the equation of any circle may therefore be put into the form

$$aX + bY + cZ + dW = 0 \dots\dots\dots(1),$$

where $X, Y, Z, W=0$ are the equations of four circles. And any equation of the first order in $XYZW$ represents a circle.

The main object of the following paper is to discuss the general equation of the *second* order in $XYZW$. Now if we write

$$\left. \begin{aligned} X &\equiv \frac{x-a_1}{r_1} + \frac{y-\beta_1}{r_1} - r_1^2 \\ Y &\equiv \frac{x-a_2}{r_2} + \frac{y-\beta_2}{r_2} - r_2^2 \\ Z &\equiv \frac{x-a_3}{r_3} + \frac{y-\beta_3}{r_3} - r_3^2 \\ W &\equiv \frac{x-a_4}{r_4} + \frac{y-\beta_4}{r_4} - r_4^2 \end{aligned} \right\} \dots\dots\dots(2),$$

then the equation (1) has an obvious geometrical meaning; viz. X means the squared tangent from the point (xy) to the first circle, or, as it is called, the *power* of that point in respect of the circle; and we have learned that if the squared tangents drawn from a point to four fixed circles satisfy an equation of the first degree, the locus of that point is a circle. We thus come to regard the quantities $XYZW$ as a sort of coordinates; and a set of values of these coordinates may represent a point.

But a point is determined by *two* coordinates; and therefore in order that a set of values of these *four* quantities may represent a point, they must satisfy two equations identically. I shall now prove that one of these equations is non-homogeneous and of the first degree, while the other is homogeneous and of the second degree.

In fact, if we write A for the determinant

$$\begin{vmatrix} 1 & \beta_2 & \gamma_2 \\ 1 & \beta_3 & \gamma_3 \\ 1 & \beta_4 & \gamma_4 \end{vmatrix},$$

and B, C, D for the three similar determinants that can be formed from the other triads of the four circles, we shall have

$$AX + BY + CZ + DW = (1234) \dots\dots\dots(3)$$

where (1234) denotes the determinant

$$\begin{vmatrix} 1 & \alpha_1 & \beta_1 & \alpha_1^2 + \beta_1^2 - r_1^2 \\ 1 & \alpha_2 & \beta_2 & \alpha_2^2 + \beta_2^2 - r_2^2 \\ 1 & \alpha_3 & \beta_3 & \alpha_3^2 + \beta_3^2 - r_3^2 \\ 1 & \alpha_4 & \beta_4 & \alpha_4^2 + \beta_4^2 - r_4^2 \end{vmatrix}.$$

This is the non-homogeneous equation of the first degree.

If we choose to concern ourselves only with the ratios of the quantities $X : Y : Z : W$, then the equation

$$AX + BY + CZ + DW = 0$$

will represent the line (or, as we may say, circle) at infinity. I shall call the form $AX + BY + CZ + DW$ the *first absolute*, and denote it by the symbol ω .

Again, we have in the equations (2) $X : Y : Z : W$ exhibited as linear functions of $x^2 + y^2, x, y, 1$. It is therefore possible, by solving the equations, to exhibit $x^2 + y^2, x, y, 1$ as proportional to linear functions of X, Y, Z, W . That is to say, we must have

$$\frac{x^2 + y^2}{P} = \frac{x}{Q} = \frac{y}{R} = \frac{1}{S},$$

where P, Q, R, S are four linear functions of X, Y, Z, W . But

$$(x^2 + y^2) \cdot 1 = (x)^2 + (y)^2,$$

therefore

$$PS = Q^2 + R^2,$$

a homogeneous equation of the second degree in X, Y, Z, W which must be identically satisfied if X, Y, Z, W are the coordinates of a point. The form $PS = Q^2 + R^2$ I shall call the *second absolute*, and denote by the symbol ϕ .

The question now naturally arises; supposing that the coordinates $XYZW$ do not satisfy the equation $\phi = 0$, and therefore do not represent a *point*, what do they represent? This question I proceed to answer.

Let k be the distance between the centres of two circles, r and r_1 their radii. The quantity $k^2 - r^2 - r_1^2$ I call the *power of one circle in respect of the other*. If one of them becomes a point, so that $r_1^2 = 0$, the power is $k^2 - r^2$, which is the squared tangent from that point to the other circle; so that this definition agrees with the previous one of the word *power*. The power is also equal to

$2r_1 \cos \theta$, where θ is the angle of intersection of the circles; for, in the figure, $\angle C'B$ is this angle (nearly the angle through which (1) must be turned about the point C' , in order that its concavity may coincide with the concavity of (2)), and we have at once

$$k^2 - r^2 - r_1^2 = 2r_1 \cos \theta,$$

which establishes the equivalence in question. [Fig. 119.]

Let us now examine the effect of supposing X, Y, Z, W to denote the powers of a circle in respect of the four fixed circles.

First, the equation (3) is still satisfied; for the coordinates $XYZW$ of the circle are got from the coordinates $X_1 Y_1 Z_1 W_1$ of its centre by subtracting r^2 from each of them; we have then only to prove that

$$(A + B + C + D)r^2 = 0,$$

which is obvious, for $A+B+C+D$ is the coefficient of (x^2+y^2) on the left-hand of the equation, and the right-hand is constant.

Next, what is meant by an equation of the first order? Let us write, a little more generally,

$$a_1X = a_1(x^2 + y^2) + 2b_1x + 2c_1y + d_1,$$

and let the equation of our new circle be

$$0 = a(x^2 + y^2) + 2bx + 2cy + d,$$

then we shall have

$$\delta^2 = \left(\frac{b}{a} - \frac{b_1}{a_1} \right)^2 + \left(\frac{c}{a} - \frac{c_1}{a_1} \right)^2,$$

$$r^2 = \frac{b^2 + c^2 - ad}{a^3},$$

$$r_1^2 = \frac{b_1^2 + c_1^2 - a_1d_1}{a_1^3},$$

and therefore

$$\delta^2 - r^2 - r_1^2 = \frac{ad_1 + a_1d - 2bb_1 - 2cc_1}{aa_1} \dots\dots\dots (4).$$

Now if the power of two circles vanishes, they cut at right angles; for the power is $-2rr_1 \cos \theta$, and $\cos \theta = 0$ means that θ is a right angle. We see thus that the condition for two circles to cut at right angles is

$$ad_1 + a_1d - 2bb_1 - 2cc_1 = 0,$$

which is linear in the coefficients of each. And further, any linear relation among the coefficients of a circle expresses that it cuts some fixed circle at right angles. For the relation

$$la + mb + nc + sd = 0$$

expresses that the circle

$$a(x^2 + y^2) + 2bx + 2cy + d = 0$$

cuts at right angles the circle

$$s(x^2 + y^2) - mx - ny + l = 0.$$

It follows that if the coordinates of a circle satisfy an equation of the first degree, the circle cuts at right angles a certain fixed circle. For the equation

$$lX + mY + nZ + sW = 0$$

implies a linear relation among $a b c d$, since $X Y Z W$ are proportional to linear functions of these quantities by the equation (3). And this circle is precisely the one that we before represented by this equation, viz. when we considered therein X, Y, Z, W as functions of x, y determined by equations (2). For the circle is cut orthogonally by all circles whose coordinates satisfy the equation; but if these circles become points, they must be points on the circle which they cut orthogonally.

These results I sum up as follows:

The coordinates of a circle are four quantities proportional to its powers in respect of four fixed circles not having the same radical centre.

The coordinates of all circles which cut at right angles a given circle C satisfy a homogeneous equation of the first degree, which is called the equation of the circle C .

The coordinates of all points satisfy a homogeneous equation of the second order.

The Orthogonal System.

The following theorem includes about one-third of the metrical properties of points, lines, and circles.

If there are five circles, 1 2 3 4 5, and five other circles, 1' 2' 3' 4' 5', and if we form the determinant whose constituents are the powers of the first set of circles in respect of the second set, arranged as in a multiplication table so as to be represented in the umbral notation by the symbol $\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1' & 2' & 3' & 4' & 5' \end{smallmatrix}$, this determinant vanishes identically.

For if we multiply together the two matrices

$$\begin{array}{ll} 1, 2b_1, 2c_1, d_1 & d_1', -b_1', -c_1', 1 \\ 1, 2b_2, 2c_2, d_2 & d_2', -b_2', -c_2', 1 \\ 1, 2b_3, 2c_3, d_3 & d_3', -b_3', -c_3', 1 \\ 1, 2b_4, 2c_4, d_4 & d_4', -b_4', -c_4', 1 \\ 1, 2b_5, 2c_5, d_5 & d_5', -b_5', -c_5', 1 \end{array}$$

we shall form a determinant which, by the theory of [matrices], vanishes identically; while its constituents are of the type

$$d' + d - 2bb' - 2cc',$$

which, as we have already seen, means the power of the circles

$$x^2 + y^2 + 2bx + 2cy + d = 0,$$

$$x^2 + y^2 + 2b'x + 2c'y + d' = 0.$$

Reserving for the present the further discussion of this theorem and of a similar one, I proceed to consider a particular case. Let us take for the first four circles of each set the fundamental circles $XYZW$, and let the coordinates of the circles 5 and 5' be represented by x, y, z, w and x', y', z', w' . Then the identity

$$\begin{array}{l} XYZW5 \\ XYZW5' \end{array} = 0$$

becomes

$$\left| \begin{array}{cccccc} -2r_1^2, & (XY), & (XZ), & (XW), & x & \\ (XY), & -2r_2^2, & (YZ), & (YW), & y & \\ (XZ), & (YZ), & -2r_3^2, & (ZW), & z & \\ (XW), & (YW), & (ZW), & -2r_4^2, & w & \\ x', & y', & z', & w', & (5'5') & \end{array} \right| = 0.$$

By this equation the power of two circles is expressed in terms of their coordinates. Let the expression be denoted thus:

$$(5'5') = (* \text{X} x, y, z, w \text{X} x', y', z', w').$$

then clearly the equation of the circle whose coordinates are $x y z w$ is.

$$0 = (* \text{X} x, y, z, w \text{X} X, Y, Z, W).$$

And further, if we suppose the circles $5, 5'$ to coincide in a circle of radius r we have

$$-2r^2 = (* \text{X} x, y, z, w)^2,$$

which gives the radius of a circle in terms of its coordinates. It follows that the Second Absolute is

$$\Phi = (* \text{X} X Y Z W)^2.$$

By dividing the first four rows and columns of the determinant by r_1, r_2, r_3, r_4 respectively, we may reduce this to the simpler form

$$\begin{vmatrix} 1 & \cos XY & \cos XZ & \cos XW & \frac{X}{r_1} \\ \cos XY & 1 & \cos YZ & \cos YW & \frac{Y}{r_2} \\ \cos XZ & \cos YZ & 1 & \cos ZW & \frac{Z}{r_3} \\ \cos XW & \cos YW & \cos ZW & 1 & \frac{W}{r_4} \\ \frac{X}{r_1} & \frac{Y}{r_2} & \frac{Z}{r_3} & \frac{W}{r_4} & 0 \end{vmatrix}$$

which it is worth stating is an identical relation connecting the equations of any four circles.

It is obvious that all these formulæ will be immensely simplified if we take for our fundamental circles four circles cutting each other orthogonally. In this case the quantities $(XY), (XZ), \dots$ &c. are all zero; the Second Absolute becomes

$$\Phi \equiv \frac{X^2}{r_1^2} + \frac{Y^2}{r_2^2} + \frac{Z^2}{r_3^2} + \frac{W^2}{r_4^2},$$

the radius of $(x y z w)$ is

$$\frac{1}{2} \left(\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} + \frac{z^2}{r_3^2} + \frac{w^2}{r_4^2} \right);$$

the power of $(x y z w)$ in respect of $(x' y' z' w')$ is

$$-\frac{1}{2} \left(\frac{xx'}{r_1^2} + \frac{yy'}{r_2^2} + \frac{zz'}{r_3^2} + \frac{ww'}{r_4^2} \right),$$

and consequently the equation of $(x y z w)$ is

$$\frac{x}{r_1^2} X + \frac{y}{r_2^2} Y + \frac{z}{r_3^2} Z + \frac{w}{r_4^2} W = 0.$$

I shall simplify these expressions still further by taking as coordinates of a circle not the powers $X Y Z W$ themselves, but the powers each divided by the radius of the corresponding fundamental circle, viz. the quantities

$$\frac{X}{r_1}, \frac{Y}{r_2}, \frac{Z}{r_3}, \frac{W}{r_4}.$$

Calling these quantities X_1, Y_1, Z_1, W_1 , we find for the absolute merely

$$X_1^2 + Y_1^2 + Z_1^2 + W_1^2,$$

and the equation of (Umn) is

$$lX_1 + mY_1 + nZ_1 + sW_1 = 0.$$

This step, it will be seen, is precisely analogous to the step from *areal* to *collinear* coordinates in the geometry of straight lines. It must be remembered that the only speciality which has been introduced is that our four fundamental circles form an orthogonal system. One of them is therefore imaginary.

The radius of the circle

$$lX_1 + mY_1 + nZ_1 + sW_1 = 0$$

vanishes when

$$P + m^2 + n^2 + s^2 = 0.$$

It then appears that a point (or circle of no radius) is a circle which satisfies the analytical condition of touching the Second Absolute. This remark will be found useful in the sequel.

Equation of an Anallagmatic.

The general equation of the second order in power-coordinates represents an *anallagmatic* curve of the fourth order, i.e. a quartic curve having a double point at each of the circular points at infinity. For if in the expression

$$(\infty) X_1, Y_1, Z_1, W_1^2$$

we substitute for X_1, Y_1, Z_1, W_1 their values in terms of $x^2, y^2, x, y, 1$, it becomes

$$a(x^2 + y^2)^2 + L(x^2 + y^2) + P,$$

where L is of the first degree in x, y , and P of the second. Now this equated to zero is the general equation of an anallagmatic quartic; and it contains one constant less than the former, which is thus (if we bear in mind the existence of the Second Absolute) just sufficiently general to represent all such anallagmatic curves.

I say, "if we bear in mind the existence of the Second Absolute" for this reason. The equation $\Phi = 0$ is satisfied identically by the coordinates of every point. Now if we take $\Phi = 0$ for an equation of the second order in X_1, Y_1, Z_1, W_1 , the equation

$$\Theta + \lambda\Phi = 0$$

must represent exactly the same curve as $\Phi = 0$; for the equation is satisfied by the coordinates of all points which satisfy this latter equation. Out of the nine apparently arbitrary constants, therefore, in the expression Θ , one is at our disposal independently of the determination of the curve; and the real number of constants is then four only.

In fact, if we regard X, Y, Z, W as the coordinates of a point in space, then the Second Absolute represents a quadric surface; and every anallagmatic curve is represented in the first instance by another quadric surface, but in the second instance (viz., when we remember the remarks just made) by the curve of inter-

section of this surface with the Second Absolute. Now a quadri-quadric curve depends upon eight constants only, and so fitly represents a general anallagmatic curve.

Bitangent Circles.

Consider therefore any quadric Θ and the second absolute Φ . We know that by a linear transformation it is possible in one and only one way to reduce these simultaneously to the canonical form. That is to say, there is precisely one set of four circles $X Y Z W$ such that when we express Θ and Φ in terms of them they take the forms

$$\begin{aligned}\Phi &\equiv X^2 + Y^2 + Z^2 + W^2, \\ \Theta &\equiv \frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} + \frac{W^2}{d^2}.\end{aligned}$$

The first of these indicates that the circles $X Y Z W$ are mutually orthogonal. They may be called the *principal circles* of the curve. (Montard.)

The quantities a^2, b^2, c^2, d^2 , are not necessarily positive; they are written in this form for the sake of subsequent convenience.

It is now possible, with the aid of the second absolute, to express the equation of the curve in terms of any three of the four principal circles. In fact, eliminating W between the equations $\Theta = 0, \Phi = 0$, we have

$$d^2\Theta - \Phi \equiv \frac{X^2}{a^2}(d^2 - a^2) + \frac{Y^2}{b^2}(d^2 - b^2) + \frac{Z^2}{c^2}(d^2 - c^2) = 0,$$

an equation which, if X, Y, Z were trilinear coordinates, would represent a conic referred to a self-conjugate triangle.

A circle which satisfies the analytical condition of touching this curve is a bitangent circle of the anallagmatic. For if

$$\begin{aligned}x &\equiv lX + mY + nZ, \\ y &\equiv l'X + m'Y + n'Z, \\ u &\equiv l_1X + m_1Y + n_1Z,\end{aligned}$$

where x and y are two such tangents, then we know the equation of the curve can be reduced to the form

$$xy = ku^2,$$

where it is clear that, for instance, the circle x touches the curve at the two points where it meets u . All these circles are orthotomic of W_1 for since XYZ are orthotomic of W , any circle $lX + mY + nZ$ is so. Hence we may enunciate the following propositions:—

The bitangent circles of an anallagmatic curve arrange themselves into four systems, all the circles of each system cutting orthogonally one of the principal circles.

The four points of contact of two circles x, y , of the same system lie on a circle u (circle of contact), and the product of the tangents drawn from any point of the curve to the circles x, y is in a constant ratio to the square of the tangent drawn to the circle u .

Two bitangent circles of a given system can be drawn through an arbitrary point.

If two bitangent circles be drawn through a point on u , their circle of contact is concyl with u, g .

Further, let x, y, z be any three bitangent circles of the system W , where

$$x = F(X) + G(Y) + H(Z),$$

then we know from the theory of conics that the equation of the anallagmatic can be written in the form

$$\alpha X^2 + \beta Y^2 + \gamma Z^2 = 0,$$

where α, β, γ are constants. Hence

The tangents drawn from any point of the curve to three bitangent circles of the same system satisfy a linear relation.

A focus of the curve is a bitangent circle of evanescent radius. Since the two quadrics

$$\frac{X^2}{a^2} (d^2 - a^2) + \frac{Y^2}{b^2} (d^2 - b^2) + \frac{Z^2}{c^2} (d^2 - c^2) = 0,$$

$$X^2 + Y^2 + Z^2 = 0,$$

have just four common tangents, and since a circle of evanescent radius is one satisfying the analytical condition of touching the absolute, it follows that:

There are four foci on each principal circle.

The distances of any point on the curve from three foci of the same system are connected by a linear relation.

Any two foci of the same system have a contact circle, such that the product of the distances of any point of the curve from the two foci is proportional to the power of the point in respect of this circle.

Conformal Curves.

The equation

$$\frac{X^2}{a^2} + \theta + \frac{Y^2}{b^2} + \theta^3 + \frac{Z^2}{c^2} + \theta^5 + \frac{W^2}{d^2} + \theta^7 = 0$$

(in which θ is a variable parameter) represents a series of conformal anallagmatics; for it is readily observed that the equations for determining the foci involve only the differences of the quantities a^2, b^2, c^2, d^2 . In fact, the foci are common bitangents of the curve and of the absolute; now the equation above written obviously represents a curve touching all the bitangents common to

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} + \frac{W^2}{d^2} = 0,$$

and

$$X^2 + Y^2 + Z^2 + W^2 = 0.$$

Two confocals cut at right angles. For let $X_1 Y_1 Z_1 W_1$ be a point of intersection. The bitangent circles of the W -system at that point are

$$\frac{XX_1}{a^2} (d^2 - a^2) + \frac{YY_1}{b^2} (d^2 - b^2) + \frac{ZZ_1}{c^2} (d^2 - c^2) = 0,$$

$$\frac{XX_1}{a^2 + \theta} (d^2 - a^2) + \frac{YY_1}{b^2 + \theta} (d^2 - b^2) + \frac{ZZ_1}{c^2 + \theta} (d^2 - c^2) = 0,$$

which cut at right angles if

$$\frac{X_1^2}{a^2 (a^2 + \theta)} (d^2 - a^2)^2 + \frac{Y_1^2}{b^2 (b^2 + \theta)} (d^2 - b^2)^2 + \frac{Z_1^2}{c^2 (c^2 + \theta)} (d^2 - c^2)^2 = 0.$$

But this equation follows readily from the three

$$\frac{X_1^2}{a^2} + \frac{Y_1^2}{b^2} + \frac{Z_1^2}{c^2} + \frac{W_1^2}{d^2} = 0, \dots\dots\dots (\text{A}),$$

$$\frac{X_1^2}{a^2 + \theta} + \frac{Y_1^2}{b^2 + \theta} + \frac{Z_1^2}{c^2 + \theta} + \frac{W_1^2}{d^2 + \theta} = 0, \dots\dots\dots (\text{B}),$$

$$X_1^2 + Y_1^2 + Z_1^2 + W_1^2 = 0, \dots\dots\dots (\text{C}).$$

It is worth while to write down the ratios of $X_1 Y_1 Z_1 W_1$ which are deducible from these. They are

$$kX_1 = \frac{aa'}{\sqrt{(a^2 - b^2)(a^2 - c^2)(a^2 - d^2)}},$$

$$kY_1 = \frac{bb'}{\sqrt{(b^2 - c^2)(b^2 - d^2)(b^2 - a^2)}},$$

and so on, where a'^2 has been written for shortness instead of $a^2 + \theta$.

The conditions that a circle

$$lX + mY + nZ + sW,$$

shall be a bitangent of the system W to the curve (A) are $s = 0$ and

$$\frac{l^2 a^2}{d^2 - a^2} + \frac{m^2 b^2}{d^2 - b^2} + \frac{n^2 c^2}{d^2 - c^2} = 0.$$

Call this $U = 0$, and let

$$V \equiv \frac{l^2}{d^2 - a^2} + \frac{m^2}{d^2 - b^2} + \frac{n^2}{d^2 - c^2},$$

then the equation $U = 0$ can be written in either of the forms

$$\frac{U - Va^2}{a^2} = \frac{U - Vb^2}{b^2} = \frac{U - Vc^2}{c^2}.$$

I shall now prove that the quantities $U - Va^2$, $U - Vb^2$, $U - Vc^2$ are proportional to the products of the powers of (*lms*) in respect of pairs of foci of the systems $X Y Z$ respectively.

In fact, if we write

$$P^2 \equiv (a^2 - b^2)(c^2 - d^2),$$

$$Q^2 \equiv (a^2 - c^2)(d^2 - b^2),$$

$$R^2 \equiv (a^2 - d^2)(b^2 - c^2),$$

then it is easily seen that the coordinates of the sixteen foci are

four of the X -system $0, \pm P, \pm Q, \pm R,$

four of the Y -system $\pm P, 0, \pm R, \pm Q,$

four of the Z -system $\pm Q, \pm R, 0, \pm P,$

four of the W -system $\pm R, \pm Q, \pm P, 0.$

Now, for instance,

$$U = \frac{m^2 (b^2 - c^2)}{d^2 - b^2} + \frac{n^2 (c^2 - a^2)}{d^2 - c^2},$$

$$= \frac{(mP + nQ)(mP - nQ)}{(d^2 - b^2)(d^2 - c^2)},$$

$$= \frac{(-mP - nQ)(-mP + nQ)}{(d^2 - b^2)(d^2 - c^2)},$$

which verifies the assertion.

From these equations we deduce the following propositions:—

Any bitangent circle of the W -system is so related to the four foci of the X -system that the product of its powers in respect of a certain two of them is equal to the product of its powers in respect of the other two.

These products in respect of the X , Y , and Z -systems have constant ratios for all bitangent circles of the W -system.

[The following Notes, "Theory of Powers," preceded the above paper in the Note-book, but appear to have no connection with it. They are very fragmentary and, in places, apparently inaccurate, but it has been thought desirable to print them almost as they were left by the writer. It is not easy to see how the equations in I and 4 are got, nor how the other equation in I. contains a linear relation between the powers of a point with respect to a , &c.]

Let a, b, c be three points in a right line: the rectangle or product $ab \cdot ac$ is called the power of the point a in respect of the point-pair bc .

I. Let $a, a; b, \beta; c, \gamma$ be pairs of points dividing harmonically the length XY , then $(m, n, p, o$ being the middle points respectively)

$$aa + XY = [2]m \cdot o,$$

$$b\beta + XY = [2]n \cdot o,$$

$$c\gamma + XY = [2]p \cdot o,$$

multiply by $n - p, p - m, m - n$ and add; then

$$n - p, aa + p - m, b\beta + m - n, c\gamma + o,$$

Now the origin is arbitrary, and the distances $n - p, p - m, m - n$ are constant: hence

The powers of any point whatever, in respect of three pairs in involution, satisfy a certain linear relation.

2. Let a, b correspond uniquely to α, β respectively, and let X, Y be the double points of the correspondence.

Then because any two corresponding points make the same anharmonic ratio with the double points, $(aaXY) = (b\beta XY)$, or

$$\frac{aa \cdot XY}{aX \cdot aY} = \frac{b\beta \cdot XY}{bX \cdot \beta Y}.$$

Also because the anharmonic ratio of four points = anharmonic ratio of corresponding points,

$$(abXY) = (\alpha\beta XY),$$

or

$$\frac{ab \cdot XY}{aY \cdot bX} = \frac{\alpha\beta \cdot XY}{\alpha Y \cdot \beta X}.$$

Multiply these equations together; then

$$\frac{aa \cdot ab}{aX \cdot bY} = \frac{b\beta \cdot a\beta}{\beta X \cdot \beta Y}, \text{ or } \frac{aa \cdot ab}{\beta a \cdot \beta b} = \frac{aX \cdot \alpha Y}{\beta X \cdot \beta Y};$$

that is, if two points be taken one from each system of an unique correspondence, the ratio of their powers in respect of the points corresponding to them is equal to the ratio of the powers in respect of the double points.

If A, B, C are three lines through a point, $\sin AB \cdot \sin AC$ is called the power of A in respect of the pair BC .

It will follow in a similar manner that the corresponding proposition is true about an unique correspondence of lines.

3. Let now pOp', qOq' be two chords of a conic, meeting in O . Let the lines $qq', qp, p'q', p'p$ be called a, α, b, β respectively; it is clear that the directions a, b correspond uniquely to α, β and that the double lines are the asymptotes. Now

$$\frac{Oq}{Op} = \frac{\sin Opq}{\sin Oqp} = \frac{\sin b\beta}{\sin ab},$$

and

$$\frac{Oq'}{Op'} = \frac{\sin Op'q'}{\sin Oq'p'} = \frac{\sin \alpha\beta}{\sin aa};$$

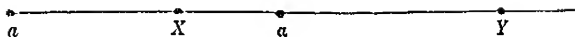
$$\therefore \frac{Oq \cdot Oq'}{Op \cdot Op'} = \frac{\sin \beta b \cdot \sin \beta a}{\sin ab \cdot \sin aa} = \frac{\sin \beta X \cdot \sin \beta Y}{\sin \alpha X \cdot \sin \alpha Y},$$

by the previous proposition, [Fig. 119.]

Whence the ratio of the product of segments of chords drawn through any point is independent of the position of the point, and depends only on the direction of the chords relative to the asymptotes. Also the product of segments of any chord drawn through a point, divided by square of parallel semidiameter, is independent of the direction of the chord: this I call the *power* of the point in respect of the conic. Further

The square of any semidiameter is inversely proportional to the products of the sines of the angles it makes with the asymptotes.

4. Properties of Harmonic Section,



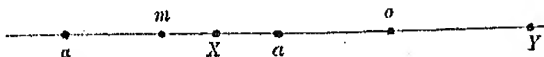
If a, a are harmonic of X, Y , we have by definition

$$\frac{aX}{aY} = -\frac{aX}{aY}, \text{ or } aX \cdot aY + aX \cdot aY = 0;$$

$$\begin{aligned} \therefore aX \cdot aY + aX \cdot aY &= (aa + aX) aY + (aa + aX) aY \\ &= aa (aY - aY) = aa^2. \end{aligned}$$

Therefore if two points are harmonic in respect of X, Y the sum of their powers in respect of X, Y is equal to the square of their distance.

Hence the sum of squares of reciprocals of two conjugate diameters of a conic varies as squared sine of angle between them.



This proposition may also be got from the one concerning the middle points. Thus

$$aa + XY = 2mo,$$

whatever be the origin. Take a and a successively for origin, then

$$aX \cdot aY = 2am \cdot ao = aa \cdot ao,$$

$$aX \cdot aY = 2am \cdot ao = -aa \cdot ao;$$

$$\therefore aX \cdot aY + aX \cdot aY = aa (ao - ao) = aa^2,$$

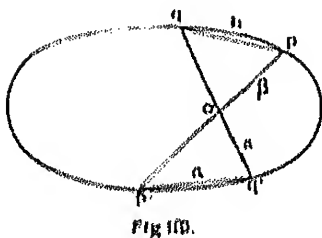
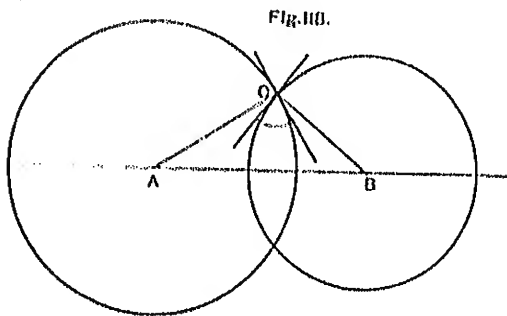
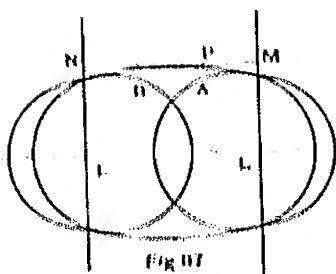
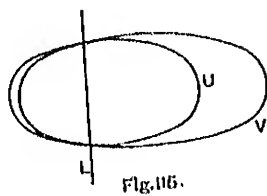
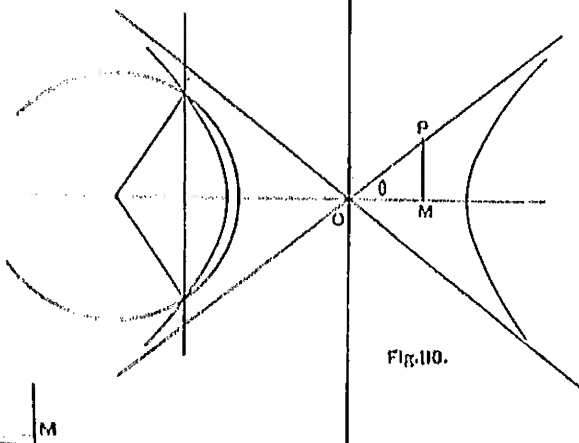
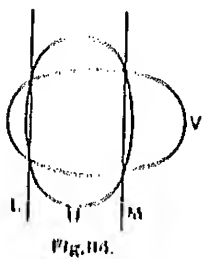
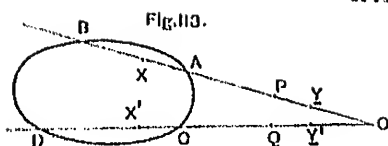
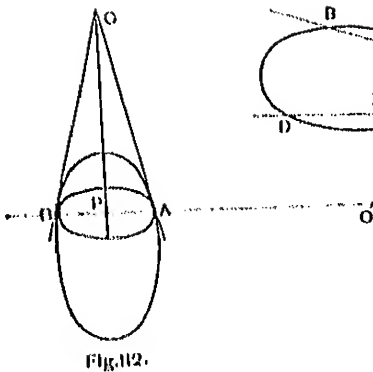
$$\begin{aligned} \text{also } aX \cdot aY - aX \cdot aY &= -aa^2 \cdot ao \cdot ao = -aa^2 \cdot oX^2 \\ &= -\frac{1}{4} \cdot aa^2 \cdot XY^2: \end{aligned}$$

that is, product of powers of a, a in respect of XY = squared distance of a, a \times squared distance of X, Y . Hence product of conjugate diameters into sine of angle between them is constant.

Farther, by division,

$$\frac{1}{aX \cdot aY} + \frac{1}{aX \cdot aY} = -\frac{4}{XY^2},$$

therefore sum of squares of conjugate diameters is constant.



REVIEWS.

I.

A Budget of Paradoxes. By AUGUSTUS DE MORGAN (Reprinted, with the Author's additions, from the *Athenæum*. London: Longmans, 1872^o).

It is an opinion current among librarians, that there is no such thing as trash; that the most foolish and unimportant flyleaf treating of nothing at all should be preserved and bound up with other such flyleaves, not in view of any possible future hypothesis to whom it may be resold among quacks, but because it is right that this thing should be done. No doubt a very great good comes of this absolute universalism in the conscience of all kinds of collectors. But still, for the purposes of the outer world, it remains true that there are books and books. It is obvious of some kind of literary or scientific work that if A. B. had not done it, C. D. would have taken his place; and that at no loss to the world, even though C. D. were a person of mean capacities. The fishbaker who was to lay a certain course of bricks may fall out a ladder and yet the house be no worse in the end; while the alchemist never who sowed a gangue may have something which represents merely his dirt's value but himself (invaluable) so long as some shall last, and therefore some thing which no other man could exactly produce. The book belongs to us essentially as gangue. It is by no means the most valuable kind of the *opus* individual, not merely in its own singularity as a book, but as presenting with a notable degree of clearness and exactness the personality of some human being, who is never among men, but always a man among other men.

The paradoxes begin treated of are those set forth by people ignorant of mathematics, who think themselves qualified to show each us are not ignorant where they have gone wrong. It might well have been conceived that a large book on such a subject would have been the work of the dull, that it would appeal only to mathematical novices, and even to them only for so long a time as the follies exposed in it were of a sort latent. No anticipation could be more thoroughly wrong. The fault of the book is in the direction of a too incessant playfulness, it is a brilliant protegee, which is only to be borne because it is so clearly an outgrowth of the beautiful. For while the pretensions here slaughtered are for the most part indeed pretensions, whose only use is for an example and a warning, while these pretensions, some not to be excepted are small jokes, and such as to like chiefly as you fill the measure, while even the character which so clearly shines through these pages, great and lovable

as it is, is yet rather singular than pre-eminent, a study for comparative psychology rather than an ideal for the world to come; while all this is true, the book is an endeavour and a stretching forth towards right thinking and a protest against wrong thinking which is of infinite solemnity and weight to us of this present time. For we have no right to conclude that these paradoxers upon whom the *Budget* has fallen have been sinners above all that prate; we ought rather to learn that except we mend our ways we shall all likewise perish.

The word *paradox* is unfortunate; it includes under one name a rare thing and a common thing, and it brings upon the rare thing which is good some of the discredit that belongs to the common thing which is bad. "A paradox is something which is apart from general opinion, either in subject-matter, method, or conclusion." The "general opinion" must be that of people who have an opinion; not of all people indiscriminately, including those who have never considered the subject. The common form of paradox consists in ignorance of the subject-matter, powerlessness in the method, or incapacity to understand the conclusion. The rare form of paradox is an addition to the reasonable part of general opinion, which happens to contradict some of the unreasonable part. The older use of the word was strictly impartial, and it might be applied without any want of respect; De Morgan says the change came in the seventeenth century. It is certain that at present the epithet is a disparaging one; the overplus of wrong thoughts included under it has slowly sapped the moral constitution of the word, and it now sometimes stands in the way of a right appreciation of the nobler form of paradox.

For as it is hardly possible to lay too great a stress on the weight and worthiness of thought which diverges from the general opinion on account of its greater strength, which by its continual work in the world has in fact built up the present mind of man; so it is before all things necessary here to distinguish carefully from it that other divergence which comes of weakness and goes to destruction. It is true in all departments of human action that reform is the most precious and sacred prerogative of a citizen; but in order to have that prerogative one must be a citizen, not an alien; and one must act like a citizen in a legitimate and constitutional way. A man who should find an error in the value of π —even in the six hundredth place—would have all honour paid him as a true reformer by the brotherhood; but to this two things are necessary: he must not be ignorant of trigonometry, and he must work out the calculation. The belief of the weak paradoxer, on the contrary, is that things can be done by a flash; that a discovery is to start from his ignorant and untried mind like Pallas from the brain of Zeus. We know, of course, that the great discoveries—the true and noble paradoxes—have always come from men who by long apprenticeship have so far mastered the tools forged by their fathers that they were not tied down to one particular way of using them; we know that Jove's head cannot crack with Minerva unless he have previously swallowed Metis. The time taken by distant discoveries—gravitation for instance, is foreshortened by perspective; but we have good cases immediately before our eyes. In Maxwell's theory of Electricity we have as instructive an example of the paradox of

right thinking as might well be; a conclusive victory over rival doctrines won by twenty years' patient proving (and improving) of the weapons where-with previous battles had been gained; a testimony to all time that genius is a capacity for taking an infinite amount of the right sort of trouble. But your paradoxer of the *Budget* will master by a *coup-d'état* the republic of science, which allows no masters, but proved comrades only; he will climb by the back stairs into the house of knowledge, that has no back stairs. If there be any reward in the penal incurable blindness that follows such sacrilege, verily he has his reward.

And here is another important difference between the two kinds of heretics. The strong heretic is so because his ideas are living and plastic, and have an internal motion whereby they adapt themselves continually to new work; so that no man is so perfectly open to conviction as he is. But the weak heretic is so from the very narrowness of his range, which cannot grasp even established demonstration; he is hermetically sealed against all possible argumentative germs that might bring into his mind the lower forms of life.

In drawing this sharp distinction between two habits of mind, however, we must not forget what the *Budget* is specially calculated to impress upon us in a terrible and alarming manner; the exceedingly gradual transition from one to the other, and the possible coexistence of both in the same person in regard to different subjects. De Morgan has some very good remarks on the value of a study of logic in helping us to extend the habits of right thinking which we have got by practice in one subject over the whole range of our knowledge. A good specialist who is also a good logician can hardly be betrayed into gross paradox out of his proper range; for his special knowledge will make him cautious about facts, and his logic about conclusions. No man could have greater advantages in this respect than the author of the *Budget*, who had himself made important additions to logic, and was an excellent mathematician. And yet this is the solemn warning of the book—he has in one case fallen into a sin to which we are all tempted, whether by the uncompromising precepts of theological systems, or by the insidious seductions of scientific text books; the sin of making assumptions and then hiding from ourselves that they are assumptions and that we have no right to believe in them. Apropos of "From Matter to Spirit," he says that he refers certain phenomena "*either to unseen intelligence or something which man has never had any conception of.*" This apparently suspended judgment involves and hides the assumption that the said phenomena cannot possibly be referred to certain well known and commonly conceived things—the art of the conjuror, and the delusion of contagious excitement. This enormous assumption is, of course unconsciously, introduced and hidden under a brilliant display of candid impartiality and cautious scepticism. We point to this, not as throwing a stone thereat; but desiring that it should indicate the great and serious importance of the *Budget of Paradoxes*. To sum up, this is a book that should be read by those who care about circle-squarers and all manner of jokes, mathematical and other; by those who care to make the acquaintance of Augustus de Morgan, which it is well worth while to do; but above all by those who care to be led into right thinking and warned from wrong.

will suppose that they are here reading the latest developments of an obscure subject, and may profit largely by every paragraph. But the freshness and originality of treatment which are here to be found will prove an effective stimulus to those who have endeavoured to keep up with the rapid progress of science. The geometrical applications of Elliptic and Abelian functions which have chiefly interested later geometers are those in which the function of a complex argument is used to represent the aggregate of the real and imaginary points of a curve. Such applications bear upon the projective theory of curves, and it is rarely that any metrical properties emerge from them. But Dr Booth considers the representation of functions of a real argument by the rectification and quadrature of curves; and it is well worth while to be recalled in a time to this aspect of the elliptic functions, which is, of course, the most important for physical purposes. The study of these chapters in the light of new ideas and with the help of later notation cannot fail to be highly profitable. With a small example of this we shall conclude a very meagre notice.

Of the two modes of representation of elliptic integrals which Dr Booth uses, that by the area of quadrupedal curves is undoubtedly the more important; but considerable interest attaches to the expression of elliptic integrals of the first kind by the area of the negative pedal of an ellipse in regard to the centre. If we $OP = a$, $OP' = r$, $OC = a$, angle $OC'Z = \phi$, then

$$\phi = \arcsin \left(\frac{a}{r} \sqrt{1 - e^2} \right),$$

the modulus e being the eccentricity of the ellipse. The quantity $a - r$ is called the residual u , and a method is given for finding a curve whose residual arc shall represent any given integral. In fact, $a - r = \int p d\phi$, if $p = OCZ$; so that the

tangential polar equation of the curve which represents $\int f(\phi) d\phi$ by its residual arc is $p = r(\phi)$. There are two special cases of interest. When $e = 0$, the ellipse becomes a circle, and the negative pedal coincides with the circle itself; in this case $r = 0$, and the equation becomes $a\phi = s$. If, while a/r is kept constant, e is made to increase from 0 to 1, the ellipse will lengthen out until it becomes the pair of tangents at the extremities of the minor axis. The negative pedal then becomes two parabolas, and it is clear that we need only attend to one of them. The elliptic amplitude becomes a function which is called by Cayley the Gudermannian (*Elliptic Functions*, p. 50), and which may be thus defined: If $\cos \phi = \cos(u) = 1$, then $\phi = \text{gd } u$. It follows that

$$\tan \phi = i \sinh(u), \quad \text{whence } u = \int_0^\phi \sec \phi d\phi = \log \tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right).$$

We have

$$\begin{aligned} \tan \frac{1}{2}(\text{gd } u + \text{gd } v) &= i \sinh(u + iv) = i \sinh(u) \cosh(iv) + i \cosh(u) \sinh(iv) \\ &= \tan \frac{1}{2} u \sec \text{gd } v + \tan \frac{1}{2} v \sec \text{gd } u. \end{aligned}$$

Dr Booth writes this formula thus

$$\tan (\phi + \chi) = \tan \phi \cos \chi + \tan \chi \cos \phi,$$

so that

$$\phi + \chi = \phi \cos \chi + \phi \sin \chi \tan \chi,$$

similarly,

$$\chi + \chi = \chi \cos \phi + \chi \sin \phi \tan \phi.$$

Observe that the operations $\phi + \chi$ satisfy the commutative and associative laws.

This notation appears convenient for some purposes, and the reader will find a number of very interesting developments admitted by means of it. It seems worth considering whether a similar notation might not be applied with advantage to the more general functions $\sin \alpha$, while ϕ , like $\phi \sin \alpha$, has no proper addition theorem.

PROBLEMS AND SOLUTIONS*.

1863. For every point A on a rectangular hyperbola, there exists a straight line BC , passing through the centre, such that if through any other point D on the curve, lines be drawn parallel to the asymptotes, cutting BC in B, C , the intercept BC subtends a right angle at A . [March, 1863, solved May, 1863. The proposer's solution, bearing date March 30, 1863, is given Reprint, Vol. xxvii, p. 32.]

Let O be the centre of the hyperbola; join OA , and draw BOC perpendicular to it. Let DB, DC , parallel to the asymptotes, cut them in E, E' ; and draw AM perpendicular to either of the asymptotes.

Then we have $CO \parallel OB \parallel CE \parallel ED$,

$$A, CDE; CO, OB \parallel OE, EC; OE, ED \dots\dots\dots (1).$$

But, by similar triangles OCE, OAM ,

$$CO \parallel OE, EC \parallel OM; OM, MA \dots\dots\dots (2);$$

therefore, comparing (1) and (2), since

$$OE, ED \parallel OM, MA, OA^2 \parallel CO, OB,$$

or BAO is a right angle.

The property holds for any hyperbola, but the line BC does not always pass through the centre; if it cut the asymptotes in P and Q , the angles APQ, AQP , are each equal to the angle between the asymptotes, and to BAC . I have not been able to find a geometrical construction for the line.

1873. Given a circle (C) and any point A , either within or without the circle, through A draw BCD cutting the circle in B, D . Then it is required to find another point E , such that, if ECM be drawn cutting the circle in E, M , we may always have $AE^2 \parallel EC, E M \parallel BE, AM$. [Proposed by T. T. Wilkinson, F.R.S.B. Solution sent 1863, printed in Reprint, Vol. xxvii, p. 22.]

* This is the *Educational Times*, and from *Mathematical Questions with their Solutions from the Educational Times*. This latter work, which is cited under the title "Reprint," contains also many papers and solutions not published in the *Educational Times*. Professor Clifford contributes a solution to two questions which appeared in a weekly journal, *The Key*. They will be found I. Vol. ix, No. 21, August 27th, p. 111, No. 22, August 28th, 1862, p. 140 (this last solution is given I. Rept. The solutions are not in a regular list). (

1. For the upper sign, when A is *without* the circle, and the lower, when A is *within*, E lies on the polar of A . [TEP ; AE perpendicular to CEG .]

$$\begin{aligned}\text{For } AE^2 &= AP^2 + PE^2 = AC^2 + CE^2 + EP^2 \\ &= AC^2 + CE^2 + (CP^2 + PE^2 - 2CP \cdot CE) \\ &= AC^2 + CE^2 + 2CP^2 \\ &= BA \cdot AD = LE \cdot EM,\end{aligned}$$

when E is within the circle, as in the figure; when E is without the circle, $AE^2 = LE \cdot EM + BA \cdot AD$. By interchanging A and E , we get the second case.

2. For the same, E may obviously lie on a circle with CA as diameter.

3. For the upper sign, when A is *within* the circle, and the lower, when A is *without*; bisect CA in P , and with P as centre describe a circle whose radius is $\sqrt{CP^2 + 3CP^2}$; E may lie on this circle. For, if EP be perpendicular to AC ,

$$\text{then } CP^2 + 3CP^2 = PE^2 = PC^2 + CE^2,$$

$$\text{or } 2CP^2 = \frac{1}{2}(AC^2 + CE^2 + AP^2 - 2AP \cdot CP),$$

$$\text{therefore } AE^2 = (CP^2 + CE^2) + (CP^2 + CA^2) = LE \cdot EM + BA \cdot AD.$$

4. In the same cases, E may evidently lie on a straight line through A perpendicular to CA .

1376. A *tangent* to an ellipse is a *chord* of a concentric circle, whose radius is equal to the distance between the ends of the axes of the ellipse; show that the straight lines which join the ends of the chord to the centre are conjugate diameters. [April, 1863, solved June, 1863. Proposer's solution sent May 14th, 1863, printed in Reprint, Vol. XXXI, p. 31.]

Let the equations to the *ellipse*, the *circle*, and the *chord*, be respectively

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x^2 + y^2 = a^2 + b^2, \quad \frac{xh}{a^2} + \frac{y^k}{b^2} = 1, \dots \dots (1, 2, 3).$$

$$\text{Then the equation } x^2 + y^2 = (a^2 + b^2) \left(\frac{xh}{a^2} + \frac{y^k}{b^2} \right)^2 \dots \dots \dots (4)$$

represents the two straight lines passing through the origin and the intersection of (2) and (3). If these are conjugate diameters, we must have

$$\frac{h^2}{a^2} + \frac{k^2}{a^2 b^2} = (a^2 + b^2) \frac{h^2}{a^2}$$

which may easily be shown to be the case, since (h, k) is on the ellipse, and therefore $a^2 h^2 + b^2 k^2 = a^2 b^2$.

If we change k^2 into $-k^2$, we obtain a similar theorem for the hyperbola; but the conjugate diameters will be imaginary, if

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} < \frac{1}{a^2 + b^2}.$$

1370. [If a curve of the third order have a double point A , and be cut by any straight line in B, C, D ; and if, when ABC is taken as triangle of reference, the tangents at A are represented by the equation

$$P^2\beta + Q\beta\gamma + R\gamma^2 = 0,$$

and the tangents at B, C , by the equation

$$Pa + N\gamma = 0, \text{ and } M\beta + Ra = 0;$$

show that the equation to the straight line AD is

$$N\beta + M\gamma = 0,$$

and find the equation of the curve. Proposed by * * *, April, 1863, solved June, 1863.]

Since $P^2\beta + Q\beta\gamma + R\gamma^2 = 0$ and $Pa + N\gamma = 0$ are the tangents at the points where $\gamma = 0$ meets the curve, its equation must be of the form

$$(P^2\beta + Q\beta\gamma + R\gamma^2)(Pa + N\gamma) + \gamma^3\phi;$$

also of the form

$$(P^2\beta + Q\beta\gamma + R\gamma^2)(M\beta + Ra) + \beta^2\chi.$$

It will be found that the equation

$$\alpha(P^2\beta + Q\beta\gamma + R\gamma^2) + \beta\gamma(N\beta + M\gamma) = 0 \dots\dots\dots (1)$$

is of both these forms. For clearly the lines $P^2\beta + Q\beta\gamma + R\gamma^2 = 0$ only meet the curve at the point $(\beta\gamma)$, and the lines $Pa + N\gamma = 0$, $M\beta + Ra = 0$, touch the curve at the points $(a\gamma)$, $(q\beta)$. It is now obvious that the other point where α meets the curve is in the line $N\beta + M\gamma = 0$, which is therefore the line AD .

We may notice that the tangents drawn from D to the curve are represented by

$$(Q + 2\sqrt{PR})\alpha + N\beta + M\gamma = 0 \dots\dots\dots (2);$$

there are only two because there is a double point. The lines drawn from A to their points of contact are represented by $\beta\sqrt{PA} + \gamma\sqrt{R} = 0$; hence these form an harmonic pencil with AB, AC .

The equation to the tangent at D is

$$N^2M\beta + M^2N\gamma + (M^2P + N^2R + MNQ)\alpha \dots\dots\dots (3),$$

and that to the line joining A with the other point where (3) meets the curve is $MP\beta + NR\gamma = 0$; hence the condition that D may be a point of inflexion is

$$PM^2 = RN^2,$$

1387. Four common tangents are drawn to a circle and an ellipse which passes through the centre (O) of the circle; if A, B be opposite intersections of the tangents, show that OA and OB are equally inclined to the tangent at O to the ellipse. [May, 1863, corrected to the above form in July, 1863, solved September, 1863; Reprint, Vol. 1, pp. 19, 33.]

We use rectangular tangential coordinates (Feyrer, *Tril. Co.*, p. 130; Salmon, *Higher Plane Curves*, p. 2). It is easily shown that the sum of the squares of the reciprocals of the intercepts made by any tangent to a circle on two diameters

ters at right angles be constant. Hence the equation to a circle whose centre is the origin is

$$\xi^2 + \eta^2 = k^2. \quad (1)$$

The points $\xi = 0, \eta = 0$, are at an infinite distance, one on each of the axes, and $k=0$ (where k is a constant) represents the origin. From this it follows that the equation

$$\xi^2 + \eta^2 + k(\eta + \eta') = 0 \quad (2)$$

(where the k may be left out at pleasure) represents a circle touching the axis of ξ at the origin. For if we seek the tangents drawn from $k=0$ to the curve, we find that they both coincide with the line $\xi\eta$, that is with the axis of ξ . Now if we put

$$\xi^2 + \eta^2 = r^2, \quad \xi^2 + \eta^2 + k(\eta + \eta') = 2,$$

it is clear that these equation N and N' contain complete circles, one of the second class, touching all the common tangents of N and N' . The discriminant of (2) in equation is of the third degree in N , hence there are three values of N for which $N(N')=0$ represents two points. But one is a case the result of η is zero, which is just the condition that the line joining the origin to the three points (which are evidently opposite poles) makes \angle the common tangent that should be equally inclined to the axis of ξ . For if $\eta' = 1$ for the equation of a point, (x, y) are the ordinary rectangular co-ordinates, η' is in the tangent of the angle which the line joining it to the origin makes with the axis of ξ . Hence if two points (x, y) and (x', y') are equally inclined to the point η , we must have

$$\frac{y}{x} = \frac{y'}{x'}, \quad \text{or } yx' = xy',$$

but (as a test) is the condition of η in the product of η and $\eta' = 0$ in $N(N')=0$. The theorem is thus proved.

It will be observed that the discriminant being of the third degree in N , must always have one real root, and three will be imaginary unless the circle only when the centre is on a hyperbola lying outside the two points.

It appears therefore that any two circles have two real hyperbolas of real or imaginary common tangents, passing through the vertices of a triangle of four circles.

By projection we may show that "the straight line A joins the poles of H with respect to two circles, then the line joining the two poles of H to any point of intersection forms of common tangents, L and L' , with A , L and L' are known to be parallel."

And by superposition, "If a point of the H is taken, then the poles of H with respect to two circles, and the line joining the two poles of H to any point of intersection forms of common tangents, L and L' , with A , L and L' are known to be parallel."

[A solution of this case, and theorems, is given in the same paper, and in the known theorem "If a line segment is bisected in the same point, then the tangents from any point to the two circles form a parallelogram," is given by Prof. Cayley in the same volume of the *Mathematical Papers*, p. 29.]

But [A remark of the *Mathematical Papers*, consisting of three superposition theorems, is given so as to touch the poles of any two circles and the poles of the points

Three three points are joined so as to form a new equilateral triangle. Show that if PA, PB, PC be the perpendiculars from any point P on the curve upon the sides of one equilateral triangle, and PD, PE, PF the perpendiculars from the same point on the sides of the other equilateral triangle, then the ratio vol. PA, PB, PC : vol. PD, PE, PF is constant, wherever P be taken on the curve. Proposed by " * * ". Reprint, Vol. I, p. 10, *E. T.* July, 1893.]

The general equation to a cubic touching $B+C=0, C+A=0, A+B=0$, where they meet $A=0, B=0, C=0$, is evidently

$$ABC=k(B+C)(C+A)(A+B) \dots\dots\dots (1).$$

In the case supposed, let $B+C=a, C+A=\beta, A+B=\gamma$, then (1) becomes

$$(a+\beta-\gamma)(a-\beta+\gamma)(-a+\beta+\gamma)=k\alpha\beta\gamma,$$

which expresses the property in question.

The asymptotes are parallel to $\alpha\beta\gamma$, and there are three points of inflexion, all at an infinite distance.

The proof holds if the cubic touch the sides of any triangle in three points, each that the lines joining them to the opposite vertices meet in a point.

1891. From a point A two chords are drawn meeting a conic section in four points B , joined also by four straight lines a . These intersect two and two in two points P lying on the polar of A . At the points B are drawn four tangents b , which intersect in six points, two of which are on the polar of A , and the others lie two and two on the two straight lines AP . These tangents intersect the original chords in four points, which may be joined by four straight lines intersecting by pairs in the points P . The lines a and b intersect in eight points C , which may be joined by twenty lines α ; four of these pass through A , and the others may be divided into groups of four. Each group has six intersections, two of which lie on the polar of A , and the others lie two and two on lines through A . Any two groups intersect in eight points, having properties like those of the points C . [June, 1893. Reprint, Vol. I, p. 14, *E. T.* August, 1893.]

Take the chords through A for the sides β, γ of the triangle of reference, one of the straight lines a as that of α , and let another of them be represented by

$$lx+my+nz(-\delta)=0.$$

Let the equation to the conic be

$$ax^2+by^2 \dots\dots\dots (1),$$

then the four lines a are

$$\left. \begin{aligned} a_1 \dots\dots\dots ax=0 \\ a_2 \dots\dots\dots by=0 \\ a_3 \dots\dots\dots lx+my=0 \\ a_4 \dots\dots\dots lx+nz=0 \end{aligned} \right\} \dots\dots\dots (2).$$

The polar of A is evidently

$$2ax+my+nz=0 \dots\dots\dots (3).$$

The equations to the tangents b are

$$\left. \begin{aligned} b_1 \dots m\alpha - k\gamma &= 0 \\ b_2 \dots m\alpha - k\beta &= 0 \\ b_3 \dots m\delta + lk\gamma &= 0 \\ b_4 \dots n\delta + lk\beta &= 0 \end{aligned} \right\} \dots\dots\dots (4).$$

For instance, assume $\delta = p\gamma$ for the equation to b_3 ; then from (1) we find that this meets the curve where it meets $p\alpha = k\beta$; but as b_3 is a *tangent*, this must coincide with a_3 , or $mp = -lk$; and thus b_3 becomes

$$m\delta + lk\gamma = 0.$$

The intersections of b_1b_2 and of b_3b_4 lie on $m\beta - n\gamma = 0$; of b_1b_4 and of b_2b_3 on $m\beta + n\gamma = 0$; and of b_1b_3 and b_2b_4 on $2l\alpha + m\beta + n\gamma = 0$.

The equations of the four lines joining intersections of the lines b with β and γ are

$$\left. \begin{aligned} mn\delta + lk(m\beta + n\gamma) &= 0 \\ mna - k(m\beta + n\gamma) &= 0 \\ mn(l\alpha + n\gamma) - lk(m\beta - n\gamma) &= 0 \\ mn(l\alpha + m\beta) + lk(m\beta - n\gamma) &= 0 \end{aligned} \right\} \dots\dots\dots (5).$$

The first pair meet $2l\alpha + m\beta + n\gamma = 0$ where it meets $m\beta + n\gamma = 0$; the second pair where it meets $m\beta - n\gamma = 0$.

The eight points C may be represented as follows:

$$\begin{array}{cccccccc} a_3b_3, & a_3b_4, & a_2b_3, & a_1b_3, & a_3b_1, & a_3b_2, & a_1b_4, & a_2b_1 \\ E & F & G & H & K & L & M & N. \end{array}$$

With this notation, the five groups of lines C are

$$\begin{aligned} (EK, FL, GM, HN); & (EF, GH, KL, MN); & (EG, HL, KM, NF); \\ (EM, FH, NL, GK); & (EM, FG, ML, KH). \end{aligned}$$

All the lines of the first group pass through A , and have for equations

$$\left. \begin{aligned} EK, \dots\dots\dots m^2\beta + kl\gamma &= 0 \\ FL, \dots\dots\dots n^2\gamma + kl\beta &= 0 \\ GM, \dots\dots\dots n^2\gamma + (mn + kl)\beta &= 0 \\ HN, \dots\dots\dots m^2\beta + (mn + kl)\gamma &= 0 \end{aligned} \right\} \dots\dots\dots (6),$$

which may be easily verified.

As an example of the others, take the equations of the lines of the fourth groups:

$$\left. \begin{aligned} EM, \dots (l^2k^2 + 2klmn)(l\alpha + n\gamma) &= (lk + mn)(m\delta + kl\gamma)n \\ FH, \dots (l^2k^2 + 2klmn)(l\alpha + m\beta) &= (lk + mn)(n\delta + kl\beta)m \\ NL, \dots k(mn + kl)\delta &= mn^2(m\alpha - k\gamma) \\ GK, \dots k(mn + kl)\delta &= m^2n(n\alpha - k\beta) \end{aligned} \right\} \dots\dots\dots (7).$$

The intersections of EM and FH , and of NL and GK , lie on $m\beta - n\gamma = 0$; those of EM and NL , and of FH and GK , on $m\beta + n\gamma = 0$; those of EM and GK , and of NL and FH , on $2l\alpha + m\beta + n\gamma = 0$; and so with each of the other groups.

I was wrong in saying that *any* two groups intersect in eight points, &c.; it is true of the last four, for it will be found that any two of these form two midlines, the vertices of one resting on the sides of the other, two diagonals each passing through A , and the others being identical ($2\alpha + m\beta + n\gamma = 0$); now, by the converse of the first part of the equation, a conic may be inscribed in one so as to circumscribe the other, and the preceding reasoning applies. At the first group is an exception; there are only four new points formed by uniting it with any of the others, and these may be joined by four lines meeting two and two in the polar of A .

All these theorems may be proved very readily by projecting the conic into a circle whose centre is the projection of A .

1993. [A shell formed of two equal paraboloids of revolution, having a common axis, is fixed with its vertex downwards, and axis vertical; and a heavy uniform rod of given length rests within it, in a vertical plane through the axis. Compare the pressures on the lower surface of the shell. Proposed by Mr J. R. Wilson, Jesus College, Cambridge, Reprint, Vol. i. p. 27, *N. T.* September, 1863.]

Let QPR be the rod, of length $4a$; draw tangents QT, RT , and normals OQ, OR , to the outer parabola. We know by Geometry that $QP = PR$, and therefore TP is vertical. Now the rod is kept at rest by four forces, two of which, viz., gravity and the resistance at P , pass through P ; therefore the resultant of the pressures at Q, R acts along OP . But OP , bisecting QR , is half the diagonal of the completed parallelogram (OQ, OR); hence the resistances at Q, R , are as the normals OQ, OR ; that is, as $\sin ORQ : \sin OQR$, or as $\cos TRP : \cos TQP$. Now draw a tangent (ECF) to the outer parabola at the point (C) where TP meets it; then, putting $AB = k$, $4a =$ principal parameter of BRQ , and $\angle ECF = \theta$, the equation of BRQ , referred to CP, CQ , will be

$$y^2 = 4a \cos \theta x^2 \dots \dots \dots (1).$$

At the point P , $x = k$, $y = 2a$,

$$\therefore a^2 \sin^2 \theta = ak \dots \dots \dots (2).$$

The tangents QT, RT are represented by

$$k^2 y^2 = a^2 (x + a)^2 \dots \dots \dots (3).$$

Therefore by the usual formula for oblique axes

$$\frac{\cos TPT}{\cos TEC} = \frac{(x + k \cos \theta)}{(x + k \sin \theta)} \left(\frac{a^2 + k^2 + 2ck \cos \theta}{a^2 + k^2 - 2ck \cos \theta} \right)^{\frac{1}{2}}.$$

This, therefore, is the ratio required.

[The following general remarks accompany a solution of 1418, Reprint, Vol. i. p. 30, *E. T.* November, 1863.]

Consider any two conics, U, V ; any point (ξ, η, ζ) has two polars, $\Delta U, \Delta V$, where Δ stands for

$$\xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{dz}.$$

These meet in a point which we shall call the *polar opposite* of (ξ, η, ζ) . Similarly, the line joining the poles of a right line may be called its *polar opposite*.

Now consider the equations $S=0$, $S=LM$, where L , M are common chords.

The polars are $\Delta S=0$, $\Delta S=L\Delta M+M\Delta L$.

If $\Delta L=0$, these meet on L ; that is,

(a) If any point lie on a common chord, its polar opposite lies on the same chord.

If also $\Delta M=0$, they coincide, or

(β) The intersection of a pair of common chords has only one polar. It is easily shown that this is a line joining intersections of common tangents.

It follows that

(γ) The polar opposite of any point in a straight line with two opposite intersections of common tangents is an intersection of common chords.

Let $\Delta(L+KM)=0$, then the polars meet on $L+KM=0$. That is

(δ) Lines joining two polar opposites to an intersection of common chords, form, with the chords, an harmonic pencil.

Next, let the equations be $L\bar{M}+N^2$, $L\bar{M}+R^2$, so that L , M are common tangents. The polars are now

$$\begin{cases} (L\Delta M+M\Delta L)+2N\Delta N=0 \\ (L\Delta M+M\Delta L)+2R\Delta R=0 \end{cases}$$

If $\Delta N=0$, these intersect on R , or

(e) If a point lie on one chord of contact of a pair of common tangents, its polar opposite lies on the other.

If $\Delta(N+KR)=0$, the polars meet on $R+KN=0$; or

(f) If the locus of a point is a line through the intersection of the chords of contact of a pair of common tangents, the locus of its polar opposite is another line through the same intersection.

Thirdly, consider the case of double contact, S , $S'+I^2$. Here the polars are ΔS , $\Delta S'+2L\Delta L$. These always meet on L , showing that

(g) If two conics have double contact, the polar opposite of any point whatever lies on the chord of contact.

If $\Delta L=0$, they coincide, or

(h) A point on the chord of contact has only one polar, which is also the locus of its polar opposites.

(i) In general, if the locus of a point be a straight line,

$$lx+my+nz=0,$$

the locus of its opposite is the conic

$$\begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ l & m & n \end{vmatrix} = 0,$$

which we may call the *polar conic* of the line (lmn). As the discriminant is of a third degree in (lmn), it appears that the envelop of lines whose polar conics take up into two right lines is a curve of the third class.

1409. For every point A on a conic section there exists a straight line BC , not meeting the curve, such that, if through any other point on the conic there be drawn any two straight lines meeting BC in B , C , and the curve in D , E , the angles BAC , DAE are either equal or supplementary. [July, 1863. Reprint, Vol. I. p. 33, *E. T.* December, 1863.]

Take the point A for origin, and the rectangular tangential equation used in question 1397 [cf. *supra*], but in the more convenient form

$$(\xi - a)^2 + 4b(\eta - c) \dots\dots\dots (1),$$

which is evidently equivalent to the one there given.

The line BC is represented by

$$\xi - a + 4b(\eta - c) \dots\dots\dots (2);$$

it always passes through the pole of the normal, and is in fact the polar of the point of intersection of chords subtending a right angle at A .

If we assume for the general equation of a point on the curve

$$\xi - a = m(\eta - c) + \beta \dots\dots\dots (3),$$

then the equation

$$\{m(\eta - c) + \beta\}^2 + 4b(\eta - c)$$

must have equal roots for η , which gives $\beta = \frac{b}{m}$. We shall call this the point m .

Let the intersection of BD , CE , be the point m_1 , and the points D , E , m_2 , m_3 . The equations of B , C , will therefore be

$$\left(m_1 + \frac{1}{m_2}\right) \{(\xi - a) - m_1(\eta - b - c)\} = \left(m_1 + \frac{1}{m_1}\right) \{(\xi - a) - m_2(\eta - b - c)\} \dots\dots (4),$$

$$\left(m_2 + \frac{1}{m_3}\right) \{(\xi - a) - m_1(\eta - b - c)\} = \left(m_1 + \frac{1}{m_1}\right) \{(\xi - a) - m_2(\eta - b - c)\} \dots\dots (5).$$

These equations may be easily verified.

The angle between AB and the axis of ξ is therefore

$$\tan^{-1} \frac{m_2 \left(m_1 + \frac{1}{m_1}\right) - m_1 \left(m_2 + \frac{1}{m_2}\right)}{\left(m_2 + \frac{1}{m_3}\right) - \left(m_1 + \frac{1}{m_1}\right)} = \tan^{-1} \frac{m_1 + m_2}{m_1 m_2 - 1}.$$

Hence the angle BAC is

$$\tan^{-1} \frac{m_1 + m_2}{1 - m_1 m_2} = \tan^{-1} \frac{m_1 + m_2}{1 - m_1 m_2}.$$

That is, its tangent is equal to that of the angle between the lines joining the points m_2 , m_3 to the origin. For the latter angle is clearly

$$\tan^{-1} \frac{m_2 + m_3}{1 + m_2 m_3}.$$

A , B , and draw OP perpendicular to AB , and PM , PN perpendicular to the axes. Then we have two systems of coordinates; (1) when $\frac{1}{OA}$, $\frac{1}{OB}$ are the coordinates of the point P , (2) when PM , PN are the coordinates of the point P . The formulae of transformation, between the first and Cartesian, and between the second and Tangential, coordinates, are

$$(x^2 + y^2)(x^2 + y^2) = 1, \quad xy = px \dots \dots \dots (5).$$

These represent the operation of inversion in the two cases. But it is important to remember that, in *Tangential inversion*, the tangents, not the points, are inverted; that is, to every tangent of the primitive corresponds a line parallel to it, such that the rectangle under their distances from the origin is constant. Now let $U = 0$ be an equation in x , y , and constants; and let OU denote the curve which is represented by $U = 0$, when we interpret x , y as Cartesian coordinates, TU when we interpret x , y as *Tangential* coordinates, UU according to the *first* system of this article, and NU according to the *second*. Then, for instance, $TU = R$, UU , or, by separation of symbols, $T = RC$. In this way we have the equations

$$M = JC, \quad N = RJC = RM = RRT,$$

which serve to connect any two systems.

6. It appears from (4) that if the equation of any curve be written

$$u_n + u_{n-1} + \dots + u_2 + u_1 + u_0 = 0,$$

then the equation of the inverse is

$$u_n + u_{n-1}(x^2 + y^2) + u_{n-2}(x^2 + y^2)^2 + \dots + u_0(x^2 + y^2)^n = 0.$$

This is of degree $2n$ in general, but reduces when the curve is circular, and when the origin is on the curve. If the curve be circular in the degree f , that is, if its equation be of the form

$$v_n - 2f(x^2 + y^2) + v_{n-2f+1}(x^2 + y^2)^{f-1} + \dots + v_1 + v_0 = 0,$$

the degree of the inverse is reduced $2f$, and if the origin be a multiple point of the order p , or if

$$u_n = u_{n-1} = \dots = u_{n-p+1} = 0,$$

the degree is reduced by p . Hence generally the degree of the inverse is $2(n - f) - p$. It follows by reciprocation that if n be the class of any curve, and if the lines joining the origin to the circular points at infinity are multiple tangents of the order f , and if the line at infinity be a multiple tangent of the order p , then the degree of the first positive pedal is $2(n - f) - p$.

And again, if a curve has p points at infinity distinct from the two circular points at infinity, and has a multiple point of the order ϕ at the origin, being of degree n , then the *class* of the first negative pedal is $n - \phi + p$. This is easily obtained by inverting the result just proved; it being remarked that the inverse of a curve circular in the degree f has a multiple point of the order $n - 2f$ at the origin. The *degree* of the negative pedal in the *class* of the inverse, and consequently in the same as the number of circles which can be drawn through

an arbitrary point (ξ, η) and the origin to touch the primitive curve. To find this number, we must eliminate between

$$U=0 \dots\dots\dots (1).$$

$$x^2 + y^2 + 2Ax + 2By = 0 \dots\dots\dots (2),$$

$$(x+A) \frac{dU}{dx} = (y+B) \frac{dU}{dy} \dots\dots\dots (3),$$

A and B being converted by the linear relation

$$\xi^2 + \eta^2 + 2A\xi + 2B\eta = 0 \dots\dots\dots (4).$$

The degree of the eliminant in A and B , which is the degree of the first negative pedal, is in general $n(n+2)$, but will of course be reduced by peculiarities in the form of U .

1819. [It is announced at p. 205, Vol. II., 12th ed., Davies's Hutton, that "if a tetrahedron be drawn, formed of four tangent planes to a paraboloid, the sphere described about it will pass through the focus of the paraboloid." Prove or disprove this. Proposed by N'Importe. Reprint, Vol. I. p. 45, E. T. February, 1864.]

The statement is not true.

If perpendiculars be drawn from the foci of a conicoid of revolution on any tangent plane, the rectangle of these perpendiculars is equal to the square of the minor axis. If then a conicoid of revolution having foci $\alpha\beta\gamma\delta$, $\alpha_1\beta_1\gamma_1\delta_1$, touch the faces of the fundamental tetrahedron, we must have

$$\alpha\alpha_1 = \beta\beta_1 = \gamma\gamma_1 = \delta\delta_1 = b^2.$$

So that if one of the foci lies in the plane

$$la_1 + m\beta_1 + n\gamma_1 + r\delta_1 = 0,$$

the locus of the other will be the surface of the third degree

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} + \frac{r}{\delta} = 0 \dots\dots\dots (1),$$

which is otherwise interesting (Frost and Wolstenholme's *Solid Geometry*, p. 289). A particular case is when the surface of revolution is a paraboloid, one of whose foci lies on the plane at infinity,

$$A\alpha + B\beta + C\gamma + D\delta = 0,$$

and the other on the surface

$$\frac{A}{\alpha} + \frac{B}{\beta} + \frac{C}{\gamma} + \frac{D}{\delta} = 0 \dots\dots\dots (2),$$

where $ABCD$ are the faces of the tetrahedron.

Now if the theorem of Davies's Hutton were true, we should have found for the locus the equation of the circumscribing sphere.

I write down one or two other instances of the application of this principle. (See Salmon's *Conics*, 4th ed., p. 261, Ex. 13, 15.)

Given five planes connected by the identical relation

$$a\alpha + b\beta + c\gamma + d\delta + e\epsilon = 0,$$

oid of any conicoid of revolution touching the "frustum" will lie in surface of the fourth degree,

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} + \frac{d}{\delta} + \frac{e}{\epsilon} = 0.$$

Given one focus and the intersection of the focal tangents of a parabola of third class inscribed in the triangle of reference; the other focus moves on a conic (which is never a circle) circumscribing the triangle.

Given four tangents, a focus, and the intersection of the focal tangents, a curve of the third class; the other two foci move on a curve of the third class.

The general extension is sufficiently obvious.

421. [If by the Harmonic centre, relative to a fixed plane, of A, C , points line meeting the fixed plane in D , be understood a point B between A and C such that A, B, C, D form an harmonic system; prove that if through the harmonic centre of either diagonal of any of the three quadrilateral faces of a frustum of a triangular pyramid, and the harmonic centres of the two diagonals which meet but are not in the same face with that diagonal, a plane be drawn, the six planes thus obtained will all pass through one and the same point. Proposed by Professor Sylvester, F.R.S. Reprint, Vol. I. pp. 45, 46, 1. February, 1864.]

The proposer has shewn, in the *Philosophical Magazine* for September, that the theorem is true when we put the arithmetical centre for the harmonic. The proof, by Cartesian coordinates, is exceedingly simple and elegant; but it will be found that the attempt to prove the same case by quadriplanar coordinates involves an enormous amount of algebraic work. However, it is clear that if the requisite operations were performed, the result must be the same as by the other method. Now when we find the arithmetic centre of a tetrahedron by quadriplanar coordinates, the process is simply to find its harmonic centre with respect to the plane at infinity $A\alpha + B\beta + C\gamma + D\delta = 0$. But the proposer can make no mention of the meaning of $ABCD$, since the thing proved is general for any tetrahedron, and does not depend at all upon the areas of the faces. Therefore the proof holds, whatever interpretation we give to the letters $ABCD$; that is to say, whatever plane is represented by

$$A\alpha + B\beta + C\gamma + D\delta = 0.$$

This principle is evidently identical with the method of Projections in plane geometry. (See Salmon's *Higher Plane Curves*, Art. 246). Quadriplanar coordinates are not connected with the absolute form of the fundamental tetrahedron, but hold good whatever tetrahedron we choose. Now it is analytically possible to choose a tetrahedron with reference to which a given conicoid shall be represented by a given equation. For a conicoid is determined by nine conditions, but each of the four planes involves three independent constants. We may, therefore, in addition, choose the tetrahedron so that the plane at infinity shall be represented by a given equation. Thus any property proved of any one conicoid and a plane, when expressed in quadriplanar coordinates, is true of any other conicoid and plane. The only limitation is that connecting

properties which can be expressed in the coordinates will be retained in transformation. Again, we may analytically transform real lines and planes into imaginary, and *vice versa*, without loss of continuity. Now, ruled conicoids only differ from others in containing real lines instead of imaginary; therefore the distinction between ruled and unruled conicoids is lost in transformation.

In studying, then, the properties of any figure, the principal point will be the reduction of the figure to what (by an extension of the term) may be called the *canonical form*. For instance, the canonical form of a quadrilateral is a parallelogram; of a conic, a circle; of a conicoid, a sphere; and so on. In particular, we wish to find the canonical form of a tetrahedral frustum, which is the figure formed by the intersection of five planes $ABCDE$. For a quadrilateral $ABCD$ we proceed in this way; we join the vertex A to the vertex D and project the joining line to infinity. Hence by analogy in the frustum we join the vertex ABC to the edge DE , and project the joining plane to infinity. The figure is thus reduced to three parallel straight lines cut by three parallel planes. As an instance of the use of these canonical forms, we will prove the following properties, which may be easily proved: "From a point O , three chords o are drawn to a conicoid, meeting it in six points A . These are joined again by four pairs of planes α , each pair intersecting in one of the lines β on the polar plane of O . At the points A six tangent planes γ are drawn; if any three of these (whose points of contact are not in one line through O) intersect in X and the other three in Y , then X , Y , O are on a straight line. Again, the planes δ will intersect by pairs in three lines γ on the polar plane of O , and these will pass through the three intersections of the lines β with the other three. If the chords o cut the polar plane in three points, these will lie in three straight lines through the same intersection. There are thus three coaxial triangles on the polar plane, and their common pole is on the line joining O and two of the intersections XY . The planes δ cut the chords o in twelve new points C , four of which lie on each chord. Consider the eight C -points lying on two chords o ; they are divided into two groups, each group having two points on each of the four chords. Lines joining points in either group intersect on the polar plane of O . Each group has for these two intersections, (1) a vertex of one of the three triangles, (2) the point where the opposite side cuts the common axis. Lines in different groups may be joined by eight lines, intersecting in four points lying on the polar plane of O , and eight points lying on four lines through O and so on.

"If a straight line be drawn through the vertex of either of the two tangent cones of two conicoids having double contact, to meet either of the planes of common section, and the two conicoids, it will be cut in two points so that the equal anharmonic ratios of the involution are constant."

1443. [Shew that the locus of the centres of all the conics circumscribed about a given quadrilateral is an ellipse if the quadrilateral is re-entrant, a hyperbola if it is convex. Shew further that two real parabolas may be drawn through the angles of any convex quadrilateral. Proposed by Sylvester, F.R.S. Reprint, Vol. i. pp. 51—54, E. T. March, 1864.]

1. First, the locus of the centre is a conic.

Let U, V be the tangential equations of two conics through the four points; then the general equation of a conic through the points is

$$l^2pU + lmV + m^2qV = 0 \dots\dots\dots(1),$$

where p, q are the discriminants of U, V and F is the conic touched by all the tangents to U and V at the four points of intersection; thus, if

$$U \equiv ax^2 + by^2 + cz^2, \quad V \equiv a'x^2 + b'y^2 + c'z^2,$$

$$\text{then } F \equiv aa' (bc' + b'c) x^2 + bb' (ca' + c'a) y^2 + cc' (ab' + a'b) z^2.$$

Now let (ξ, η, ζ) be a fixed straight line, and write Δ for

$$\xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{dz};$$

then the pole of (ξ, η, ζ) with respect to (1) is

$$l^2p\Delta U + lm\Delta V + m^2q\Delta V = 0 \dots\dots\dots(2),$$

whose locus is $4pq\Delta U, \Delta V = (\Delta F)^2$, a conic section.

We obtain the locus of centres by simply putting $\xi = \eta = \zeta$.

2. Solution by trilinear coordinates.

The equation to the conic is

$$\frac{l}{a} + \frac{m}{\beta} + \frac{n}{\gamma} = 0 \dots\dots\dots(3),$$

subject to the condition

$$\frac{l}{f} + \frac{m}{g} + \frac{n}{h} = 0 \dots\dots\dots(4).$$

The polar of a point (ξ, η, ζ) with respect to (3) is

$$\frac{a}{\xi} \left(\frac{m}{\eta} + \frac{n}{\zeta} \right) + \frac{\beta}{\eta} \left(\frac{n}{\zeta} + \frac{l}{\xi} \right) + \frac{\gamma}{\zeta} \left(\frac{l}{\xi} + \frac{m}{\eta} \right) = 0.$$

If this coincide with a fixed line $ax + y\beta + z\gamma = 0$, we must have

$$\frac{\frac{m}{\eta} + \frac{n}{\zeta}}{\xi x} = \frac{\frac{n}{\zeta} + \frac{l}{\xi}}{\eta y} = \frac{\frac{l}{\xi} + \frac{m}{\eta}}{\zeta z} = \frac{\frac{l}{\xi} + \frac{m}{\eta} + \frac{n}{\zeta}}{\xi x + \eta y + \zeta z};$$

$$\therefore \frac{\frac{l}{\xi}}{-\xi x + \eta y + \zeta z} = \frac{\frac{m}{\eta}}{\xi x - \eta y + \zeta z} = \frac{\frac{n}{\zeta}}{\xi x + \eta y - \zeta z} \dots\dots\dots(5).$$

Substitute these values in (4), and we have for the equation to the locus

$$\frac{\xi}{f} (-\xi x + \eta y + \zeta z) + \frac{\eta}{g} (\xi x - \eta y + \zeta z) + \frac{\zeta}{h} (\xi x + \eta y - \zeta z) = 0 \dots\dots\dots(6).$$

To find the locus of centres, we may either consider the coordinates trilinear, and put a, b, c for x, y, z ; or we may consider them triangular, and put $x=y=z=1$.

It is clear that by varying the condition (4) we may easily find the locus in other cases. Thus, for instance, "the locus of the centres of all conics passing

through three given points and touching a given straight line ($fa + g\beta + h\gamma = 0$), is the curve of the fourth degree,

$$\sqrt{\{f\xi(-\xi + \eta + \zeta)\}} + \sqrt{\{g\eta(\xi - \eta + \zeta)\}} + \sqrt{\{h\xi(\xi + \eta - \zeta)\}} = 0,$$

the coordinates being triangular." (*Cambridge and Dublin Math. Journal*, Vol. v. p. 148.)

Another solution by trilinear coordinates has been proposed in the *Messenger of Mathematics*, Vol. II. p. 169; we give it here in order to notice one of the theorems which may be deduced from it. Consider the equations in Art. 1 as trilinear; then it may easily be proved that the locus of the pole of a line L , or $\xi x + \eta y + \zeta z = 0$, with respect to all the conics, $lU + mV = 0$, is the Jacobian of U, V, L , that is,

$$\begin{vmatrix} \frac{dU}{dx} & \frac{dU}{dy} & \frac{dU}{dz} \\ \frac{dV}{dx} & \frac{dV}{dy} & \frac{dV}{dz} \\ \xi & \eta & \zeta \end{vmatrix} = 0, \dots\dots\dots (7).$$

But this is precisely the equation which has been elsewhere obtained (see Art. 4, [p. 572, 1418]) as the locus of "polar opposites" of points in the line (ξ, η, ζ) ; that is, the polars of any point in this line with respect to all the conics $lU + mV$ pass through a fixed point in (7). The sides and diagonals of the quadrilateral are evidently cut harmonically by any line and its polar conic; and since the "locus of centres" is the polar conic of the line at infinity, it must coincide with the "nine-point conic," which bisects the sides and diagonals, besides passing through the points E, F, G (Fig. 120). That the conic (7) always does circumscribe the common self-conjugate triangle of U, V , may be shown by putting it in the form

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ \frac{\xi}{x} & \frac{\eta}{y} & \frac{\zeta}{z} \end{vmatrix} = 0, \dots\dots\dots (8).$$

It appears then that the nine-point conic possesses the following property: the polars of any point of it, with respect to all the conics circumscribing the quadrilateral, are parallel. And conversely, all the diameters conjugate to a fixed straight line pass through a fixed point on the nine-point conic.

The "curves of the third class," mentioned at the end of the solution of 1418 [p. 573] as the envelop of lines whose polar conics degenerate, is no other than the three vertices of the common self-conjugate triangle, as readily appears from geometrical considerations. By taking, then, the discriminant of the Jacobian (7) with respect to (x, y, z) , we obtain a contravariant expression for these vertices, which enables us at once to reduce two quadrics to the canonical form.

3. Construction for the directions of the asymptotes. Let $ABCD$ (Fig. 121) be the four given points. Draw any line KL parallel to AB , meeting AD, BC in K, L respectively. Then DL, CK are parallel to the asymptotes of a certain conic through A, B, C, D . This follows immediately from Pascal's theorem. (See Gaskin's *Construction of a Conic Section*, &c., p. 89, Cor. 5.)

4. Construction for the *centre* of the last conic. Describe about $ABCD$ the parallelogram $a\beta\gamma\delta$, having its sides parallel to DL, CK . Let $EFGH$ be the bisections of the sides of $ABCD$.

Then $aE, \beta F, \gamma G, \delta H$ will meet in a point X , which is the centre of the conic.

It may be observed that KL, MN, OP, QR are respectively parallel to BA, AD, DC, CB .

5. Construction for the directions of the axes of the two *parabole* through the four points. This would clearly be accomplished if we could draw KL so that CK, DL should be parallel. Let then the circle through DAB (Fig. 122) meet UBC in S , and the circle through ABC meet UAD in T . Take $UK^2 = US \cdot UC$, and $UL^2 = UD \cdot UT$; then KL is parallel to AB , and DL to CK . By treating V in the same way, we get another direction; but if the quadrilateral be re-entrant, it is easily seen that the construction fails.

This immediately determines the *species* of the conic found in Arts. 1 or 2. For if two parabolas can be described through the four points, the locus of centres must have two points at infinity, that is, it must be an hyperbola. If no parabola can be so described, the locus has no point at infinity; that is, it is an ellipse.

6. We can now easily find a construction for the locus of centres when the quadrilateral is convex.

For the asymptotes are parallel to the axes of parabolas found in Art. 5 and the locus must pass through the intersections of AB, CD , of AD, BC , and of AC, BD , each pair of lines being a conic through the four points. We have then the three finite points, and two points at infinity. Construct for the centre by Art. 4 (hence "if on the three sides of any triangle as diagonals, parallelograms be described, having their sides parallel to two given lines, the other diagonals of these parallelograms will meet in a point"); thus we can draw the axes and asymptotes; construct by Pascal's theorem the points where the curve meets the major axis, and the thing is done. Or, of course, the length of the axis may be found more simply by performing the geometric operations indicated by the equation

$$CA^2 = CN^2 - \left(PN \cdot \frac{CA}{CB} \right)^2;$$

P being one of the three given finite points.

7. It appears from Art. 2 that the locus will break up into two straight lines, if E or F (Fig. 120) be at infinity, that is, if two sides of the quadrilateral are parallel. This is also clear from the fact that, when a conic becomes two parallel straight lines, any point midway between them is a centre. The line at infinity is itself part of the locus when it contains two of the points A, B, C, D . If one of these four is at infinity, only one parabola (or rather two coincident parabolas) can be drawn through them, and the locus of centres is a parabola.

When the quadrangle can be inscribed in two *differ-*
boly, the locus of centres is a circle; and when it can b

the locus of centres is an equilateral hyperbola, whose asymptotes are equally inclined to any pair of opposite sides, and to the two diagonals. If two equilateral hyperbolæ intersect in A, B, C, D , then AB is perpendicular to CD , AC to BD , and AD to BC , each of the points A, B, C, D , being, in fact, the intersection of perpendiculars of the triangles formed by the other three. It is easily seen that the locus of centres is, in this case, the nine-point circle of any of the four triangles ABC, BCD, CDA, DAB . See Note to solution of 1408 [Reprint, Vol. I. p. 28].

8. If we write the equation of the conic in the form $\alpha\beta = \mu\gamma\delta$, it is clear that we shall pass from a possible to an impossible region by changing the sign of one or three of the quantities $\alpha\beta\gamma\delta$. Attention to this and to Fig. [120] will shew that a given sign of μ the curve must lie wholly in the shaded regions, or wholly in the unshaded regions. We proceed to trace the cyclic succession of these curves, beginning with the pair of straight lines AB, CD , considered as the limit of a hyperbola. It is clear that these may separate into a finite hyperbola in two ways; so as to lie in the shaded regions, or in the unshaded regions. We begin with the former. One branch of the hyperbola lies entirely in (8), where also the centre is; the other branch lies in (6), (7), (10), (11), having its infinite parts in (7). The branch in (8), with the centre, moves off rapidly from E , and when the centre is at an infinite distance, we have a parabola in (6), (7), (10), (11), the infinite parts being in (7). When the parabola closes up into an ellipse, the centre reappears from the infinity of (7), and finally passes into (1). The ellipse again elongates itself, but in the direction of (6), into which the centre passes. In the limit we get another parabola, the centre going off to the infinity of (6). As the parabola merges into a hyperbola, the centre reappears from the infinity of (6), and the limit of the hyperbola is the pair of straight lines EA, EB , the centre being at E . We now pass into the unshaded regions, beginning with a small hyperbola, one branch lying in (2), (1), and (4), and the other in (3), (1), and (5). The centre is in (11), and moves down into (1). The parts of the hyperbola in (1) gradually approximate, giving as a limiting form, the lines AC, BD , when the centre is at G . The branches separate again in the other direction, one lying in (2), (1), and (5), and the other in (3), (1), and (4). The centre moves from (1) into (10), and gradually approaches the point E , where the hyperbola again becomes two straight lines. This is the point from which we started. The points E, F, G lie on the same branch of the hyperbola which is the locus of centres, and no part of the locus lies in the regions (2), (3), (4), (5).

The re-entrant quadrangle $ACEF$ may be treated in the same way; this case is simpler, all the conics of the series being hyperbolæ.

9. It may be worth while to notice a property of the nine-point conics of the quadrilateral faces of a tetrahedral frustum. With Prof. Sylvester's own notation, let $Oabc$ be a tetrahedron, the axes of coordinates being Oa, Ob, Oc ; and let the plane $\alpha\beta\gamma$ cut off the frustum $\alpha\beta\gamma abc$. Put $4a$ for Oa , and similarly for the others; and consider the quadrilateral $ab\beta a$ in the plane of $\alpha\beta$. Its nine-point conic is easily found to be

$$\frac{x}{a\alpha} \{x - 2(a + \alpha)\} = \frac{y}{b\beta} \{y - 2(b + \beta)\},$$

and we draw through this a cylinder whose generating lines are parallel to the axis of z . There are three such cylinders, and they evidently have common to them the curve section

$$\frac{x}{a\alpha} \{x - 2(\alpha + a)\} = \frac{y}{b\beta} \{y - 2(b + \beta)\} = \frac{z}{c\gamma} \{z - 2(c + \gamma)\}.$$

We are obviously entitled to conclude that, if we take instead the polar conics of the lines in which the faces are cut by any plane, and draw cones to the points where that plane cuts the opposite edges, these three cones will have a common section.

10. The theorem stated incidentally in Art. 6 is a particular case of Brianchon's theorem. It may be put a little more generally as follows:—

Take two points P , Q , and through each of them draw three straight lines. These triads will intersect in nine points, as in the following scheme,

$$\left. \begin{array}{c} P \\ \overbrace{A \ B \ C} \\ D \ E \ F \\ G \ H \ K \end{array} \right\} Q.$$

Take now three points, one from each of the P -lines, and one from each of the Q -lines, as, for instance, B , F , G . To each pair of these take the opposite diagonal of the quadrilateral, *e.g.*, to BF corresponds CE ; then these three lines CE , DK , AH will meet in a point. There are six such systems of lines.

The points P , Q may be considered as a conic inscribed in the hexagon $CKIHEDA$, of which CH , DK , AH are the diagonals. The theorem is thus seen to be a particular case of Brianchon's. It will be found to involve also the following theorem of determinants: viz., the determinant whose constituents are the nine determinants,

$$\left\| \begin{array}{ccc} C & \frac{a}{b} & A \\ c & \frac{A}{B} & a \end{array} \right\|$$

$$\left\| \begin{array}{ccc} A & \frac{b}{c} & B \\ a & \frac{B}{C} & b \end{array} \right\|$$

$$\left\| \begin{array}{ccc} B & \frac{c}{a} & C \\ b & \frac{C}{A} & c \end{array} \right\|$$

vanishes identically.

1410. [Show that the area of the perspective representation, in a given picture, of a triangle of given area in a fixed plane, varies as the product of the

distances of the angles of the perspective representation from the vanishing line. Proposed by Prof. Sylvester, F.R.S. Reprint, Vol. i. p. 77, E. T. June, 1864.]

Let ABC be the perspective representation of the triangle, DE the vanishing line. Let BC meet DE in D , and join AD . If abc is the triangle represented, AD is the picture of a line through a parallel to bc . If therefore B and C are fixed, the point A can only move along AD . But the area ABC varies as the perpendicular from A on BC , which is in a constant ratio to the perpendicular from A on DE , because A lies on a fixed line through the intersection of BC and DE . Since then when two of the perpendiculars on the vanishing line are fixed, the area varies directly as the remaining one; therefore when all vary, the area varies as the product of the three.

1479. Prove that the ordinary inverse of the *Tangential inverse* is the second positive pedal; and that the *Tangential inverse* of the ordinary inverse is the second negative pedal of the primitive.

[February, 1864. Solved, Reprint, Vol. i. p. 78.]

1497. (1) Given three points by equations of the form $lx + my + nz = 0$, prove that the area of the triangle contained by them is

$$(l_1 m_2 n_3) \div (l_1 + m_1 + n_1)(l_2 + m_2 + n_2)(l_3 + m_3 + n_3),$$

that of the triangle of reference being unity.

(2) Also, if (123) denote the area of the triangle contained by the points 1, 2, 3, and so on, prove that

$$(123)(456) \equiv (156)(423) + (164)(523) + (145)(623).$$

[April, 1864. Solved, Reprint, Vol. i. p. 79; Proposor's solution, Reprint, Vol. iv. p. 53; September, 1865 (?).]

1. The area of the triangle formed by three points vanishes only when they are in a straight line, and becomes infinite only when one of them is at infinity. The condition that they may be in a straight line is

$$J \equiv \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0, \text{ or } (l_1 n_2 n_3) = 0;$$

and the condition that one of them may be at infinity is

$$P \equiv (l_1 + m_1 + n_1)(l_2 + m_2 + n_2)(l_3 + m_3 + n_3) = 0.$$

Now the expression for the area must be of no dimensions in the coefficients, since we are only concerned with their ratios; and the equation obtained by equating this expression to a constant must be of the first order in each set of coefficients; since, two of the points being fixed, the locus of the other is then a straight line. The expression for the area is therefore some numerical multiple

of $\frac{J}{P}$. By putting $x, y, z = 0$ for the three points, we find that the area of the fundamental triangle, on the same scale, is unity.

3. To prove the second proposition, take (450) for the fundamental triangle. Then, by applying the above interpretation to the well-known theorem

$$(l_1 m_2 n_3) - l_1(m_2 n_3) + m_1(n_2 l_3) + n_1(l_2 m_3),$$

we find it equivalent to

$$(123)(450) - (150)(123) + (164)(523) + (145)(623),$$

the factor P dividing out on both sides.

[Solutions were also sent to *E. T.* 1373, 1373, June, 1863; Reprint, Vol. 1, 1300, p. 10; 1366, 1366, p. 9; 1403, p. 10; 1404, 1405, p. 21; 1406, p. 22; 1411, p. 43.]

1506. [If $P, Q, 1, 2, 3, 4$ be points on a conic, then the four points $P1, Q2; P2, Q1; P3, Q4; P4, Q3$ lie on a conic passing through the points P and Q . Proposed by Prof. Cayley. Reprint, Vol. II, pp. 9, 10, *E. T.* July, 1864.]

1. Let the four points $P1, Q2; P2, Q1; P3, Q4; P4, Q3$ be called S, T, U, V respectively; then

$$\{P, 1234\} = \{Q, 1234\};$$

but $\{P, 1234\} = \{P, STUV\},$

and $\{Q, 1234\} = \{Q, TSVU\},$

also $[TSVU] = [STUV]^*$,

therefore $\{P, STUV\} = \{Q, STUV\},$

which proves that the six points P, Q, S, T, U, V lie on a conic.

2. Let $A = 0, B = 0, C = 0, D = 0$, denote respectively the pairs of right lines $(P1, Q1), (P2, Q2), (P3, Q3), (P4, Q4)$. Then we shall prove presently that there is a bilinear relation

$$A + B + C + D = 0,$$

constant multipliers being supposed.

3. The Jacobian of any three of the four conics A, B, C, D is obviously the original conic $PQ1234$, together with the straight line PQ . Now the conic $A + B = 0$ is identical with $C + D = 0$ (by Art. 1); and it passes through all the intersections of A with B , and of C with D . It must therefore be the very conic $PQSTUV$. And there are clearly two more conics, namely, $A + C$ or $B + D$, and $A + D$ or $B + C$, obtained just in the same way. It may be as well to remark that the three are represented by the equation

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} = 0.$$

Moreover, the original conic may be reproduced by treating $STUV$ in the same way as we treated 1234 . We have therefore four conics derived from the two points P, Q in a symmetrical manner. Each of these conics is the Jacobian of the other three; the line PQ being of course added. For the three conics $A + B, B + C, C + A$ have the same Jacobian as A, B, C ; that is, the original conic and the line PQ .

The pole of PQ , with regard to the circle $ABCD$, is the intersection of AD and BC . For the chord AD is divided harmonically by PQ and EF , and the chord BC by PQ and AD . Hence the poles of AD , with regard to any three of the circles, form a self-polar triangle with regard to the fourth. For the poles with regard to AB are B , D , C , A , and the poles with regard to AC , AD , AB , BC , which form a self-polar triangle with regard to BD .

4. By projecting the points P and Q from the same line parallel to itself, we may prove M. Laguerre's theorem, by projecting the two circles from the line AB . (March, 1864, pp. 141, 142, question 108.)

"Two plane circles equidistant from a given circle, whose common tangents touch a third circle, are self-polar with regard to each other, and the circle is the orthogonal circle."

It follows that when four circles are equidistant from a circle, each circle is the orthogonal of any three of the remaining three circles. Hence one of the circles must be orthogonal to the other three.

5. We now proceed to prove the stated theorem.

Lemma. When four circles have the same radical line, then each circle's equation may be written $1, 2, 3, 4$ as the following relations.

Any four circles can be referred to a common radical line AB :

$$\begin{aligned} x_1^2 + y_1^2 + 2x_1 + 2y_1 + 1 &= 0 \\ x_2^2 + y_2^2 + 2x_2 + 2y_2 + 1 &= 0 \\ x_3^2 + y_3^2 + 2x_3 + 2y_3 + 1 &= 0 \\ x_4^2 + y_4^2 + 2x_4 + 2y_4 + 1 &= 0 \end{aligned}$$

where x, y is a coordinate. This can be done by projecting the circles into AB from the determinant $(x_1, y_1, 1, x_2, y_2, 1, x_3, y_3, 1, x_4, y_4, 1)$. In the same way, the radical line

$$\begin{aligned} x_1 - x_2 &= x_3 - x_4 \\ y_1 - y_2 &= y_3 - y_4 \end{aligned}$$

and if these are all identical, we must have $(x_1, y_1, 1, x_2, y_2, 1, x_3, y_3, 1, x_4, y_4, 1) = 0$ on the radical line, and hence AB must be the radical line.

For a metric interpretation, see the diagram above, where AB is the radical line.

1511. If P be the pole of the radical line of the three given circles, and if the general equation of any circle be $x^2 + y^2 + 2x + 2y + 1 = 0$, then the pole of the circle BC is (x_1, y_1) , the pole of the circle AD is (x_2, y_2) , the pole of the circle AB is (x_3, y_3) , the pole of the circle CD is (x_4, y_4) , the pole of the circle AC is (x_5, y_5) , the pole of the circle BD is (x_6, y_6) , the pole of the circle AD is (x_7, y_7) , the pole of the circle BC is (x_8, y_8) , the pole of the circle AB is (x_9, y_9) , the pole of the circle CD is (x_{10}, y_{10}) , the pole of the circle AC is (x_{11}, y_{11}) , the pole of the circle BD is (x_{12}, y_{12}) , the pole of the circle AD is (x_{13}, y_{13}) , the pole of the circle BC is (x_{14}, y_{14}) , the pole of the circle AB is (x_{15}, y_{15}) , the pole of the circle CD is (x_{16}, y_{16}) , the pole of the circle AC is (x_{17}, y_{17}) , the pole of the circle BD is (x_{18}, y_{18}) , the pole of the circle AD is (x_{19}, y_{19}) , the pole of the circle BC is (x_{20}, y_{20}) , the pole of the circle AB is (x_{21}, y_{21}) , the pole of the circle CD is (x_{22}, y_{22}) , the pole of the circle AC is (x_{23}, y_{23}) , the pole of the circle BD is (x_{24}, y_{24}) , the pole of the circle AD is (x_{25}, y_{25}) , the pole of the circle BC is (x_{26}, y_{26}) , the pole of the circle AB is (x_{27}, y_{27}) , the pole of the circle CD is (x_{28}, y_{28}) , the pole of the circle AC is (x_{29}, y_{29}) , the pole of the circle BD is (x_{30}, y_{30}) , the pole of the circle AD is (x_{31}, y_{31}) , the pole of the circle BC is (x_{32}, y_{32}) , the pole of the circle AB is (x_{33}, y_{33}) , the pole of the circle CD is (x_{34}, y_{34}) , the pole of the circle AC is (x_{35}, y_{35}) , the pole of the circle BD is (x_{36}, y_{36}) , the pole of the circle AD is (x_{37}, y_{37}) , the pole of the circle BC is (x_{38}, y_{38}) , the pole of the circle AB is (x_{39}, y_{39}) , the pole of the circle CD is (x_{40}, y_{40}) , the pole of the circle AC is (x_{41}, y_{41}) , the pole of the circle BD is (x_{42}, y_{42}) , the pole of the circle AD is (x_{43}, y_{43}) , the pole of the circle BC is (x_{44}, y_{44}) , the pole of the circle AB is (x_{45}, y_{45}) , the pole of the circle CD is (x_{46}, y_{46}) , the pole of the circle AC is (x_{47}, y_{47}) , the pole of the circle BD is (x_{48}, y_{48}) , the pole of the circle AD is (x_{49}, y_{49}) , the pole of the circle BC is (x_{50}, y_{50}) , the pole of the circle AB is (x_{51}, y_{51}) , the pole of the circle CD is (x_{52}, y_{52}) , the pole of the circle AC is (x_{53}, y_{53}) , the pole of the circle BD is (x_{54}, y_{54}) , the pole of the circle AD is (x_{55}, y_{55}) , the pole of the circle BC is (x_{56}, y_{56}) , the pole of the circle AB is (x_{57}, y_{57}) , the pole of the circle CD is (x_{58}, y_{58}) , the pole of the circle AC is (x_{59}, y_{59}) , the pole of the circle BD is (x_{60}, y_{60}) , the pole of the circle AD is (x_{61}, y_{61}) , the pole of the circle BC is (x_{62}, y_{62}) , the pole of the circle AB is (x_{63}, y_{63}) , the pole of the circle CD is (x_{64}, y_{64}) , the pole of the circle AC is (x_{65}, y_{65}) , the pole of the circle BD is (x_{66}, y_{66}) , the pole of the circle AD is (x_{67}, y_{67}) , the pole of the circle BC is (x_{68}, y_{68}) , the pole of the circle AB is (x_{69}, y_{69}) , the pole of the circle CD is (x_{70}, y_{70}) , the pole of the circle AC is (x_{71}, y_{71}) , the pole of the circle BD is (x_{72}, y_{72}) , the pole of the circle AD is (x_{73}, y_{73}) , the pole of the circle BC is (x_{74}, y_{74}) , the pole of the circle AB is (x_{75}, y_{75}) , the pole of the circle CD is (x_{76}, y_{76}) , the pole of the circle AC is (x_{77}, y_{77}) , the pole of the circle BD is (x_{78}, y_{78}) , the pole of the circle AD is (x_{79}, y_{79}) , the pole of the circle BC is (x_{80}, y_{80}) , the pole of the circle AB is (x_{81}, y_{81}) , the pole of the circle CD is (x_{82}, y_{82}) , the pole of the circle AC is (x_{83}, y_{83}) , the pole of the circle BD is (x_{84}, y_{84}) , the pole of the circle AD is (x_{85}, y_{85}) , the pole of the circle BC is (x_{86}, y_{86}) , the pole of the circle AB is (x_{87}, y_{87}) , the pole of the circle CD is (x_{88}, y_{88}) , the pole of the circle AC is (x_{89}, y_{89}) , the pole of the circle BD is (x_{90}, y_{90}) , the pole of the circle AD is (x_{91}, y_{91}) , the pole of the circle BC is (x_{92}, y_{92}) , the pole of the circle AB is (x_{93}, y_{93}) , the pole of the circle CD is (x_{94}, y_{94}) , the pole of the circle AC is (x_{95}, y_{95}) , the pole of the circle BD is (x_{96}, y_{96}) , the pole of the circle AD is (x_{97}, y_{97}) , the pole of the circle BC is (x_{98}, y_{98}) , the pole of the circle AB is (x_{99}, y_{99}) .

It is required to prove,

(1) That the circle which passes through the points of intersection of the radical lines of the three given circles, and the radical line of the three given circles, is the radical line of the three given circles.

(2) That the circle which passes through the points of intersection of the radical lines of the three given circles, and the radical line of the three given circles, is the radical line of the three given circles.

(7) That the self-conjugate and circumscribing circles of the triangle, the circle which bisects its sides, the circle upon PG as diameter, the circle Σ , and the director of the maximum ellipse that can be inscribed in the triangle, all pass through the same two points, real or imaginary. [Proposed by J. Griffiths, M.A. Reprint, Vol. II, p. 37, E. T. August, 1884.]

$$\text{Let} \quad U = a^2yz + b^2zx + c^2xy, \\ V = (x + y + z)(b \cos A + x + ca \cos B + y + ab \cos C + z);$$

then $U + kV$ represents respectively (the coordinates being triangular)

- (1) when $k = \frac{1}{2}$, the *nine-point* circle of the triangle of reference;
- (2) when $k = 0$, the *circumscribing* circle;
- (3) when $k = 1$, the *self-conjugate* circle;
- (4) when $k = \frac{1}{2}$, the *director of the maximum ellipse*;
- (5) when $k = -\frac{1}{2}$, the *circle on PG as diameter*.

The last two are the only ones which present any difficulty.

In (4) the tangential equation of the ellipse and the circular points at infinity are respectively

$$yz + zx + xy = 0,$$

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bc \cos A + yz - 2ca \cos B + zx - 2ab \cos C + xy = 0.$$

Forming, by the rule, the *harmonic curve* of these two, we at once write down equation (1). To find equation (5), we make use of the following

THEOREM. If $A = 0$, $B = 0$, $C = 0$, $D = 0$ are any four lines in a plane, and if $\Phi, (AB) = 0$ denote the condition that A and B may be at right angles, then, the equation to the circle whose diameter is the line joining (AB) , (CD) is

$$\begin{vmatrix} \Phi, AC & \Phi, AD & A \\ \Phi, BC & \Phi, BD & B \\ C & D & 0 \end{vmatrix} = 0.$$

In the present case, take the four lines

$$A = x + y = 0;$$

$$B = y + z = 0;$$

$$C = bz \cos A + x + ca \cos B + y = 0;$$

$$D = cx \cos B + y + ab \cos C + z = 0;$$

then the condition of perpendicularity of $(l_1m_1n_1)$, $(l_2m_2n_2)$ being

$$a^2l_1l_2 + b^2m_1m_2 + c^2n_1n_2 - bc \cos A (m_1n_2 + m_2n_1) - \Delta \sin A = 0,$$

we have

$$\Phi, AC = \Phi, AD = abc (a \cos A + b \cos B + c \cos C) + 2abc + a \sin B \sin C,$$

$$\Phi, AB = \Phi, BC = -a^2c + b \sin C \sin A + \frac{1}{2} \Phi, CC.$$

Expanding then the determinant, it becomes simply

$$3 (AC + BD) + AD + BC = 0,$$

and, substituting the values of A , B , C , D , we get equation (5).

1486. If two transversals ABC , DEF cut the sides of any triangle, then AE , BF , CD cut the sides in three points on a straight line X . If moreover the triangle touch a cubic at A , B , C , and cut it in D , E , F , the lines AE , BF , CD meet the curve in three points on a straight line Y , and the lines X , Y , ABC , meet in a point.

[E. T. March, 1864. Solved, Reprint, Vol. II. p. 40.]

1519. [ABC is a triangle having the three real points (P , Q , R) of inflexion of a cubic on the sides BC , CA , AB respectively, each of which also passes through two imaginary points of inflexion. The tangents at Q and R meet in D , those at R and P in E , and those at P and Q in F . Shew that AD , BE , CF meet in a point which is fixed for all the cubics having the same nine points of inflexion. Proposed by F. D. Thomson, M.A. Reprint, Vol. II. p. 48, N. T. September, 1864.]

Taking ABC for triangle of reference, the equation of the cubic is

$$a^3x^3 + b^3y^3 + c^3z^3 - 3dxyz = 0,$$

which may also be written in the form

$$\left(\frac{d}{bc}x + by + cz\right)\left(\frac{d}{bc}x + \theta by + \theta^2cz\right)\left(\frac{d}{bc}x + \theta^2by + \theta cz\right) = \frac{d^3 - a^3b^3c^3}{b^3c^3}x^3,$$

(where θ is an imaginary cube root of unity) showing that the tangents at P , Q , R are

$$\frac{d}{bc}x + by + cz = 0,$$

$$ax + \frac{d}{ca}y + cz = 0,$$

$$ax + by + \frac{d}{ab}z = 0.$$

The equation to AD is therefore $by = cz$, so that AD , BE , CF meet in the point $ax = by = cz$.

Now all the points of inflexion are on the axis $xyz = 0$; consequently any other cubic having the same points of inflexion can only differ from the above in the coefficient of xyz , of which the point $ax = by = cz$ is independent.

This point is the pole of the line PQR with respect to the triangle ABC .

1517. [If each edge of a tetrahedron is perpendicular to the non-conterminous edge (it being observed that if two pairs of such edges be mutually perpendicular, the third pair will be so too); prove that the nine-point circles of the three triangular faces lie on a sphere; also that the nine-point circle of any triangular face, and the three points of intersection of the perpendiculars of the other three triangular faces, lie on a sphere; and find the equations of all these spheres. Proposed by H. R. Greer, M.A. Reprint, Vol. II. p. 70, November, 1864.]

The triangular equation of the nine-point circle is

$$a^2yz + b^2zx + c^2xy = 2(x + y + z)(bc \cos A \cdot x + ca \cos B \cdot y + ab \cos C \cdot z) \dots (I).$$

1. two opposite edges of a tetrahedron are perpendicular, a plane through either perpendicular to the other, and will therefore perpendiculars from the extremities of the first upon the second.
 2. if a tetrahedron has *two* pairs of perpendicular edges, the perpendiculars from the vertices to opposite faces will meet in a point, and the foot of one will be the intersection of perpendiculars of the face in which it lies. It is obvious that, if $ABCD$ be such a tetrahedron,

then $BD \perp AC$, $AD \perp BC$, $CD \perp AB$, $AD \perp BC$, $AB \perp CD$, $AC \perp BD$, suppose.

It is at once that the nine-point circles of the four faces lie on a sphere; and the equation of this, the "twenty-four-point sphere," is

$$x^2 + y^2 + z^2 + w^2 - 3(x + y + z + w) \{ (A)x + (B)y + (C)z + (D)w \} + \dots (2).$$

Now to the sphere which contains the nine-point circle of the face a , the center of the other three faces, is got by changing the sign of (A) in (2). The equation of the self-conjugate sphere is

$$x^2 + y^2 + z^2 + w^2 - 3(x + y + z + w) \{ (A)x + (B)y + (C)z + (D)w \};$$

and, passing through the intersection of the circumscribed and the pedal spheres.

With the middle point of each edge of a tetrahedron a line be drawn to the opposite edge, the tetrahedron will be reproduced in an inverted position. In the present case, the two tetrahedra will have the same twenty-four-point sphere, and the sphere self-conjugate to one will circumscribe the

former in his solution refers to an article on this species of tetrahedron, Wolstenholme (*Quarterly Journal of Mathematics*, Vol. III.)]

Required a *direct* proof that an ellipse and its osculating circle have contact of the third order at the ends of its axes; also prove that the deviations of the circle osculating it most closely at the ends of its axes are as inversely as the seventh powers of the axes. Proposed by Matthew A. Reprint, Vol. II, pp. 82-86, December, 1861.]

know that if the osculating circle at a point P meet the ellipse again at the tangent at P are equally inclined to the axes. (Salmon's *Conics*, p. 244; Taylor's *Geometrical Conics*, p. 85). This shows that the equation of the osculating circle at (ξ, η) may be written in either of the forms

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) - \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \left(\frac{x\xi}{a^2} + \frac{y\eta}{b^2} - 1 \right) \left\{ \xi \left(\frac{x-\xi}{a^2} - \frac{\eta}{b^2} \right) \right\} \dots (1),$$

$$+ \frac{\eta^2}{b^4} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) = \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \left\{ \xi^2 \left(\frac{x-\xi}{a^2} - \frac{\eta}{b^2} \right)^2 - \frac{\eta^2}{b^4} (y-\eta)^2 \right\} \dots (2).$$

If $\eta = \pm b$, (3) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \left(\frac{1}{a^2} - \frac{1}{b^2} \right) (y-\eta)^2 = 0 \dots (3).$$

If $\xi = \pm a$, (2) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \left(\frac{1}{a^2} - \frac{1}{b^2} \right) (x-\xi)^2 = 0 \dots (4).$$

whence, putting $a^2 + b^2 = c^2$, we readily obtain the equation of the osculating circle in the form

$$x^2 + y^2 + 2c^2 \left(\frac{x}{a} \cot^3 \phi + \frac{y}{b} \sin^3 \phi \right) + c^2 \sin^3 \phi + b^2 \cos^3 \phi + c^2 \cos 2\phi, \dots (11).$$

4. If we put $x = x_1 + \alpha$, $y = y_1 + \beta$, the equation of the tangent will still be of the form

$$l(x_1 + \alpha) + m(y_1 + \beta) = \sqrt{(l^2 a^2 + m^2 b^2)}, \dots (12),$$

and the line through (α, β) perpendicular to this is clearly $\frac{x_1 - \alpha}{l} = \frac{y_1 - \beta}{m}$.

The locus of the foot of this perpendicular is therefore obtained by writing (x_1, y_1) for (l, m) in (12).

Hence, if the equation of the tangent to any curve can be put in the form

$$lx + my = F(l, m),$$

then the equation of the pedal with the point (α, β) for origin is

$$x(x - \alpha) + y(y - \beta) = F(x + \alpha, y + \beta) \dots (13).$$

For instance, the locus of the foot of the perpendicular from the centre of the ellipse on the common chord (7) is

$$(x^2 + y^2)^2 (a^2 x^2 + b^2 y^2) = (a^2 x^2 + b^2 y^2)^2 \dots (14).$$

All that has been here said may be applied to the hyperbola by changing the sign of b^2 .

5. The first pedal of the cycloid may be simply obtained by the method of Art. 4. For let the circle begin to roll on the axis of x at the origin, and consider the tangent at a point corresponding to a revolution ϕ of the circle. If its equation be written

$$lx + my = F(l, m),$$

we must have

$$\phi = 2 \cot^{-1} \left(\frac{l}{m} \right),$$

now the tangent passes through the point $2x = a\phi$, $y = a$, where a is the diameter of the circle; whence the equation may be written

$$lx + my = a \left(m + l \cot^{-1} \frac{l}{m} \right),$$

and we immediately get the pedal with origin (α, β) ; viz.,

$$x(x - \alpha) + y(y - \beta) = a \left\{ (y - \beta) + (x - \alpha) \cot^{-1} \frac{x - \alpha}{y - \beta} \right\}.$$

Putting $\alpha = 0$ and $\beta = 0$, we get the pedal from the origin, viz.,

$$x^2 + y^2 = a \left(y + x \cot^{-1} \frac{x}{y} \right),$$

or, in polar coordinates,

$$r = a (\sin \theta + \theta \cos \theta).$$

1468. Given the centre of a conic, and a conjugate triad; to construct for the directions of the asymptotes. [Proposed, *E. T.* January, 1864: solution, February, 1865.]

[If O be the given centre of the conic; A, B, C the three points of the self-conjugate triad; and OX, OY, OZ the three lines through O parallel to BC, CA, AB respectively; the two double rays (real or imaginary) OM and ON of the involution determined by the three angles AOX, BOY, COZ are the two asymptotes required. For the three pairs of conjugates, OA and OX, OB and OY, OC and OZ , determining the involution, being evidently pairs of conjugate diameters of the conic, divide, therefore, harmonically the angle (real or imaginary) MON determined by the two asymptotes OM and ON .]

The same construction (with some slight and obvious modifications) applies also to the following more general problem, of which the above is evidently a particular case: viz., given a point and a line, pole and polar with respect to a conic and a conjugate triad; to construct the two tangents (real or imaginary) from the point to the curve, and the two intersections (real or imaginary) of the line with the curve. For if P and L be the point and line; A, B, C , as before, the three points of the triad; X, Y, Z and X', Y', Z' the six intersections of L with BC, CA, AB , and with PA, PB, PC , respectively; then, as X and X', Y and Y', Z and Z' are evidently pairs of conjugate points with respect to the conic, the two double points M and N of the involution determined by the three segments XX', YY', ZZ' , as cutting them all harmonically, are the two intersections required; and as PX and PX', PY and PY', PZ and PZ' , are evidently pairs of conjugate lines with respect to the conic, the two double rays PM and PN of the involution determined by the three angles XPX', YPY', ZPZ' , as cutting them all harmonically, are the two tangents required.

The two corresponding problems in geometry of three dimensions, viz., Given, of a quadric, the centre and a self-conjugate tetrahedron, to construct the asymptotic cone of the surface; or, more generally: Given, of a quadric, a point and plane, pole and polar to each other, and a self-conjugate tetrahedron, to construct the tangent cone from the point to the surface, and the conic of intersection of the plane with the surface, may be readily solved by application of the above.]

Corollary i. Let any two straight lines parallel to two conjugate diameters be called *conjugate* with respect to a conic; then it is shown above that the pairs of lines 12, 84; 13, 24; 14, 23, joining the points 1234, are conjugates with respect to the conic which has the point 1 for a centre, and 234 for a conjugate triad. But the symmetry of this statement shows that they are also conjugates with respect to the conic which has any other of the five points for centre, and the remaining three for a conjugate triad. We may draw four such conics; and since the asymptotes are determined in direction by two pairs of conjugates, it follows that these four conics are all similar and similarly situated. So, in the more general case, we shall have four conics intersecting in two points on the given straight line.

Corollary ii. Let a straight line and plane, drawn parallel to any diameter and its conjugate diametral plane, be called *conjugates* with respect to a conic.

coid. Then if we are given five points 12345, of which 1 is the centre, and 2345 a self-conjugate tetrahedron of a given conicoid, it is evident that since 2 is the pole of the plane 345, 12 is conjugate to 345, and so on. We thus get four pairs of conjugates. Again, since 23 is the polar line of 45, 123 is conjugate to 45, and 145 to 23, and so on. This gives us six more pairs of conjugates. But this amounts to saying that if we join any three of the five points by a plane, and the other two by a line, the line and plane are conjugate. This statement makes no mention of the particular point taken for centre; and we conclude as before, that if five conicoids are drawn, by taking each of five points in succession for centre, and the remaining four for a self-conjugate tetrahedron, these five conicoids will be similar and similarly situated. A line and its conjugate plane cut the plane at infinity in a point and line which are pole and polar with respect to the section which the plane at infinity makes of the conicoid. The problem is therefore equivalent to that of describing a conic, being given the poles of certain lines. Three points and their polars are sufficient to determine a conic; for let A, B, C be the points, and let AB meet the polar of A and B in P, Q respectively. Then the foci of the involution determined by AP, BQ , are evidently points on the conic. In this way we can determine six points on the sides of the triangle ABC , and six more on the sides of the reciprocal triangle; and it would be interesting to prove *a priori* that these twelve points must lie on the same conic, when the triangles are in perspective.

Corollary III. In the plane case we are given three pairs of conjugates to determine two points at infinity; and we conclude that any transversal is cut in involution by the six lines joining four points. Similarly we conclude from the solid problem that "if five points in space are joined every way by ten lines and ten planes, the system will be cut by any plane in a system of points and lines which are poles and polars with respect to a certain conic." The analogy of this relation of points and lines with involution, may be illustrated analytically. Let $U^2 = ax^2 + by^2 = 0$ be a pair of points; then if we put Δ for

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} \right),$$

a point and its harmonic conjugate will be represented by the equations

$$\begin{vmatrix} x & y \\ \xi & \eta \end{vmatrix} = 0, \text{ and } \Delta U = 0.$$

And a system of such harmonic conjugates is of course a system in involution. Next let U represent a conic, and Δ stand for

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{dz} \right),$$

then a point and its polar will be represented by the equations

$$\begin{vmatrix} x & y & z \\ \xi & \eta & \zeta \end{vmatrix} = 0, \text{ and } \Delta U = 0,$$

and the analogy is obvious.

Corollary IV. Lastly, the four conics mentioned in Cor. I. are all similar to the nine-point conic of the quadrangle, or locus of centres of all conics through

the four points. This proposition was set in a problem paper, at St John's College, Cambridge, in Dec. 1862; but I do not know to whom it is due.

It follows at once from the equation to the nine-point conic given in Art. 2 of the solution to Question 1443 [Reprint, Vol. i. p. 51, *supra*, p. 580]; for an equation of the second degree in $x, y, 1$, in which the coefficient of xy is zero, obviously represents a conic with respect to which the axes are conjugates. Thus the lines 12, 34; 13, 24; 14, 23, are conjugates with respect to the nine-point conic, and therefore its asymptotes are parallel to those of the other four. These, therefore, are ellipses when the quadrangle is re-entrant, and hyperbolas when it is convex.

[Reprint, Vol. III, pp. 35, 36. Prof. Townsend and the proposer are credited with the solution. I felt sure that the Corollaries were Clifford's work, and on asking Mr Miller I find that the part enclosed in brackets above was due to Prof. Townsend, and that Clifford instead of having his solution, a short one, printed, added the above Corollaries to the "proof."]

1652. Through the angles A, B, C of a plane triangle three straight lines Aa, Bb, Cc are drawn. A straight line AR meets Cc in R ; RR meets Aa in P ; PC meets Bb in Q ; QA meets Cc in r ; and so on. Prove that, after going twice round the triangle in this way, we always come back to the same point.

Shew that the theorem is its own reciprocal. Find the analogous properties of a skew quadrilateral in space, and of a polygon of n sides in a plane. [*E. T.* February, 1865. Reprint, Vol. III. p. 66, *E. T.* April, 1865, which contains a solution by Prof. Cayley.]

Let x, y, z be the sides of the triangle ABC , and let $ay=z, bz=x, cx=y$ be the three lines drawn through them. Start with the line AR or $y=z$, which meets Cc or $cx=y$ on $cx=z$, which meets $ay=z$ on $cx=ay$, which meets $bz=x$ on $cbz=ay$, which meets $cx=y$ on $bz=ax$, which meets $ay=z$ on $by=x$, which meets $bz=x$ on $y=z$; so that we have come round again. The extension of this is most easy; I write down two enunciations:—

Consider a plane polygon of an odd number of sides; let the two sides adjacent to any given side be produced to meet, and through their intersections let an arbitrary line be drawn; then treating these lines in the same way as Aa, Bb, Cc were treated in the case of the triangle, we may go twice round the polygon, and shall always come back to the same point.

Let $ABCD$ be a skew quadrilateral in space, and through the four sides AB, BC, CD, DA let arbitrary planes be drawn; let any line through A meet the plane through CD in a ; aB meets the plane DA in b ; and so on; after going three times round the quadrilateral we shall come back to the same point.

The theorem is not true for a plane polygon of an even number of sides; I have not been able to find an analogue in this case.

1679. [To find the envelope of the straight line joining the feet of the perpendiculars drawn on the sides of a triangle from a point in the circumference of the circumscribed circle. Proposed by H. R. Greor, B.A. Reprint, Vol. III. pp. 81, 2, *E. T.* May, 1865.]

The line in question is known to be the tangent at the vertex of a parabola which touches the sides of the triangle. Now this straight line, being always at right angles to the axis of the parabola, determines on the line at infinity a series of points in involution with the series determined by the parabola itself; we have then a series of conics touching four given lines, and a series of points on one of the lines, homologous with the series of conics; and we want to find the envelope of the remaining tangent, drawn from each point to its corresponding conic. Let then $U = AP$ be the tangential equation of the series of conics, and $V = A'Q$ of the series of points. We obtain the required envelope by eliminating A ; it is $UV = VP$, a curve of the third class touching the common tangents of U and V , and the line PQ . When, as in the case we are considering, the line PQ coincides with one of the common tangents of U , V , then it is a double tangent to the curve $UV = VP$, and the points of contact are the double points of the involution; in this case, the cusp-points at infinity. Since the curve is of the third class, and has one double tangent (that is, all it can) it is of the fourth order; and because the double tangent has imaginary contacts, the curve has three real cusps. To determine the position of these cusps, and the general form of the curve, we have to study a most singular figure.

Consider four points, 1, 2, 3, 4, such that each is the intersection of perpendiculars of the triangle formed by the other three. About the triangles 234, 341, 412, 123 describe circles; it is known that these circles are all equal, and that their centres 1', 2', 3', 4' form another quadrangle, exactly similar and equal to 1 2 3 4, but in an inverted position, their centre of (inverse) similitude being the centre of the nine-point circle. Now suppose that the feet of the perpendiculars from any point in the circle 234 to the sides of the triangle 234 are joined by a line X . Then I say that *if at the points where the line X cuts the six connectors of the quadrangle 1234, perpendiculars be drawn to these six connectors respectively, the perpendiculars will concur three by three, in four points, 1'', 2'', 3'', 4'', situate one on each of the four circumscribing circles, and forming a quadrangle equal, similar, and similarly situated to 1' 2' 3' 4'. And the centre of (inverse) similitude of 1 2 3 4 and 1'' 2'' 3'' 4'' is situated on the line X , and bisects the segment determined on it by any pair of connectors.* Hence we see (1), that the line X is connected with the whole quadrangle, and not with three particular points of it; (2), it is cut by the connectors in an involution, one double point of which is at infinity; and then four is an *asymptote* of some circle passing through the points 1, 2, 3, 4.

Now, take any connector 1 2, and find a point on it, symmetrical in respect of 1, 2, with the point where it is cut by 34. Then the envelope of X touches all the connectors at the points thus determined.

While writing the above, I have read a paper on the subject by Steiner, in the fifth volume of the *Berlin Journal* *. He asserts that the curve is a hypocycloid of three branches, and gives a simple construction for the cusps.

The property of a quadrangle mentioned above, is in fact this:—If four parabolas be drawn, having their axes parallel, each inscribed in one of the four triangles determined by a quadrangle, these four will have a common tangent:

* An abridgement of Steiner's paper is printed on pp. 37-39 of this Vol. *etc.* of the *Reprint*.]

which is at once seen to be a particular case of the reciprocal of this;—The four circles, each circumscribing one of the triangles determined by a quadrilateral, have a common point. And this again is a particular case of that wonderful proposition, the involution of cubics:—All the cubics which pass through eight fixed points pass also through a ninth point.

Finally, reciprocate the whole figure in respect of the self-conjugate circle of any of the triangles 234, &c. We then get the locus of a point where the normal at (1) meets again a rectangular hyperbola circumscribing the quadrangle; it is a cubic having its asymptotes parallel to the sides of 234, and with a double point at (1), the tangents to which are the asymptotes of the polar circle. In fact, this problem is rather easier than its reciprocal.

1680. [(1) Prove that the envelope of the asymptotes of a rectangular hyperbola described about a given triangle is a curve of the third class, touching the sides of the triangle, the three perpendiculars, lines through the feet of the perpendiculars parallel to the opposite sides of the triangle formed by joining them, and also the line at infinity.

(2) Prove that the envelope of the asymptotes of a conic inscribed in a given quadrilateral, is a curve of the third class touching the sides and diagonals of the quadrilateral, the line at infinity, and the line joining the middle points of the diagonals. Proposed by F. D. Thomson, M.A. Reprint, Vol. III. pp. 82, 3, E. T. May, 1865.]

(1) It is shewn in the solution of 1679 that the line whose envelope is there considered is an asymptote of *some* rectangular hyperbola circumscribing the quadrangle; whence the two envelopes must be identical. This may also be proved thus: the proposition is that a rectangular hyperbola may circumscribe any triangle which circumscribes a parabola, and have for an asymptote the tangent at the vertex of the parabola. Let β be the axis of the parabola, α the tangent at its vertex, γ the line at infinity; then the respective equations to the hyperbola and parabola are

$$\gamma^2 + 2p\alpha\gamma = 2\mu\alpha\beta, \quad \beta^2 = 2\lambda\gamma\alpha;$$

whence

$$\Theta = -p^2, \quad \Theta' = 2p\lambda, \quad \Delta = -\mu^2, \quad \Delta' = -\lambda^2,$$

and the condition $\Theta'^2 = 4\Theta\Delta'$ is satisfied. In fact, the triangle ($\alpha\beta\gamma$) is inscribed in the hyperbola, and circumscribes the parabola.

Hence (i) the envelope of the asymptotes of all conics through four given points is a three-cusped quartic touching the six connectors of the given points, and the line at infinity at the points of contact of the parabola through them. (ii) If two tangents to a three-cusped quartic divide harmonically the double tangent, their intersection lies on a conic through the points of contact of the double tangent. This conic touches the quartic in three points. (iii) If a chord of a nodal cubic subtend harmonically the double point, its envelope is a conic touching the tangents at the double point, and the curve itself in three points.

M. Chasles gets the result (i) by his method of characteristics (Theor. xvi.). The envelope of the asymptotes is in general of class $\mu + \nu$, and has a ν -ple tangent at infinity; where μ is the number of conics of a system that can be

drawn through a given point, and x the number that can be drawn to touch a given line.

(2) Here again M. Chasles's method shows that the envelope is of the third class, and touches once the line at infinity. Let U, V be two inscribed conics, and (ξ, η, ζ) the coordinates of the line at infinity; and write also Δ for $(\xi_2 + \eta_2 + \zeta_2)$; then a conic of the system is $U + kV$, the centre $\Delta U = k\Delta V$, and the envelope required $U\Delta V = V\Delta U$, which is of the third class, touching the sides of the quadrilateral, and the line $\Delta U = 0, \Delta V = 0$, which joins the middle points of the diagonals. If for U, V we write AB, CD , the equation is

$$AB(U + \Delta D + D + AC) = CD(U + \Delta B + B + \Delta A),$$

showing that the curve touches the lines $(A = 0, B = 0)$ and $(C = 0, D = 0)$; that is the diagonals of the quadrilateral.

1731. The equations of three conics being given in the following forms, viz.,

$$a_1x^2 + b_1y^2 + c_1z^2 + d_1xyz = 0,$$

$$a_2x^2 + c_2z^2 = 0,$$

$$a_3x^2 + c_3z^2 = 0,$$

where

$$x + y + z + w = 0,$$

show that a straight line $(\xi x + \eta y + \zeta z + w = 0)$ will be cut in involution by them, if

$$2 \cdot \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} + (\xi - \eta)(\xi - \zeta)(\xi - w) + (\text{to four terms}) = 0.$$

[Proposed, *P. T. Math.*, 1903. Reprint, Vol. iv, p. 53.]

1732. To find the area of a triangle, the equations of whose sides in trilinear coordinates are

$$L_1x + m_1y + n_1z = 0, \quad L_2x + \dots = 0, \quad L_3x + \dots = 0.$$

Proposed by W. A. Whitworth, M.A. Reprint, Vol. iv, pp. 55, 6, *E. T.* September, 1905.]

Call the three lines 1, 2, 3. Then we have to find the area of the triangle included by the points $(21), (31), (12)$; that is, by the points

$$L_1x + M_1y + N_1z = 0, \quad L_2x + \dots = 0, \quad L_3x + \dots = 0;$$

where L_1, \dots are the first minors of the determinant $(L_1m_2n_3)$. But by the Solution to 1407 [p. 524], this area is a numerical multiple of $\frac{J}{J_1}$, where

$$J = (L_1M_2N_3) - (L_1m_2n_3)^2, \\ J_1 = (aL_1 + bM_1 + cN_1)(aL_2 + \dots)(aL_3 + \dots)$$

in the Trilinear system.

In the case of the fundamental triangle we find that $\frac{J}{J_1} = \frac{1}{abc}$. Hence the ratio has the value given in the foregoing solution* (and the area $\frac{abc}{\Delta} = J_1/J$).

* (Two other modes of solution were printed.)

The general formula for all systems (see my paper on "Analytical Metrics" [p. 90 supra]), is

$$\frac{\{J(ABC)\}^3}{J(BC\infty) \cdot J(CA\infty) \cdot J(AB\infty)},$$

where $J(ABC)=0$ is the condition that the lines $A, B, C,=0$ may meet in a point, and $J(BC\infty)=0$ is the condition that B and C may be parallel. The ratio to the area of the fundamental triangle may easily be found in any particular case by the method used above.

In the same way it may be shown that the volume of a tetrahedron is

$$\frac{\{J(ABCD)\}^3}{J(BCD\infty) \cdot J(CDA\infty) \cdot J(DAB\infty) \cdot J(ABC\infty)},$$

where $J(ABCD)=0$ is the condition that the planes $A, B, C, D,=0$ may meet in a point, and $J(BCD\infty)=0$ is the condition that B, C, D may be parallel to the same line.

In quadriplanar coordinates, for instance, if $\alpha, \beta, \gamma, \delta$, denote the areas of the faces of the fundamental tetrahedron, the equation to the plane at infinity is

$$\alpha x + \beta y + \gamma z + \delta w = 0 \dots\dots\dots (1),$$

and the above expression for the volume, if calculated by means of (1), must be multiplied by the product $\alpha\beta\gamma\delta$ to give the ratio of the volume of the given tetrahedron to that of the fundamental one.

1732. [Prove that the characteristics of a system of conics, satisfying four conditions, remain unaltered when, in place of passing through a given point, each conic is required to divide a given finite segment harmonically. Proposed by T. A. Hirst, F.R.S. Reprint, Vol. iv. pp. 56, 7, *Ed. T.* September, 1865.]

In a system of conics satisfying four conditions (Z, Z', Z'', Z''') let μ be the number of conics that pass through an arbitrary point, and ν the number that touch an arbitrary line. Suppose that the polars of a point P , in respect of all the conics of the system, envelope a curve of class α . Then from the point P , α tangents can be drawn to the curve, that is to say, there are α polars of P which pass through P . But a point which lies on its polar in respect of a given conic is a point on the conic. Therefore α conics pass through P . But μ conics (by hypothesis) pass through P ; so that $\alpha=\mu$. Thus we get Chasles's Prop. xii., —the polars of an arbitrary point, in respect of a system of conics (μ, ν) envelope a curve of class μ . It follows that there are μ polars of P which pass through another arbitrary point Q ; that is to say, there are μ conics of the system which divide harmonically a given segment PQ . This is Chasles's Prop. xxviii.

Suppose now that the condition Z''' is that the conics shall pass through a given point. Call S the condition that they shall divide harmonically the segment PQ . Then (by the above) the number of conics satisfying the conditions (Z, Z', Z'', S), and passing through the given point, is μ ; that is to say, the first characteristic (μ') of the system (Z, Z', Z'', S) is equal to the first characteristic (μ) of the system (Z, Z', Z'', Z''') where Z''' is the condition of passing through a given point. In the next place let Z''' be the condition of touching a given line. Then the number of conics which satisfy the conditions

(Z, Z', Z'', Z''', S) be the same as the number which satisfy the conditions (Z, Z', Z'', S'' , point); that is to say, the *second characteristic* (v') of the system (Z, Z', Z'', S) is equal to the *second characteristic* (v) of the system ($Z, Z', Z'',$ point). Thus neither of the characteristics is altered when we substitute for the condition of passing through a given point, the condition S of dividing harmonically a given segment PQ .

By similar reasoning it may be shown that neither characteristic is altered when we substitute for the condition of touching a given line, the condition of subtending harmonically a given angle.

1759. Given four straight lines whose equations are connected by the equation $x^2 + y^2 + z^2 + w^2 = 0$; show that the straight lines

$$L = my + nz + uw = 0, \quad X = py + rz + uw = 0,$$

will be conjugates in respect of any conic touching (x, y, z, w) , if

$$(p + 1)(x + z) + (m + 1)(x + z) + (n + 1)(x + z) + (w + 1)(x + z) = (p + 1)(x + z) + (m + 1)(x + z).$$

Show also that if a quadrangle be formed from the quadrilateral by taking the pole of each line in respect of the triangle formed by the other three; then the relation between the two figures will be reciprocal; and if two straight lines be conjugates in respect of any conic inscribed in the quadrilateral, their poles in respect of the common connection-triangle will be conjugates in respect of any conic circumscribing the quadrangle.

[*E. J. July, 1905. — Solution Reprint, Vol. iv, p. 63.*]

1775. If a straight line meet the faces of the tetrahedron $ABCD$ in the points a, b, c, d , respectively; the spheres whose diameters are Aa, Bb, Cc, Dd have a common real axis.

[*E. J. August, 1905. — Solution Reprint, Vol. iv, pp. 63–64.*]

1825. If a triangle abc be the reciprocal of $A'B'C'$ in respect of a parabola whose parameter is $4a$, and if u_1, u_2, u_3 be the normals at the vertices of diameters through $A'B'C'$, then

$$\frac{(\text{area of } abc)^2}{(\text{parameter})^2} = \frac{2a^2}{u_1 u_2 u_3} (\text{area of } A'B'C').$$

[*E. J. March, 1905. — Solution Reprint, Vol. iv, pp. 90, 1.*]

1831. The conicoids which pass through six fixed points in space, intersect one plane in a series of conics having a constant self-conjugate quadrilateral. Any four conics have a common self-conjugate quadrilateral. (Def. A quadrilateral is self-conjugate in respect of a conic which divides its diagonals harmonically.)

[*E. J. October, 1905. — Solution Reprint, Vol. iv, p. 110.*]

1856. (1) Let P be the point in a homogeneous triangular lamina ABC at which the sides subtend equal angles. Show that if the lamina be placed in a smooth plane spheroid whose long axis is vertical, it will rest in equilibrium where the point P coincides with the lower focus of the spheroid.

such conic when the cubic has a node, and none when it has a cusp. Proposed by E. de Jonquieres. *Hopaint*, Vol. v, p. 50, E. T. March, 1866.]

1. Let A be the given point on the cubic, and let F be any point of inflexion, or flex. Join AF , and let AF meet the curve again in B . Then a conic may be drawn having three-point contact with the cubic at the points A and B . For, consider these three cubic curves: (a) the cubic itself; (b) the line ABF taken three times over; (c) a conic having three-point contact at A and touching the cubic at B , together with the tangent at the flex F . The cubic (a) passes through eight out of the nine points of intersection of the cubics (a) and (b); consequently, by the theorem known as the *involution of cubics*, it passes through the ninth point. That is to say, a conic having three-point contact at A , and touching the cubic at B , will necessarily have three-point contact at B .

By joining the point A , therefore, to the nine flexes F , we shall obtain nine points B , and therefore nine conics fulfilling the required conditions; but only three of these points B will be real when the point A is real.

It remains to show that a conic having three-point contact at A passes through the three real points B . Let F_1, F_2, F_3 be the three real flexes, which are known to be in one straight line; and let B_1, B_2, B_3 be the corresponding points B . Draw a conic C having three-point contact at A and passing through B_1, B_2 . Then consider these three cubic curves: (a) the cubic itself; (b) the straight lines $AB_1F_1, AB_2F_2, AB_3F_3$; (c) the conic C and the line $F_1F_2F_3$. The cubic (a) passes through eight out of the nine intersections of the cubics (a) and (b), consequently it passes also through the ninth. That is to say, the conic C passes through the point B_3 .

A cusped cubic has only one flex; in this case, therefore, the number of conics is reduced to one.

2. Let A be the given point. By Charnot's Theorem (which again is a particular case of the *involution of cubics*), if a conic have four-point contact with the cubic at A , its remaining chord of intersection with the cubic will pass through a fixed point *when the curve*. Now the tangent at A , taken twice over, may be regarded as a conic having four-point contact at A ; whence it appears that the point M is the reciprocal tangential of A . The number of conics of the system which touch the cubic at some other point is therefore the number of tangents that can be drawn from M to the curve; that is, four in general, two when the cubic has a node, and one when it has a cusp. But in this number there is always included that conic which is made up of the tangent at A taken twice over; and this is not a proper solution.

1866. If x, y, z satisfy $A = 0, B = 0, C = 0, D = 0$ are mutually orthotomic, the x points of the cubics of x (a) $A = 0$ and (b) $A = 0$ in

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2) + (x_1 + x_2 + x_3 + x_4)^2,$$

where x_1, x_2, x_3, x_4 are the radii of A, B, C, D .

[*E. T. J. J. J. J.*, 1866. *Mathematical Repository*, Vol. vi, pp. 74, 76.]

1870. A line of length n is broken up into n pieces at random; prove that (1) the chance that they cannot be made into a polygon of n sides is n^{n-2} ; and

Therefore the chance required is

$$\frac{1}{(n-r)r+2} \left(\frac{x}{a} \right)^{r+2} \left(\frac{a-x}{a} \right)^{n-r+1} dx.$$

Now, if we integrate this with respect to x from 0 to $\frac{1}{2}a$, we shall get the entire chance that the r^{th} piece may be greater than $\frac{1}{2}a$. The integral is easily found to be $\frac{1}{2}n^{-1}$. And as there are thus n equal chances, whose events are all mutually exclusive, the chance that some one of these events will happen is $n \cdot \frac{1}{2}n^{-1}$.

4. *Third solution.* To make this clear, I will state first the previously-known analogous solutions in the cases where $n=3$ and $n=4$. When the line is divided into three pieces, call them x, y, z , and take their lengths for the co-ordinates of a point P in geometry of three dimensions. Then, since

$$x+y+z=a \dots\dots\dots (1),$$

and x, y, z are all positive, the point P must lie somewhere on the surface of the equilateral triangle determined on the plane (1) by the coordinate planes. Now, consider those points on the triangle for which $x > \frac{1}{2}a$. These are cut off by the plane $x = \frac{1}{2}a$; and it is easy to see that this plane cuts off from one corner of the triangle a similar triangle of half the linear dimensions, and therefore of the fourth the area. Now, there are three corners cut off; their joint area is therefore three-fourths of the area of the triangle; and the chance required is accordingly $\frac{1}{4}$.

When the line is divided into four pieces, take the first three pieces as the co-ordinates of a point in space. Then we have $x+y+z < a$, and x, y, z all positive; so the point must lie within the content of the tetrahedron bounded by the plane $x+y+z=a$ and the coordinate planes. Now, if $x+y+z > \frac{1}{2}a$, the fourth piece must be greater than $\frac{1}{2}a$. The points for which this is the case are cut off by the plane $x+y+z = \frac{1}{2}a$; and it is easily seen as before that this plane cuts off from one corner of the tetrahedron a similar tetrahedron of half the linear dimensions, and therefore of one-eighth the volume. So also the plane $x+z = \frac{1}{2}a$ cuts off from another corner a similar tetrahedron of half the linear dimensions. Since therefore there are four corners cut off, their joint volume is $(\frac{1}{8})$ or one-half of the volume of the tetrahedron; and the chance required is accordingly $\frac{1}{2}$.

5. Now, consider the analogous cases in geometry of n dimensions. Corresponding to a closed area, and a closed volume, we have something which I shall call a *contine*. Corresponding to a triangle, and to a tetrahedron, there is a contine with $n+1$ corners or vertices, which I shall call a *prime contine*, as being the simplest form of contine. A prime contine has also $n+1$ faces, each of which is, not a plane, but a prime contine of $n-1$ dimensions. Any two vertices may be joined by a straight line, which is an *edge* of the contine. Through each vertex pass n edges. A prime contine may be *regular*, which it is when any three vertices form an equilateral triangle; or *rectangular*, which it is when the edges through some one vertex are all equal and at right angles to one another.

To solve the question for general values of n , we may adopt as a type either of the geometrical solutions given for the cases $n=3$ and $n=4$. First, take the

lengths of the n pieces for the coordinates of a point in geometry of n dimensions. Then, since their sum is n , and they are all positive, the point must lie within a certain regular plane-rhombus of $n-1$ dimensions. The supposition that a certain piece is greater than $\frac{1}{2}$ cuts off from one corner of the rhombus a smaller rhombus of half the linear dimensions, and therefore of 2^{1-n} times the content. And so there are n corners, their point content is $n \cdot 2^{1-n}$ times the content of the rhombus; the chance required is consequently $n2^{1-n}$. (6), Take the lengths of the first $n-1$ pieces as the coordinates of a point in geometry of $n-1$ dimensions; the point will then lie within a certain certain regular rhombus of $n-1$ dimensions; and the investigation proceeds as before, the corners being cut off in the same manner.

6. It will be seen that this third solution involves in a geometrical form the assumption of which some sort of proof was given in the first solution. Let us make this extension of our fundamental definition. A point is taken at random in a finite or infinite space of n dimensions, and the chance that the point lies in a finite portion of this space varies as the contents of that portion. The assumption is that when the lengths of the pieces into which a line is broken up are taken as coordinates of a point, then if this line is broken up at random the point is taken at random, and vice versa. The proof of this assumption may be shown to involve a geometrical proposition equivalent to the integration by parts of the differential by Art. 19.

Making this assumption, we may solve the second part of the question by the method of the third solution of the first part. I will first state the previously known analogous solution of the case where $n=2$. The question is in this case, If a line of length n be divided into three pieces at random, find the chance that the sum of the squares of these pieces shall be less than $\frac{1}{2}$. Take the lengths of the three pieces for coordinates x, y, z of a point P in geometry of three dimensions, then, as before, the point must lie somewhere in the area of the equilateral triangle formed by the plane $x+y+z=n$ cut by the coordinate planes. But it also lies in the region of the plane $x+y+z=n$ where the quantity $x^2+y^2+z^2$ is less than a certain quantity m^2 . Thus the point P must lie within a certain definite rhombus in the plane $x+y+z=n$ cut by the coordinate planes. Now, in the case where $n=2$, this rhombus is the rhombus inscribed in the equilateral triangle, so that the question only requires to find the area

of this rhombus in terms of the area of the equilateral triangle, the area of the inscribed rhombus.

Now let us give little difficulty, and find the chance that $x^2+y^2+z^2 < \frac{1}{2}$. Here we shall have to take a point P in geometry of four dimensions, x, y, z, w being the coordinates in the regular tetrahedron $x+y+z+w=n$ cut by the xy plane.

$$x+y+z+w=n$$

by the coordinate hyperplane. If m^2 be the value of the quantity $x^2+y^2+z^2$ then there is a certain quantity m' , then the point P must lie within a rhombus $x_1y_1z_1w_1$ determined on the hyperplane $x+y+z+w=n$ cut by the xy plane.

$$x_1^2+y_1^2+z_1^2+w_1^2 < m'^2$$

In the particular case where m is the perpendicular from the vertex on the base of a rectangular tetrahedron, each of whose equal edges is of length a , or

$$m^2 = \frac{1}{3}a^2,$$

this sphere is the sphere inscribed in the regular tetrahedron. The question is therefore reduced to this one :

To find, in terms of the volume of a regular tetrahedron, the volume of its inscribed sphere.

Now, a similar reduction holds in the general case; viz., the question can always be reduced to this one :

To find, in terms of the contents of a regular prime cone of $n-1$ dimensions, the contents of its inscribed quasi-sphere.

This question I proceed to solve.

7. Let $n-1=p$. The perpendicular from any vertex on the opposite face of a regular prime cone in p dimensions $= \left(\frac{p+1}{2p}\right)^{\frac{1}{2}} \cdot (\text{edge})$.

For, let O be the vertex in question, OA, OB, \dots the p edges through O . Draw through each vertex A a space of $p-1$ dimensions parallel to the face opposite to A . The p spaces thus drawn will intersect in a point P , such that OP is the diagonal of a cone analogous to a parallelogram and to a parallelepiped. Then OP is p times the perpendicular from O on the opposite face of the regular cone; for the perpendicular is the projection of *one* edge at a certain angle, while OP is the projection at the same angle of a broken line consisting of p edges.

We have also

$$\begin{aligned} OP^2 &= OA^2 + OB^2 + OC^2 + \dots + 2OA \cdot OH \cos AOH + \dots \\ &= 2OA^2 + 2OA \cdot OH \text{ (since } \cos AOH = \frac{1}{p}, \text{ \&c.)} \\ &= (p + \frac{1}{2}p(p-1)) \cdot OA^2 = \frac{1}{2}p(p+1) \cdot OA^2, \\ \therefore (\text{perpendicular})^2 &= \frac{OP^2}{p^2} = \frac{p+1}{2p} \cdot (\text{edge})^2. \end{aligned}$$

[If the cone were *rectangular*, or all the angles at O right angles, we should have $\cos AOH = 0$, &c., and so

$$(\text{perpendicular})^2 = \frac{1}{p} (\text{edge})^2 = \frac{a^2}{n+1};$$

which proves that the question *does* always reduce itself to the one now under consideration].

The content of a regular prime cone in p dimensions whose edge is a , is

$$= \frac{a^p}{[p]} \left(\frac{p+1}{2p}\right)^{\frac{1}{2}}.$$

Suppose this formula true for $p-1$ dimensions; that is, let

$$V_{p-1} = \frac{a^{p-1}}{[p-1]} \left(\frac{p}{2p-1}\right)^{\frac{1}{2}}.$$

If n pieces be cut off at random, one from each of n equal lines, the chance that the pieces cannot be made into a polygon is $\frac{1}{n-1}$.

2253. If four circles have a common radical centre, it is possible to find four planes which intersect, two and two, at angles equal to those at which the circles intersect, but not otherwise.

[*E. T.* October, 1886. Reprint, Vol. vii, p. 32.]

2259. A, B, C, D are four points on a circle, and through every pair, as AB , another circle (AB) is drawn; then the pair of circles (AB), (CD) intersects the pair (AC), (DB) in four new points on a circle U ; the pair (AC), (DB) meets (AD), (BC) on a circle V ; and the pair (AD), (BC) meets (AB), (CD) on a circle W ; also the three circles U, V, W have a common radical axis. (This may be extended to spheres; and there are also analogous properties of rectangular hyperbolæ).

[*E. T.* August, 1886. Reprint, Vol. vii, p. 37.]

2255.* Reprint, Vol. vii, p. 35, is merely a repetition of 1878 (p. 566), with a different solution. It is noteworthy that there is no question no numbered in the *E. T.*, for the August (1886) No. given 2110 in succession to 2000 and 2220 next to 2110.]

1952. Required the *characteristics* of the system of conics having five-pointin contact with a curve of order m and class n .

[*E. T.* May, 1883. Reprint, Vol. vii, p. 47.]

2313. A is any point within or without a conic, B any point on its polar, CD a fixed straight line. Tangents BC, BD are drawn cutting CD in C, D . AD, AC meet BC, BD in E, F ; show that EF is a fixed straight line and meets CD on the polar of A .

[*E. T.* February, 1867. Reprint, Vol. viii, pp. 61, 6.]

2522. Prove (1) that the perpendiculars of a circular triangle have a common point; and (2) that if the perpendiculars from the piths of vertices of one circular triangle on the sides of another meet in a point, then *vice versa*. (Def. A, B, C being circles, a circle coaxal with A, B , and orthogonal to C , is called the perpendicular from AB on C .)

[*E. T.* November, 1887. Reprint, Vol. ix, p. 42.]

2661. A and B are fixed points with regard to a conic, ACD a variable straight line passing through A and cutting the curve in C, D . The polar of A meets BC, BD in E, F ; show that DE and CF meet in a fixed point G , and that ABG is a straight line.

[*E. T.* April, 1882. Reprint, Vol. x, p. 61.]

2732. An ep- or hypo-cycloid is pushed through a very short fixed tube, so as to remain in one plane, show that the locus of its centre is an ellipse.

[*E. T.* September, 1868. Reprint, Vol. x, p. 66.]

2748. If a circular cubic with a double point O be cut by a circle in four points A, B, C, D ; and if OA, OB, OC, OD cut the circle again in E, F, G, H ; shew that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O .

[*E. T.* October, 1868. Reprint, Vol. x. pp. 105, 6.]

2801. A circle is drawn so that its radical axis with respect to the focus S of a parabola is a tangent to the parabola; if a tangent to the circle cut the parabola in A, B , and if SC , bisecting the angle ASB , cut AB in C , the locus of C is a straight line.

[*E. T.* December, 1866. Reprint, Vol. xi. p. 31.]

2776. Through A , the double point of a circular cubic, draw AB perpendicular to the asymptote; if chords be drawn to the curve subtending a right angle at the double point, shew that there is a fixed point in AB at which also they subtend a right angle.

[*E. T.* November, 1868. Reprint, Vol. xi. p. 64.]

2108. Required Analogues in Solid Geometry to the following propositions in Plane Geometry:—

- (a) The perpendiculars of a triangle meet in a point.
- (b) The middle points of the diagonals of the quadrilateral are in one straight line.
- (c) The circles whose diameters are the diagonals of a quadrilateral have a common radical axis.
- (d) Every rectangular hyperbola circumscribing a triangle passes through the intersection of perpendiculars.
- (e) Every rectangular hyperbola to which a triangle is self-conjugate passes through the centres of the four touching circles.
- (f) $\sin(A+B) = \sin A \cos B + \cos A \sin B$.
- (g) The sum of the angles of a triangle = two right angles.
- (h) In any triangle

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

[*E. T.* July, 1864, where it is numbered 1526, cf. 2185, p. 607. Reprint, Vol. xi, pp. 102, 103, May, 1869.]

I have an analogue for each of the four (d), (e), (f), (g), and more than one of each of the others obtained by extensions of Mr Greco's methods*.

E.g. :—

- (e) A straight line cuts the faces of a tetrahedron $ABCD$ in a, b, c, d ; the spheres whose diameters are Aa, Bb, Cc, Dd , have a common radical axis. Hence the middle points of these four lines are in one plane.

* (? Reprint, Vol. ii, p. 801 cf. p. 598, supra.)

Let a conicoid whose asymptotic cone has three generating lines at right angles be called a rectangular conicoid.

(d), (e) Every rectangular conicoid circumscribing a tetrahedron whose perpendiculars meet in a point, passes through the point. And every rectangular conicoid to which a tetrahedron is self-conjugate, passes through the centres of the eight touching spheres.

$$(f) \quad \sin(ABC) + \sin(ABD) + \sin(ACD) + \sin(BCD) \\ + \sin(BCD) + \cos AD + \sin(ACD) + \cos BD + \sin(ABD) + \cos CD,$$

where A, B, C, D are four lines in space, and

$$\sin^2 ABC = \begin{vmatrix} 1 & \cos AB & \cos AC \\ \cos AB & 1 & \cos BC \\ \cos AC & \cos BC & 1 \end{vmatrix}.$$

(g) In the triangle case this should be written

$$(BC) + (CA) + (AB) = 0.$$

The analogue is then obviously

$$(BCD) + (CDA) + (DAB) + (ABC) = 0,$$

A, B, C, D being any four planes.

(h) In any tetrahedron,

$$AD^2 + BE^2 + CF^2 + GH^2 = ad + bc + ca + ab + 1^2 + \cos A \cos B \cos C \cos D, \\ \sin AD^2 + \sin BE^2 + \sin CF^2 + \sin GH^2 = 1^2 + \cos A + \cos B + \cos C + \cos D,$$

where a, b, c, d are the faces, and

$$\cos A = \begin{vmatrix} 1 & \cos AB & \cos AC \\ \cos AB & 1 & \cos BC \\ \cos AC & \cos BC & 1 \end{vmatrix}.$$

(AB, AC , denoting angles between a and b),

2999. θ is the angle focus of a conic subtends parabola, and from any point O three tangents are drawn to the curve; if OP, OQ, OR be perpendicular to them, show that DE and CF are equally inclined to the direction of the infinite branches.

[*E. T.* December, 1902. *Reprint*, Vol. xii, p. 224.]

3002. [Given the inscribed and circumscribed circles of a triangle, the envelope of the pedal circle is a linear equation. Proposed by the Rev. J. Wolstenholme. *Reprint*, Vol. xii, p. 224.]

Let B, C, X be the inscribed, circumscribed, and pedal circles. The circle X has to be such that a triangle self-conjugate with regard to the circle B and inscribed to C ; that is, X is subject to a first and a condition of the second degree by its coefficients. We can therefore draw this circle an arbitrary point.

3114.

Now, any series of circles, such that two of them can be drawn through an arbitrary point, is one system of bitangent circles of a bicircular quartic. For the equation of a circle of the series must contain a variable parameter in the second order; that is, it must be of the form

$$X + 2\theta Y + \theta^2 Z = 0^*,$$

where $X=0$, $Y=0$, $Z=0$ are circles. But the envelope of this is $XZ=Y^2$, a bicircular quartic.

This important remark is made by Cremona (*Teoria Geometrica*, ii, 241) in the case of a series of curves of any order and of index 2; that is, such that two of them can be drawn through an arbitrary point. The envelope is always what Prof. Cayley (*Edinb. Phil. Trans.*, 1868) calls a *trizomal* curve; viz. X , Y , Z being any three curves of the series, its equation may be written

$$\sqrt{\alpha X} + \sqrt{\beta Y} + \sqrt{\gamma Z} = 0.$$

2923. [In a bicircular quartic, the points of contact of the four single tangents drawn from the centre of a circle on which four foci lie, are on the circle, and the corresponding points of contact of double tangents also lie on a circle. Proposed by S. Roberts, M.A. Reprint, Vol. XII. p. 57. *E. T.* September, 1860.]

A bicircular quartic is its own inverse with regard to any focal circle (Montard, *Nouvelles Annales*, 1866). The bitangent circles divide themselves into four systems, all the circles of any one system being cut orthogonally by the corresponding focal circle. Through any point of the plane can be drawn two bitangent circles of each system. The two bitangent circles, then, that can be drawn through the centre of the focal circle of their system, are in fact straight lines touching the curve in two pairs of inverse points, which consequently lie on a circle.

The corresponding theory in anallagmatic surfaces is that the centre of each one of the five principal spheres is vertex of a quadric cone doubly tangent to the surface; the curve of contact being the intersection of this cone with a sphere. These five cones noticed by Montard are independently arrived at by Kummer (*Berlin. Monatsber.*) in the case of the general quartic surface with a nodal conic.

2924. [On a focal chord PSQ of a parabola are taken p , q , on opposite sides of S , such that $Sp \cdot Sq = SP \cdot SQ$, and any parabola is described through p , q , and having its axis parallel to that of the former: prove that their chord of intersection will pass through S . Proposed by the Rev. J. Wolstenholme. Reprint, Vol. XII. pp. 62, 3. *E. T.* September, 1860.]

I consider the following more general question:—

Through a point a let a line B be drawn meeting a conic in l , m ; then the quantity

$$al \cdot am \sin BP, \sin BQ, \dots \dots \dots (1),$$

* [With reference to this solution Mr Wolstenholme remarks (note p. 63), that "It is not necessary that X , Y , Z should be all circles; it is sufficient that one be a circle and the others straight lines. Thus the envelope might, so far as depends on this reasoning, be a circular cubic."]

(where P and Q are the asymptotes) is independent of the position of the line B , and may be called the *distance* of the point a from the conic. What now is the locus of a point equidistant from two given conics?

Let $C_2=0$, $D_2=0$ be the equations of the conics, and let a^2C_2 denote the result of substituting the coordinates of the point a for the variables in C_2 ; also let i, j be the circular points at infinity. Then I find that the distance (above defined) of the point a from C_2 is

$$\frac{a^2C_2}{(aij)^2 \cdot \sqrt{(i^2C_2 \cdot j^2C_2)}} \dots\dots\dots (2),$$

where, of course, (aij) means the determinant formed with the coordinates of the points a, i, j .

This being so, the equation of the required locus is

$$\frac{C_2}{\sqrt{(i^2C_1 \cdot j^2C_2)}} = \frac{D_2}{\sqrt{(i^2D_1 \cdot j^2D_2)}} \dots\dots\dots (3),$$

shewing that the locus is a conic passing through the intersecctions of the two given ones.

Now if we are using Cartesian coordinates, and if the two conics are similar and similarly situated, it is easy to see that the terms of the second order have entirely disappeared from the equation (3); which indicates that the line at infinity is part of the locus. The remainder of it is then their finite chord of intersection; which is a true radical axis, in the sense that if any line whatever is met by the radical axis in a , by C_2 in l, m , and by D_2 in l', m' , we must have always

$$al \cdot am = al' \cdot am' \dots\dots\dots (4).$$

To apply this to the question we have only to observe that parabolas with parallel axes are homothetic, or similar and similarly situated curves; and that the equation

$$Sp \cdot Sq = SP \cdot SQ$$

indicates that the focus is situated on their radical axis.

The theorem of the radical axis of two homothetic conics may of course be proved for ellipses by orthogonal projection from the circle, and then extended by the doctrine of continuity to the rest.

2446. PQ is a chord of a conic, equally inclined to the axis with the tangent at P . Any circle through PQ cuts the conic in RS . Shew that the harmonic conjugate of RS relative to P lies on the straight line joining Q to the other extremity of the diameter through P . Hence shew by inversion that, if chords be drawn to a circular cubic through the point where the asymptote cuts the curve, the locus of their middle points is a circle through the double point.

[*E. T. July, 1867.* · Reprint, Vol. xii. p. 91.]

2960. The envelope of a series of surfaces of order n , such that two of them can be drawn through an arbitrary point, is a surface of order $2n$, whose equation may be written in the form

$$\sqrt{\alpha X} + \sqrt{\beta Y} + \sqrt{\gamma Z} = 0,$$

where $X=0$, $Y=0$, $Z=0$ are equations of any three surfaces of the series.

The envelope of a net of surfaces of order n , such that two drawn through two arbitrary points, is a surface of order $2n$ referred to any four surfaces of the net is of the same form as quadric referred to four tangent planes.

[E. T. September, 1869. Reprint, Vol. XII. p. 96.]

2942. Let p, q be the foci, and P, Q the asymptotes of a conic it subtends at a point a , and $\{A\}$ the chord it cuts off from a line

1. If a line B is drawn through the point a meeting the conic

$$al \cdot am \cdot \sin BP \sin BQ = \frac{ap^3 \cdot aq^3 \cdot \sin^2 Q}{pq^3}.$$

2. If from a point b on the line A tangents L, M are drawn

$$\sin AL \sin AM \cdot bp \cdot bq = \frac{\sin^2 AP \sin^2 AQ \cdot \{A\}^2}{\sin^2 PQ}.$$

(Here al means the distance between the points a, l , and angle between the lines B, P .)

3. Find analogous propositions for a curve of any order or sphere.

[E. T. August, 1869; 4724, July, 1875. Reprint, Vol. X. E. T. November, 1869. Solutions given by the Rev. J. Walker, Mr J. J. Walker, and Prof. Clifford remarks]

The extensions for a plane are

(curve of class n)... $al \cdot am \cdot an \dots \sin BP \cdot \sin BQ \dots$

$$= \frac{(ap \cdot aq \cdot ar \dots)^{2(n-1)} \sin^2 LAM \cdot \sin^2 Q}{pq^3 \cdot pr^3 \cdot qr^3}$$

(curve of order m)... $\sin AL \cdot \sin AM \dots bp \cdot bq \dots$

$$= \frac{(\sin AP \cdot \sin AQ \dots)^{2(m-1)} \sin^2 LAM \cdot \sin^2 Q}{\sin^2 PQ \cdot \sin^2 PR \cdot \sin^2 QR}$$

where P, Q, R, \dots are the asymptotes, and p, q, r, \dots the real foci. The ideas of the "distance" of a point from a line or surface, are extended so as to give the distance of two curves from one another.

3021. The three pairs of foci of a sphero-conic are $a, a'; b, b'; c, c'$ is any point on the sphere.

Prove the formulae

$$\sin aa' \cdot \sin bb' \cdot \sin cc' = 8 \dots \dots \dots$$

$$(\sin aa')^{-2} + (\sin bb')^{-2} + (\sin cc')^{-2} = 0 \dots \dots \dots$$

$$\frac{(\sin pa \cdot \sin pa')^2}{\sin^2 aa'} = \frac{(\sin pb \cdot \sin pb')^2}{\sin^2 bb'} = \frac{(\sin pc \cdot \sin pc')^2}{\sin^2 cc'}$$

[E. T. December, 1869. Reprint, Vol. XIII. p. 50.]

2979. Two triads of points abc , $a\beta\gamma$ being taken on a line, let the two triads be called *harmonic* of one another when

$$aa \cdot b\beta \cdot c\gamma + a\beta \cdot b\gamma \cdot ca + a\gamma \cdot ba \cdot c\beta + a\gamma \cdot b\beta \cdot ca + a\beta \cdot ba \cdot c\gamma + aa \cdot b\gamma \cdot c\beta = 0;$$

then (1) the envelope of a line cut harmonically by two cubics is of the third class. (The contravariant $a11'^3$)—(2) this line is also cut harmonically by every pair of cubics through the intersections of the first two. (3) The envelope of a line cut harmonically by a given cubic and the cubic made up by the polar line and conic of a given point is the mixed concomitant $a12 \cdot a13^2$. (4) Two cubics having the same inflexions cut harmonically any line whatever.

[E. T. October, 1869. Reprint, Vol. XIII, p. 52.]

3197. If the epicycloid described by a point on the circumference of a circle rolling on an equal fixed circle be loaded with matter proportional to its curvature at every point, the centre of gravity of the whole will be at the centre of the fixed circle.

[E. T. August, 1870. Reprint, Vol. XIV, p. 98.]

3282. It is known that the circles circumscribing the triangles formed by four lines meet in a point, and that the points so belonging to the five tetragrams formed by five lines lie in a circle. Prove that the circles so belonging to the six pentagrams formed by six lines meet in a point, and so on; the series of theorems being interminable.

To every $2n+1$ lines there belongs in this way a circle. If from any point p on this circle perpendiculars be let fall on the straight lines, their feet will all lie on a curve of order n , having a $(n-1)$ -ple point at p .

[E. T. December, 1870. Reprint, Vol. XV, p. 47.]

3885. If A be the single focus of a semi-cubical parabola, there exists a straight line BC , such that if two tangents at right angles cut it in B , C , the angle BAC is also a right angle.

[E. T. October, 1872; 2674, June, 1863. Reprint, Vol. XVIII, p. 82.]

3876. [Shew that there are 6184 positions in a cubic curve such that at each of them curves of the 50th order may be drawn having 90-point contact with the cubic. Proposed by J. J. Sylvester, F.R.S. Reprint, Vol. XIX, p. 46. April, 1873?]]

The gross number of points where a curve of order n can have $3n$ -point contact with a cubic is $9n^2$. The problem is in fact the same as that of the divisions of the periods of an elliptic function by $3n$, and as there are two periods, there are $9n^2$ solutions. (Clebsch, *Anwendung der Abelschen Functionen in der Geometrie*, Grolle, LXII.) But in the case when n is a composite number, all the curves whose order is a division of n , and which have complete contact with the cubic, are included in the result. Thus each inflexional tangent, taken n times over, constitutes a curve of order n having $3n$ -point contact with the cubic; and the nine inflexional tangents are thus always

included in the $9n^2$ solutions. To obtain the number of *proper* solutions, then, we must subtract all these improper ones. When $n=80$, the result is

$$9\{30^2 - 15^2 - 10^2 - 6^2 + 5^2 + 3^2 + 2^2 - 1^2\} = 9 \times 576 = 5184.$$

Here it is to be observed that the curves of order 5 are *twice* subtracted, with the curves of order 15 and 10; so that they have to be added in again. The same remark applies to the orders 3 and 2. The curves of order 1 (inflectional tangents) having been thrice subtracted and thrice added, must finally be subtracted again.

4010. Prove that the lines of curvature of a quadric surface are projected from an umbilic on a plane parallel to its tangent plane into a series of confocal Cartesian ovals.

[E. T. March, 1873. Reprint, Vol. xix. pp. 73, 4.]

4031. Prove that the forty umbilics of a cubic surface which passes once, or of a quartic surface which passes twice, through the imaginary circle at infinity, lie by fives upon sixteen straight lines.

[E. T. April, 1873; 3308, January, 1871. Reprint, Vol. xix. p. 77.]

2020. [Reprint, Vol. xix. p. 81. The question and solution are identical with those of 1379, p. 567 *supra*: here the Authorship of the Question is ascribed to N'Importe.]

4097. If about a prolate conicoid of revolution there be described an octahedron so that its three diagonals pass through a focus, shew that they must be at right angles to each other.

[E. T. June, 1873; 1415, August, 1863. Reprint, Vol. xix. p. 100.]

2022. [Reprint, Vol. xix. p. 108. This is 1378, of which see Solution p. 554 *supra*.]

4069. 1. Curves of order $2n+1$ pass n times through each circular point and through n^2+4n+1 other fixed single points (or their equivalent in multiple points); shew that the envelope of their asymptotes is a triuncped hypocycloid.

2. Curves of order $2n+2$ pass n times through each circular point and through n^2+6n+4 other fixed points, and their real asymptotes are at right angles; shew that the envelope of their asymptotes is a triuncped hypocycloid.

[E. T. May, 1873. Reprint, Vol. xx. pp. (31) 50—53.]

Prof. Wolstenholme's remark in his solution of this question, given on p. 311 of this volume of the *Reprint*, that the first part "is not quite true as it stands," has led me to examine the whole with the help of his method; and it turns out, singularly enough, that it is the *second* part that requires correction, not the first. The way in which this comes about is instructive, and the corrected theorem leads us to consider a somewhat interesting series of curves.

1. I will first state the grounds on which I originally concluded that these theorems were true. It is required to find the envelope of the asymptotes of a pencil of curves which if of odd order have *one* real point at infinity besides

the circular points, if of even order *two* which are at right angles or harmonic of the circular points. The intersections with the line infinity at the circular points are due to multiplicity of these points, not to contact with the line infinity.

Now, first, *the line infinity is a tangent to this envelope at each of the circular points and no otherwhere.* For the line infinity can only become an asymptote by the variable one point or one of the variable two points at infinity coming to coincide with one of the circular points. In the second case the variable two points being harmonic of the circular points, if one of them coincide with a circular point, the other must coincide with it. In both cases, then, there are two curves of the pencil which have the line infinity for asymptote; and it is clear that the intersection of the line infinity with the next consecutive asymptote (i.e. its point of contact with the envelope) is the circular point at which it is an asymptote.

Next, from any point at infinity not a circular point, one tangent distinct from the line infinity can be drawn to the envelope. For there is one curve of the pencil that passes through this point.

If, then, the line infinity is an ordinary tangent at each of the circular points, we see that from any point at infinity three tangents may be drawn to the envelope; viz., the line infinity counting twice, and one other. The envelope therefore is of the third class, having the line infinity for double tangent whose points of contact are the circular points; that is to say, a hypocycloid of three branches.

In fact, the tangential equation of the curve may be at once written down. Let $i=0, j=0$ be the equations to the circular points, $k=0$ that to some other point; then the equation is $ijk + (i, j)^3 = 0$. It is, in fact, of the same form as the equation of a cubic curve having a node at the origin to which the axes are tangents. If for k we write $k + \lambda i + \mu j$, it is clear that by proper choice of λ, μ we can get rid of the two middle terms of $(i, j)^3$; the equation then becomes $ijk + \alpha i^3 + \beta j^3 = 0$, which is the same as $p = a \cos 3\theta$, where p is the distance of a tangent from the origin $k=0$, and θ the angle it makes with a fixed line. (Salmon, *Higher Plane Curves*, p. 271, Ex. 5.)

This result is true *if the line infinity is an ordinary tangent at each of the circular points.* Now this holds good in the *first case* of the question; for in this the one variable point at infinity is made to move up to a multiple point, and so only one branch acquires an ordinary contact; in virtue of this, then, the line infinity counts only once as an asymptote for each circular point. It also holds good in the already well-known case of curves of the second order, i.e. in the second case of the question when $n=0$. For in this only the two variable points at infinity coincide at a circular point, making again an ordinary contact.

But in the second case of the question, when n is not zero, something different happens. Here the two variable points at infinity simultaneously approach a circular point which is already multiple on the curve; they approach it on the same branch, and *produce a point of inflexion on that branch.* In respect of each circular point, therefore, the line infinity counts

for two asymptotes; the envelope is raised to the fifth class, and has the line infinity for inflexional tangent at each circular point.

2. This synthetic discussion shall now be confirmed by analysis. In the first case, the equation of a curve of the pencil is

$$(x + \lambda y)(x^2 + y^2)^n + k(\lambda, 1\bar{x}x, y)^3 \cdot (x^2 + y^2)^{n-1} + \dots = 0,$$

and its real asymptote is

$$(x + \lambda y)(1 + \lambda^2) + k(\lambda, 1\bar{x}\lambda, -1)^2 = 0,$$

whose envelope is of the third class, touched by $k=0$ (the line infinity) for the two values $\lambda = \pm (-1)^{\frac{1}{2}}$; whence as before.

In the second case, it is convenient to write the equation of the variable curve in the form

$$\left\{x^2 + \left(\lambda - \frac{1}{\lambda}\right)xy - y^2\right\} \cdot (x^2 + y^2)^n + k\left(\lambda - \frac{1}{\lambda}, 1\bar{x}x, y\right)^3 \cdot (x^2 + y^2)^{n-1} + \dots = 0.$$

The two real asymptotes are

$$\left(x - \frac{y}{\lambda}\right)(1 + \lambda^2)^2 + k\left(\lambda - \frac{1}{\lambda}, 1\bar{x}1, \lambda\right)^3 = 0,$$

$$(x + \lambda y)(1 + \lambda^2)^2 + \lambda k\left(\lambda - \frac{1}{\lambda}, 1\bar{x}\lambda, -1\right)^3 = 0.$$

These have the same envelope, as one equation is got from the other by writing $-\lambda^{-1}$ for λ . The envelope is of the fifth class, touched by $k=0$ twice for each of the values $\lambda = \pm (-1)^{\frac{1}{2}}$. The line infinity is therefore a double tangent with united contacts (*i.e.* an inflexional tangent, just as a cusp is a double point with united branches) at each of the circular points.

3. It remains to investigate the nature of a curve of the fifth class having the line infinity for inflexional tangent at each of the circular points. This singularity being equivalent to two inflexions and four double tangents, Plücker's equations at once tell us that the curve is of the sixth order and has five cusps and five nodes. Its tangential equation may be at once written down, being of the same form as that of a quintic curve having a quadruple point at the origin, two of whose branches coincide with each of the axes; namely, it is $i^2j^2k + (i, j)^5 = 0$, where $i=0, j=0$ are the circular points, and $k=0$ is some other point. As before, we may suppose k to have been so selected as to get rid of the two middle terms of $(i, j)^5$. Now a particular case of the equation is

$$i^2j^2k + \alpha i^5 + \beta j^5 = 0, \text{ or } p = a \cos 5\theta,$$

which represents the hypocycloid [Fig. 128] described by a point on a rolling circle whose radius is two-fifths of the radius of the fixed circle.

The general equation may be transformed into

$$p = a \cos 5\theta + b \cos 3\theta + c \sin 3\theta,$$

or, if we write

$$p_1 = 2a \cos 5\theta, p_2 = 2b \cos 3\theta + 2c \sin 3\theta,$$

the equation is

$$2p = p_1 + p_2.$$

Now p_3 and p_4 are the distances from the origin of parallel tangents to a three-cusped and a sextic five-cusped hypocycloid respectively; whence we learn that *the curve in question is the evolute of a line midway between parallel tangents to two such hypocycloids.*

These hypocycloids have only to be concentric; and their relative size and orientation are the two variable elements in the equation of our curve. The method of description by tangents, however, given us immediately a description by points, since it is clear that the point of contact of the variable tangent bisects the line joining the points of contact of the two tangents to which it is parallel and intermediate. In this way it is easy to draw roughly a few typical forms.

4. A hypocycloid in which the radii of the rolling and fixed circles are in one another as n to $2n+1$ is a curve of order $2n+2$, class $2n+1$, with $2n+1$ cusps, $(n-1)$ $(2n+1)$ nodes, and has $(n+1)$ points in contact with the line infinity at each circular point. Its tangential equation is

$$r^2 p^2 (\tan^{2n+1} \theta / p)^{2n+1} = 0;$$

or, which is the same thing, $p = a \cos(2n+1)\theta$.

To this simplest class of evolute, all whose tangential singularities are at infinity, it may be permissible to give the name "stars." Thus an ordinary triangle is a three-rayed star, the curve in Fig. [133] is a five-rayed star, and so on. We may now state the following proposition:

Every curve of class $2n+1$, which has $(n+1)$ points in contact with the line infinity at each circular point, is the envelope of a line which is the mean of the parallel tangents of a concentric star of all odd classes up to $2n+1$.

Namely, its equation is $r^2 p^2 + (f, g)^{2n+1}$,

or $p = a_2 \cos 2\theta + b_3 \sin 2\theta + \dots + a_{2n+1} \cos(2n+1)\theta$,

from which the proposition is obvious. The curve has the same number of nodes and cusps as a star of class $2n+1$; only they need not, as in the case of the star, be all real.

I remark, in conclusion, first, that the point equation of one curve of the fifth class is

$$D(x, y, z) + (y, x - iy, x + iy, x, f, g, p)^5 = 0,$$

which is worked out in Dr. Salmon's *Higher Algebra**; and secondly, that the Hessian of the tangential equation is easily calculated and shows the cuspidal tangents to be common tangents of a three- and a five-rayed star.

4201. CT is the double point of a circular cubic, and a straight line cuts the curve in DEF ; join CD , CE , CF , and on the two latter lines take A , B so that CA , $CF = CB$, CE ; then prove that AB and CT are equally inclined to the tangents at C .

[*E. T.* November, 1873; 1917, January, 1939. Reprint, Vol. xx, p. 88.]

4199. Three ternary quadrics U, V, W , break up into linear factors 1, 1'; 2, 2'; 3, 3' respectively. Prove that

$$\square(U, V, W) \equiv 123 \cdot 1'2'3' + 1'23 \cdot 12'3' + 12'3 \cdot 1'23' + 123' \cdot 1'2'3,$$

where $\square(U, V, W)$ is the coefficient of $\lambda\mu\nu$ in the discriminant of $\lambda U + \mu V + \nu W$, and 123 means the determinant formed with the coefficients of the linear factors 1, 2, 3. Required developments and interpretations.

[*E. T.* October, 1873; 1907, February, 1866. Reprint, Vol. xxi, p. 38.]

4641. If a circular cubic with a double point O be cut by a circle in four points A, B, C, D ; and if OA, OB, OC, OD cut the circle again in E, F, G, H ; shew that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O .

[*E. T.* April, 1875. Reprint, Vol. xxiii, p. 59.]

4696. Six circles pass through twelve points on a conic in the following order,

$$\begin{array}{lll} (a) & A_1A_2A_3A_4, & (b) & B_1B_2B_3B_4, & (c) & C_1C_2C_3C_4, \\ (d) & A_1A_2B_3C_4, & (e) & B_1B_2C_3A_4, & (f) & C_1C_2A_3B_4; \end{array}$$

prove that two circles and another point may be taken arbitrarily, and that the circles abc meet the circles def in six new points which lie on the circumference of another circle.

[*E. T.* June, 1875; 2281, November, 1866. Solution of the first part given in Reprint, Vol. xxiv, pp. 42, 43. The solver called in question the truth of the second part, but, on seeing the following solution on pp. 76, 77, admitted the correctness of this 'beautiful' theorem.]

Three circles taken together constitute a sextic curve passing three times through each of the circular points at infinity. Now the sextic def passes through all the twelve points of intersection of the sextic abc with the conic, which we may call k ; hence, by a well-known theorem, there must be an identical equation of the form

$$\mu \cdot def = \lambda \cdot abc + q \cdot k.$$

Here λ and μ are numerical ratios, and q is a quartic function of the coordinates. The equation may also be written

$$-q \cdot k = \lambda \cdot abc - \mu \cdot def,$$

and in this form it shews that the equation $qk=0$ represents a curve of the sixth order passing three times through each circular point. But, by hypothesis, the conic k does not pass through either circular point. The quartic curve q has therefore two triple points on the line infinity, it must therefore contain that line. The rest of it is a cubic having two double points on the line infinity; it also must therefore contain that line. The final remainder is a conic passing once through each circular point, that is to say, a circle. Calling this circle s , we reduce our equation to the form

$$s \cdot k \cdot \infty^2 = \lambda \cdot abc - \mu \cdot def,$$

which shews that the remaining six intersections of abc with def lie on a circle s .

we observed that the proof holds good if we substitute for the conic section a circular cubic or a bicircular quartic. From the latter we may obtain a transformed theorem of some interest. Invert the conic in regard to a point not in its plane; the bicircular quartic section of a sphere by an arbitrary quadric surface, and every circle section of the sphere by a plane. In this form we may substitute for any quadric surface, and the transformed theorem may then be proved as follows:

Since pass through twelve points on a quadriquadric curve in the order d , the six lines of intersection ad , af , bf , bd , cd , ce will meet every face passing through the curve in six points which lie in one plane.

It can be observed that these six lines already meet the quadriquadric curve points $A_2, A_1, B_3, B_4, C_3, C_4$. Let h_1, k_1 be two quadric surfaces through the curve; then the cubic surface abc passes through all the sections of the cubic def and the quadrics h_2, k_2 . We must therefore attach equation of the form

$$\lambda . abc = \mu . def + \nu h_2 + \tau k_2,$$

where μ and ν are numerical ratios, while λ and τ are expressions of the first coordinates. Writing this identity in the form

$$\tau k_2 = \lambda . abc - \mu . def - \nu h_2,$$

the three lines of intersection of abc and def must meet the quadric in the quadric k_2 (i.e. on the quadriquadric curve) or on the plane ν . Of these, ad , bf , cd meet the curve in two points each, and the rest, af , bd , ce in one point each; consequently these latter must meet the curve on the plane ν .

The construction of the figure depends first on that of the hexagon A_1C_3 . In the case of the plane conic the opposite sides of this hexagon are parallel, and the possibility of the construction is assured by reason. When the hexagon has been drawn, it is easy to make a chord pass through the ends of two opposite sides and intersect on the line. In this way I have drawn the figure as carefully as I can, and it seems light. In the case of the quadriquadric curve, each pair of opposite sides that a quadric surface can be drawn through them to contain

In the first instance they are given as two chords which are both common third chord; thus the lines A_1A_3, B_3C_4 are both met by A_1A_2 . To draw a straight line meeting two given straight lines and a circle twice, admits in general of eight solutions; but in this case the two given lines are chords of the curve, the four lines joining each of intersection count for two solutions each, and if there is one then there must be an infinite number; i.e. the two lines and the circle lie on the same quadric surface. The possibility of the inscription on whose opposite sides possess this property may be shown by a diagram similar to that used for the plane conic [pp. 42, 43]. The quadriquadric curve of arbitrary one, and therefore the coordinates of any point may be expressed as elliptic functions of a parameter; this may be so the sum of the parameters of four points in one plane shall be con-

gruent to zero (Clebsch "On the application of Abel's functions to Geometry," *Crelle's Journal*). Using the letters A_1, A_2 , &c., to represent these parameters, we shall have

$$A_1 + A_2 + A_3 + A_4 \equiv 0 \pmod{\omega, \omega'}, \text{ where } \omega, \omega' \text{ are the periods,}$$

$$A_1 + A_2 + B_3 + C_4 \equiv 0;$$

and therefore, $A_3 + A_4 \equiv B_3 + C_4 \pmod{\omega, \omega'}$, as the condition to be satisfied by two opposite sides of the hexagon. Now, if this condition is satisfied by two pairs of opposite sides, it will be satisfied by the third pair; for the congruence $B_4 + B_3 \equiv C_3 + A_4$ follows from the congruences

$$A_4 + A_3 \equiv B_3 + C_4, \quad C_4 + C_3 \equiv A_3 + B_4.$$

The theorem states that, when such a hexagon has been constructed, lines may be drawn through its vertices which shall meet every quadric surface passing through the curve in six points on one plane. *As the surface varies, this plane passes through a fixed line*; for

$$uh_3 + vk_2 = u(h_2 + \rho k_2) + (v - \rho u)k_2.$$

Lastly, I observe that not every skew hexagon can have a quadriquadric curve drawn through it so that each pair of opposite sides shall be generators of the same quadric passing through the curve. Let the hexagon $A_1A_2B_3B_4C_3C_4$ be given; through the lines A_1A_3 , B_2C_4 and the points B_1C_3 , a singly infinite number of quadrics can be drawn, which will intersect in a quadriquadric curve; and one condition is necessary in order that the chords A_3B_4 , C_4C_3 may possess the required property, or, which is the same thing, that the three curves which we may then get from the three pairs of opposite sides may be identical. The hexagon therefore possesses a geometrical property which can doubtless be expressed in terms of its diagonal lines or planes; this expression, however, I have not as yet been able to find.

4972. Let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be any eight tangents of a conic, and let a cubic pass through all the intersections of A 's with B 's excepting A_1B_1 , A_2B_2 , A_3B_3 , A_4B_4 . Then (1) there is a singly infinite number of such octagons inscribed in the cubic and circumscribed to the conic; (2) the groups of eight tangents form an involution of the eighth order; (3) the quadriquadrics $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are totally inscribed in a second fixed cubic; (4) the points A_1B_1 , A_2B_2 , A_3B_3 , A_4B_4 are on a fixed straight line.

[E. T. May, 1876. Reprint, Vol. xxv. p. 76.]

4996. If the series

$$1 + a \frac{1-x}{1-r} + a^2 \cdot \frac{1-x}{1-r} \cdot \frac{1-rx}{1-r^2} + a^3 \cdot \frac{1-x}{1-r} \cdot \frac{1-rx}{1-r^2} \cdot \frac{1-r^2x}{1-r^3} + \dots$$

be called $\phi(a, x)$; then prove that

$$\phi(1, a) \cdot \phi(a, x) = \phi(1, ax).$$

[E. T. June, 1876. Reprint, Vol. xxvi. p. 18.]

5304. Prove that the negative pedal of an ellipse, in regard to the centre, has six cusps and four nodes; find their positions, and the length of the arc external to the ellipse between two real cusps; and account fully for the apparent reduction of the curve to a circle and two parabolas respectively, in special cases.

[*E. T.* June, 1877. Reprint, Vol. xxix. p. 47.]

4871. Let U, V be any two cubic functions of x ; show that a quartic function $f(x)$ may always be found, such that, by the substitution $y = U : V$, the elliptic differential $dx : \{f(x)\}^{\frac{1}{2}}$ will be transformed into $Mdy : \{\phi(y)\}^{\frac{1}{2}}$, where $\phi(y)$ is a quartic function of y , and M a constant.

[*E. T.* January, 1876. Reproposed as 6475, October, 1890. Solved, Reprint, Vol. xxxv. (in progress). *E. T.* January, 1881.]

3980. It is known that if four lines be given, the circles circumscribing the four triangles so formed meet in a point; and that if five lines be given, the five points so belonging to their five tetragrams lie on a circle, (Miquel's Theorem; see *Diary* for 1861, p. 55 [viii. supra, pp. 38—54].)

Show that this series of propositions is interminable; so that if $2n$ lines be given, they determine $2n$ circles which meet in a point; and if $2n + 1$ lines be given, they determine in this manner $2n + 1$ points which lie on a circle.

[*E. T.* February, 1873; 5423, October, 1877; 6441, September, 1890. Solved, Reprint, Vol. xxxiv. p. 80, *E. T.* November, 1880; cf. 3232, p. 613.]

5626. The circles doubly normal to a bicircular quartic arrange themselves in four systems, each system cutting orthogonally a principal circle; find the envelope of all the binormal circles of one system.

[*E. T.* May, 1878. Proposed November, 1869, as 3000; July, 1873, as 4123. Reprint, Vol. xxxii. p. 17.]

4143. Three elastic strings without weight, whose natural lengths are OA, OB, OC , are joined together at O , the centre of the circumscribing circle of the horizontal triangle ABC ; and a smooth sphere of given radius and weight is placed with its centre vertically above O , and allowed to descend until the centre rests at O . Find the moduli of elasticity in the three strings.

[*E. T.* August, 1873. Reprint, Vol. xxxiii. p. 18. Originally proposed as 1460, December, 1863.]

1433. [Prove the following reciprocal cases of involution:—

a. The three sides of every triangle, and every three concurrent lines through its three vertices, intersect every axis in six points in involution.

a'. The three vertices of every triangle, and every three collinear points on its three sides, subtend every vertex in six rays in involution.

b. The six perpendiculars on the six lines from any point in the former case determine at the point a pencil of six rays in involution.

b'. The six perpendiculars from the six points upon any line in the latter case determine on the line a system of six points in involution. Proposed by W. J. C. Miller, B.A. Reprint, Vol. xxxiii. pp. 50, 51.]

Six points are in involution when the anharmonic ratio of any four is equal to that of their four conjugates.

a. Let then ABC be the triangle [Fig. 124], D the point of concurrence; and let a straight line meet the sides and corresponding lines in $a, b, c, a', \beta, \gamma$ respectively; then

$$[abc\gamma] = \{A . DCB\gamma\} = \{A . DCF\gamma\} = \{B . DCF\gamma\} = [\beta ac\gamma] = [a\beta\gamma c], *$$

which proves the proposition.

a'. Let now ADF be the triangle, C, B, E the collinear points, and take any point O ; then we have

$$\{O . ABDE\} = \{B . AODE\} = \{B . FODC\} = \{O . FBDC\} = \{O . CDBE\},$$

which proves the second case.

b. The line at infinity is cut in involution, by prop. *a*; hence, lines parallel to the given six through any point will form a pencil in involution; turn this pencil through a right angle, and it coincides with the perpendiculars.

b'. Any point at infinity is subtended in involution, by *a'*; whence the theorem immediately follows.

* [] Auctoris.

UNSOLVED QUESTIONS.

[123.] Show that

$$\int_0^{1/\sin\beta} (\sin\alpha \cos\alpha) e^{1/2 \sin\alpha} d\alpha = \frac{1}{2} \pi e^{-\alpha} (\cos\beta + \sin\beta).$$

[L. T. September, 1963. Reproposed as 2316, January, 1967; 3941, December, 1967; 1791, October, 1968; 5630, July, 1977.]

[124.] The sides of a triangle repel with a force varying inversely as the square of the distance; find the position in which a particle will rest.

Also, supposing the faces of a tetrahedron to repel according to the same law, find where a particle will rest.

[L. T. November, 1961. Reproposed as 3396, February, 1971; 4171, September, 1971, and 6120, November, 1973.]

[125.] Consider six planes $ADGDEF$, and join the point ABC to the point DEF and ACE ; we have thus ten finite straight lines, and their middle points in six planes.

[L. T. May, 1961.]

[126.] If three circles are mutually orthogonal, prove that the circles on their common chords as diameters have a common radical axis.

[L. T. October, 1961.]

[127.] Reproduce the area of the triangle included by three points in space, as a sum of equations of the form

$$x^2 + y^2 + z^2 + ax = 0.$$

[L. T. November, 1961.]

[128.] If the radii of two spheres be p_1, p_2 , and D the distance between their centres, and if a tetrahedron be inscribed in each; prove that the product of the volumes of the tetrahedra into $(D^2 - p_1^2 - p_2^2)$ may be expressed as an integral function of the squares of the distances between the vertices of the tetrahedron. Hence deduce the condition (if any) that four points in a plane may lie in a circle. If they do not lie in a circle, what is the meaning of 0?

[L. T. April, 1961.]

1724. The equations of three conics being given in the forms :

$$a_1x^2 + b_1y^2 + c_1z^2 + d_1w^2 = 0,$$

$$a_2x^2 + b_2y^2 + c_2z^2 + d_2w^2 = 0,$$

$$a_3x^2 + b_3y^2 + c_3z^2 + d_3w^2 = 0,$$

where $x + y + z + w \equiv 0$, shew that a straight line

$$(\xi x + \eta y + \zeta z + \omega w = 0)$$

will be cut in involution by them, if

$$\Sigma \cdot \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} \cdot (\xi - \eta) (\xi - \zeta) (\xi - \omega) \text{ (to four terms)} = 0.$$

[E. T. May, 1865.]

1748. Let $X, Y, Z, U, V = 0$ be the Cartesian equations, and r_1, r_2, r_3, r_4, r_5 the radii, of five spheres, cutting each other orthogonally; then identically

$$\frac{X^2}{r_1^2} + \frac{Y^2}{r_2^2} + \frac{Z^2}{r_3^2} + \frac{U^2}{r_4^2} + \frac{V^2}{r_5^2} = 0, \quad \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} + \frac{1}{r_5^2} = 0.$$

[E. T. June, 1865.]

1918. It is known that the conic of five pointic contact at any point A of a cubic meets the curve again in a point B , constructed by joining the point A to its second tangential; let this point be called the *conic tangential* of A . Then the conic tangential of B will be the second conic tangential of A , and so on. Shew how, having given the conic-tangential of any order, and also the line tangential of any order, we can construct for the original point A by the ruler alone.

[E. T. March, 1860. Reproposed as 4299, January, 1874.]

1929. 1. If 1 2 3 4 be four concyclic foci of an anallagmatic (bi-circular) quartic curve, on a plane or on a sphere, and P any point of the curve; the arc at P is equally inclined to the circles $P12, P34$.

2. The bitangent circles of an anallagmatic are arranged in four systems, orthotomic respectively of the four focal circles. Two bitangents of the same system can be drawn to cut orthogonally a given circle A . The four points of contact of these lie on a circle B_1 (*polar circle of A in that system*). There are polar circles B_2, B_3, B_4 in the other three systems, and the circles A, B_1, B_2, B_3, B_4 have a common radical axis.

3. The two bitangents of the same system through any given point are equally inclined to each of the two confocal anallagmatics which pass through that point.

4. The bitangent spheres of an anallagmatic quartic surface are arranged in five systems, orthotomic respectively of the five focal spheres. An infinite number of bitangents of the same system can be drawn to cut orthogonally a given sphere A ; these envelope a cyclide, whose curve of contact with the anallagmatic surface lies on a sphere B , polar sphere of A in that system. The five polar spheres of A have with A a common radical plane.

5. The five tangent cycloides to an anallagmatic from any point have their focal spheres touched by the three confocal anallagmatics through that point.

6. A tangent cycloide from any focus is a *Tore*.

DEF. A *cycloide* is the envelope of a sphere touching three fixed spheres. When the centres of the three fixed spheres are in one straight line, the cycloide becomes a *tore*, or anchor-ring.

[*E. T.* April, 1866. Reproposed as 4840, March, 1874.]

2229. 1. The distances (r, s, t) of a variable point on one focal curve of an anallagmatic quartic surface from any three fixed points on another focal curve of the same surface, are connected by a relation of the form $lr + ms + nt = 0$ (i.e. a relation of the same form as that which connects the distances of a variable point on a circle from three fixed points on the circle).

2. The distances of a variable point on an anallagmatic quartic surface from four fixed foci of the surface are connected by a relation of the same form as that which connects the distances of a variable point on a sphere from four fixed points on the sphere.

3. The four points in which an anallagmatic quartic curve is cut by any circle may be taken as the foci of an anallagmatic quartic curve which passes through any four concyclic foci of the original curve. (This is true on a plane or on a sphere.)

4. The curve in which any anallagmatic quartic surface is cut by any sphere may be taken as the focal curve of an anallagmatic quartic surface which passes through any one focal curve of the original surface.

5. It is required to find the property of anallagmatic quartic curves which corresponds to the property of conics, $SP \cdot HP = CD^2$.

6. If two anallagmatic quartic curves or surfaces, A and B , are such that a confocal to A can be inscribed or subscribed to B : then also a confocal to B can be inscribed or subscribed to A .

DEF. One anallagmatic curve is *inscribed* to another when it touches it in four points on a circle. One surface is *inscribed* to another when it touches it all along its curve of intersection with a sphere. One surface is *subscribed* to another when it touches it in four points on a circle.

[*E. T.* September, 1866. Reproposed as 4754, August, 1875.]

2510. If a conic be inverted into a circular cubic with a double point, the foci and directrices of the conic will invert into foci and directing circles of the cubic.

[*E. T.* October, 1867. Reproposed as 4007, May, 1875.]

2458. If the intersections of two circles $A=0, B=0$ are concentric with the antipodal of the intersections of $C=0, D=0$, then *vice versa*; and if this property hold for the pairs AB, CD , and also for the pairs AC, DB , it will hold for the pairs AD, CB .

[*E. T.* March, 1869. Reproposed as 4513, October, 1874; 5391, July, 1878.]

3255. Two planes A, B , are said to have an (x, y) correspondence, when to every point on the plane A correspond y points on the plane B , and to every point on B correspond x points on A .

On each plane there is in general a locus of points, two of whose correspondents coincide: this is called the *cross-curve* (Uebergangscurve, Clobsehl in *Math. Annalen*).

On each plane there is also a locus of these united correspondents; this is called the *node-curve*.

1. If a curve touch the cross-curve in either plane, the corresponding curve in the other plane will have a node lying on the node-curve in that plane.

2. The correspondence may be represented as a $(1, 1)$ correspondence of two multiple planes A', B' ; A' consisting of y sheets, and B' of x sheets, which are connected together along the cross-curves.

3. In a $(1, y)$ correspondence, in which to two straight lines in the plane A correspond curves of deficiency p in the plane B , the order of the cross-curves in A is $= 2(y + p - 1)$.

[*E. T.* November, 1870.]

3308. Prove that the 40 umbilici of an anallagmatic surface lie by fives on 16 straight lines.

[*E. T.* January, 1871. Reproposed as 5274, May, 1877: cf. however 4034, p. 614.]

3961. In a polyhedron having n summits and only triangular faces (Δ -faced n -acron, CAYLEY), let every plane which contains three summits, but is not a face, be called a diagonal plane; and let every plane which contains two summits, but is not an edge, be called a diagonal line: then (a) there is a surface of class $n-4$ touching all the diagonal planes, (b) this surface contains all the diagonal lines; (c) the conditions of passing through the diagonal lines and touching the diagonal planes are just sufficient to determine the surface and no more, and (d) when the surface touches the plane at infinity, the volume of the polyhedron is zero.

[*E. T.* January, 1873. Reproposed as 5210, March, 1877. Cf. xviii. *supra*, pp. 168—176.]

4819. A spherical curve of class n has in general n^2 foci. Let n foci such that no two are on a line touching the absolute be called a *set*, and denoted by p, q, r, \dots . If x be any point of the sphere, the quantity $\frac{(11 \sin xp)^{n+1}}{11 \sin^2 pq}$ is the same for all sets.

[*E. T.* November, 1875. Reproposed as 6890, November, 1881.]

4843. If, in regard to a system of n quadric surfaces, the two systems of n polar planes in regard to any two points of space are projective to one another, either the quadrics have a common Jacobian or each of them is a doubled plane.

[*E. T.* December, 1875. Reproposed as 5325, December, 1878.]

4897. Let $U, V=0$ be the point-equations, and $u, v=0$ the line-equations of the same two conics. If a tangent to U and a tangent to V are conjugate in respect of $u \pm \lambda v=0$, they will intersect on $U - \lambda^2 V=0$. This last conic passes through the points of contact of the conics $u \pm \lambda v=0$ with the common tangents of u and v .

[*N. T.* February, 1876.]

4926. Let $U, V, W=0$ be the point-equations, and $u, v, w=0$ the plane-equations of three quadrics inscribed in the same developable, and let $u+v+w$ be identically zero. Then, if a tangent plane to U , a tangent plane to V , and a tangent plane to W , are mutually conjugate in respect of

$$au + bv + cw = 0,$$

they will intersect on

$$\frac{U}{(b-c)^3} + \frac{V}{(c-a)^3} + \frac{W}{(a-b)^3} = 0,$$

which passes through the curves of contact of the developable with $au + bv + cw$ and one other quadric.

[*N. T.* March, 1876.]

4950. Prove that every matrix of the second order may be expressed in the form $aI + bJ$, where I is the matrix unity, and J a matrix such that $J^2 = -1$. Hence find an expression for any power of such a matrix. (See Cayley on Matrices, *Phil. Trans.* 1858.) Required a geometrical representation for a non-self-conjugate linear and vector function.

[*N. T.* April, 1876.]

5457. A triangle ABC has its vertices A, B , jointed to two rods AD, BE , which can turn about the fixed points D, E ; express the coordinates of the point C in terms of elliptic functions of a single parameter.

[*N. T.* November, 1877.]

[*N.B.* 1724, p. 624, has been solved, see p. 597.]

LECTURE I. ON BOUNDARIES IN GENERAL*.

Syllabus.

Every body distinguishes two adjacent regions of space, one inside and one outside.

- (a) The surface of the body is surface to both of these regions.
- (b) It takes up no solid room, or has no thickness.
- (c) When the body is moved continuously, the surface is moved continuously with it.
- (d) And yet a surface remains in the same place when the body is taken away.

Congruent regions are those which can be filled at different times by a body which does not alter in size or shape.

The remarks (a) (b) (c) (d) are true also of a line, boundary between two adjacent surface-regions, and of a point, boundary between two adjacent line-regions.

A line is also the intersection of two surfaces.

A point is the intersection of two lines, a line and a surface, or three surfaces.

A line is the path or locus of a moving point, a surface of a moving line, and a solid of a moving surface.

The number of points in a piece of line is singly infinite; the number in a piece of surface doubly infinite; and the number in a piece of solid space triply infinite.

A point on a line has one variation; on a surface, two; in solid space, three.

Questions.

1. Can two regions be partly adjacent and partly not? (Distinguish between solid, surface, and line-regions.)
2. Explain how a point is the intersection of three surfaces, and give an example.

* [From information furnished to me by Mr F. Pollock, Mr W. J. Ritchie and others, I find that these Syllabuses belong to a series of lectures given to a class of ladies at South Kensington in the spring and summer of 1869. The "proofs," for a complete set of which I am indebted to Mr C. J. Clay, bear dates ranging from April 8 to June 11, 1869. Lecture 1 is evidently that printed in "Seeing and Thinking," pp. 127—166, *Nature Series*; *Macmillan's Magazine*, Vol. XI. No. 238, pp. 359—368, Aug. 1879.]

3. State clearly what is meant by the assertion: A point can be moved continuously from one position to another, with the bodies of whose surfaces it is the intersection; and yet remains when they are taken away.

4. Is the motion of a shadow always continuous?

5. An infinite number of circles can be drawn upon a piece of paper. Is this number singly, doubly, or triply infinite?

LECTURE II. ON PLANE SURFACES AND STRAIGHT LINES.

Syllabus.

Space exists independently of the things in it, but allows them to be moved about without altering their size or shape.

A plane surface may be slid about upon itself or another plane surface, and will always fit.

It may also be turned over and applied to itself so as to fit.

A plane is of infinite extent.

The intersection of two planes (a straight line) is also the intersection of an infinite number of planes: or a plane may turn round it and slide along it.

A straight line divides a plane into two congruent regions.

A straight line is fixed by two points, a plane by three.

Two straight lines, or three planes, can meet only in one point.

Two straight lines, meeting, divide a plane into four regions congruent two and two. If all four are congruent, each is called a right angle, and the lines are perpendicular.

Only one straight line can be drawn perpendicular to a given straight line through a given point.

Only one straight line can be drawn parallel to a given straight line through a given point. It is then parallel to the given line at all points on it.

Regions at an infinite distance in opposite directions on the same straight line are adjacent and separated by one point.

All points at an infinite distance on one plane are in a straight line.

All points at an infinite distance in space are in a plane.

A straight line and a plane are respectively a line and surface of the first order.

Questions.

1. Two bodies which have plane surfaces may have those surfaces applied to each other in an infinite number of ways. Is this number singly, doubly, or triply infinite?

2. How do we know that the spaces on the two sides of a plane surface are of the same shape?

3. One plane divides another into two congruent surface regions. Explain what this means, and how you see it to be true.

4. How many straight lines can be drawn through a given point of space to meet each of two given straight lines?

5. In what sense is it true that there is only one point at an infinite distance on a straight line?

Rouché et De Comberousse, p. 12, Ex. 1—4.

Wright, p. 12, Ex. 1—4.

Wilson, p. 10, Ex. 1—7.

LECTURE III. ON THE ROTATION OF PLANE FIGURES.

Syllabus.

The properties of plane figures are divided into *projective* properties which are retained in the shadows of the figures, and *non-projective* properties, which are not so retained.

Properties connected with Rotation are non-projective.

A straight line turning about a point in it, by equal amounts of rotation generates congruent angles.

If a plane figure turn about any point in its plane, the directions of all lines in the figure are altered by the same amount.

The direction of a figure is equally altered by the same amount of turning about two different points.

Every change of position of a figure in its plane may be produced by a single rotation.

(The external angles of a polygon are together equal to four right angles. A triangle is determined by three independent elements. A parallelogram is congruent to itself in one way; a rectangle in three.)

The path of a rotating point is a *circle*. The direction of the point's motion is always perpendicular to the radius.

Equal angles at the centre of a circle cut off equal arcs of its circumference.

Questions.

1. Parallel lines (1) do not meet, (2) make congruent intersections with a third line. Shew that any two lines which possess either of these properties possess also the other.

2. Prove that the exterior angle of a regular hexagon is equal to the interior angle of an equilateral triangle.

3. Shew how to construct a triangle of which you know the height and the two base angles.

4. Can any two congruent triangles in the same plane be made to coincide by rotating one of them about some point?

5. How does it appear that the tangent at any point of a circle is perpendicular to the radius?

Rouché et De Comberousse, p. 22, Ex. 4, 7; p. 28, Ex. 1—3; p. 86, Ex. 1—6.

Weight, p. 22, Ex. 4, 7; p. 80, Ex. 1—3; p. 80, Ex. 1—7.

Wilson, p. 17, Ex. 1—7; p. 84, Ex. 1—7.

LECTURE IV. ON SIMILAR FIGURES.

Syllabus.

When a figure is enlarged so as to remain still of the same shape, every straight line in it remains a straight line, and every angle remains congruent to itself.

All the parts of the figure are equally enlarged.

When one figure is an enlarged copy of another, the two are said to be *similar*.

The degree of enlargement necessary to make one figure equal to the other is called their *ratio of similitude*.

The ratio of two lines in the one figure is equal to the ratio of the two corresponding lines in the other.

* If four quantities are proportional, and the first is greater or less than any fraction of the second, the third is greater or less than the same fraction of the fourth.

If two quantities are so connected that each, being given, determines the other: and if to the sum of two values of one corresponds the sum of the two corresponding values of the other: then the ratio of any two values of the first quantity is equal to the ratio of the two corresponding values of the second.

Triangles are similar which have (1) their sides proportional, (2) an angle in one equal to an angle in the other, and the sides about them proportionals.

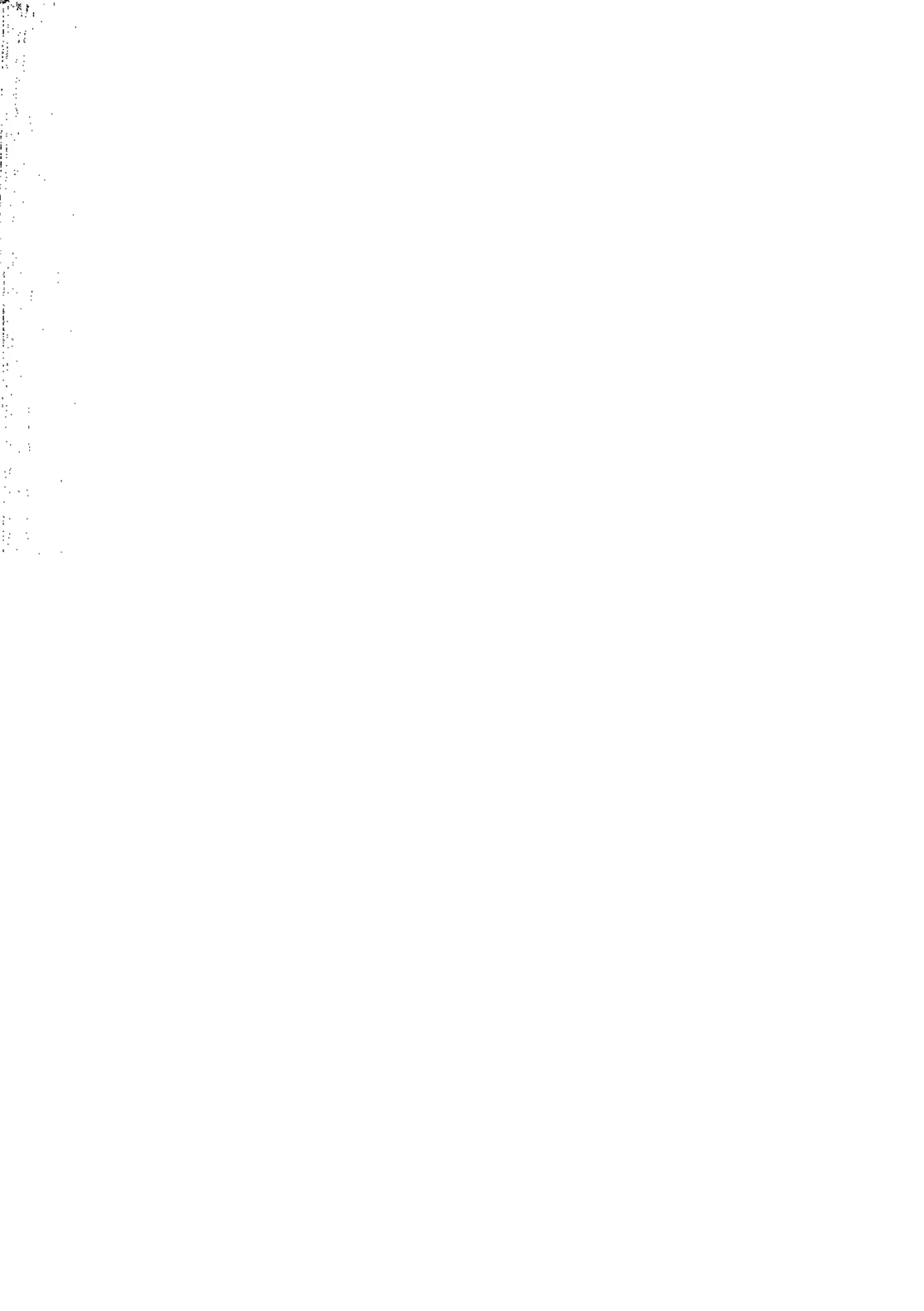
Similar rectilinear figures are made up of similar triangles.

Questions.

1. Prove that all circles are of the same shape.

2. If two parallelograms have the angles of one equal respectively to the angles of the other, and if the ratio of the two diagonals of one is equal to the ratio of the two diagonals of the other, the parallelograms are similar.

3. Show that two regular polygons of the same number of sides are similar figures.



2. What is the locus of the centres of all circles that pass through two fixed points?
3. And what is the locus of the intersection of the two tangents to each of the circles at those points?
4. Tangents are drawn to a circle at the angular points of an inscribed rectangle. Prove that the quadrilateral thus formed has all its sides equal.
5. AF , BG , CH are the perpendiculars of a triangle ABC , meeting in O . Shew that circles can be described about the quadrilaterals $AGOH$, $BHOE$, $CFOG$; and that the angles BFH , CFG are equal to one another.

LECTURE VIII. THE PROPERTIES OF ONE CIRCLE—continued.

Syllabus.

The following are different statements of the same projective property of a circle:—

1. If a series of lines are drawn through a fixed point to meet a circle, and if at the two points of intersection of each of these lines with the circle tangents are drawn; the intersections of all these pairs of tangents will lie on a fixed straight line.

2. If a series of points are taken on a fixed straight line, and from each of them two tangents are drawn to the circle; the several chords of contact of these pairs of tangents will all pass through a fixed point.

Two points A and B are called *inverse* points in respect of a circle whose centre is O , when OAB is a straight line, and the rectangle OA , OB is equal to the square of the radius.

A line through B perpendicular to OAB is then called the *polar* of the point A ; and a line through A perpendicular to OAB is called the *polar* of the point B .

When a line is the polar of a point, the point is called the *pole* of the line.

If a point L lies on the polar of M , then M lies on the polar of L .

The polar of a point outside the circle is the chord of contact of tangents from it to the circle.

Questions.

1. Two inverse points in respect of a circle are distant from the centre two inches and eight inches respectively; what is the length of the radius?
2. A circle has its radius one foot long, and a certain straight line is sixteen inches distant from the centre; how far is this straight line from its own pole?
3. Explain in what way every diameter is the polar of some point at infinity.

LECTURE X. ON THE ORDER OF GEOMETRICAL PROBLEMS.

Syllabus.

When a point has to be found on a straight line, and the problem of finding it is of the second order, so that it may have two solutions; then in certain particular cases these two solutions coincide and become one, and in other cases disappear altogether.

Geometrical problems are accustomed to say in the first case that there are two coincident solutions, rather than one solution; and in the second case that the two solutions have become imaginary or invisible, rather than that they have altogether ceased to exist.

This language contains a reference to a class of problems different from the one directly considered, in which lengths have to be measured, not on a line in one of two directions only, but in a plane and in any direction. In these problems the solutions never disappear, and the number of them is always exactly equal to the order of the problem.

By this use of language belonging to a different branch of science, not only is great generality introduced into the enunciations and proofs of theorems, but an explanation is afforded of the very different forms under which the same theorem presents itself.

[Hibbs's chair, loc. cit. see below.]*

[The following, Prof. Huxley tells me, are notes of a course of lectures on Synthetic Geometry and Geometrical Optics. I have thought them worthy of a place here as the treatment is somewhat novel, at least, to English readers.]

- I. Principles of a circle, of parallel lines, of perspective ranges and pencils of lines and planes, of range perspective to itself with two points interchanged.
1. Is called *harmonic*, construction by quadrilateral.
- II. Perspective ranges and pencils (assuming only Euclid).
2. If two point ranges are projective in one position they are in all positions (two cases).

Projective correspondence.

Case 1. Projective correspondence is determined by three pairs of corresponding points.

Case 2. Two on same line may have two united points but not more.

III. Two ranges projective to same range are projective to each other.

* I have looked at Mr H. H. Coxeter for a reference to Huxley's English *Fish Lore*, p. 204. Mr Huxley speaks of *Harporichthys* *Regius* 1791, loc. cit. Vol. 4, p. 167, and Harland and Wilkinson's *Lancashire Fish Lore*, 1863, p. 448, which I believe consists of two Hibbs's Chairs. The former work describes the chair of the *Yarrowdale House*, at *Starrus* *Starrus*, Northumberland. "It is preserved in the vestry of the Church, within all Hibbs's equal immediately in the paragon service is over, to rest themselves upon it. The chair, which is very wide and substantial, is made of oak, is 4 feet 10 inches high; having no upright back, and sides that slope off for the arms." Mr Coxeter also points out, in a figure, the resemblance of the diagram to the constellation of Cassiopeia, or 'the lady in her chair.'

16. From any point on tangent one and one only other tangent can be drawn.
 Construction of simultaneous points and tangents; definition of conic.
 19. Pascal, Brianchon,
 VII. Poles and Polars,
 VIII. Metrical properties (centre and diameters),
 IX. System of conics, foci, etc.
 Preceded by retreatment of conics by (1, 1) correspondence.
 X. Surfaces, corresponding to VII. and VIII.

Appendix A. On the representation of Solid Figures, commonly called Descriptive Geometry.

Appendix B. On Graphical Calculation.

Part 2. Graphical Statics,

- I. Polygon of forces, plane and space, stresses on frame-work by method of sections (3).
 II. The Polygon (plane only).
 III. Moments and Parallel Forces. Couples.
 IV. Centre of Parallel Forces.

For Appendix B. On Graphical Calculation. (Integration of x^n .)

1. Sum of Geometrical Series by anti-parallelism *

$$\begin{array}{ccccccc} 12 & + & 23 & + & 34 & + & 45 & + & 56 & + & 12 & = & 0a = 0a \\ 12 & & & & & & & & 23 & + & 12 & = & 0b = 0a \end{array}$$

which is obvious.

2. Differentiation of $y = ax^n$

$$\frac{y_1 - y_2}{x_1 - x_2} = a \frac{x_1^n - x_2^n}{x_1 - x_2} \quad (\text{in limit}) a \cdot nx^{n-1}.$$

3. Idem for a fraction.

4. If ordinate of one curve represents area of another, $\frac{\delta y}{\delta x}$ of this curve will represent ordinate of other.

5. Area for $y = ax^n$.

* (Draw 102, 123, 4 horizontal and 0, 1, 2, 3, 4, 5, 6 inclined to it, so that 1a, 2b, 3c, 4d, 5e form one set of parallels and a2, 13, c4, d5, e6 the other set of parallels.)

NOTES.

The paper *XVI. was printed before I had examined the question 2042 (p. 612) and the solutions of it. I have since submitted it to Mr J. J. Walker, who has placed at my service the following remarks :

Greater clearness is attained, and some errors are avoided, in establishing these fundamental formulae by starting at once with homogeneous co-ordinates $a_1 a_2 a_3$ of the point a . The constant $(\Delta \div R)$

$$a_1 \sin \alpha + a_2 \sin \beta + a_3 \sin \gamma = \frac{1}{3} \begin{vmatrix} a_1 & a_2 & a_3 \\ -1 & e^{-i\gamma} & e^{i\beta} \\ -1 & e^{i\gamma} & e^{-i\beta} \end{vmatrix}$$

so that calling i the point whose co-ordinates are proportional to

$$-1, \quad e^{-i\gamma}, \quad e^{i\beta},$$

and j the point whose co-ordinates are proportional to

$$-1, \quad e^{i\gamma}, \quad e^{-i\beta},$$

the constant may be written symbolically

$$\Delta \div R = \frac{1}{3} (aij).$$

The numerator in Fauro's expression for \overline{ab}^3 : viz.

$$\overline{ab}^3 = \frac{\Sigma (a_2 b_3 - a_3 b_2)^2 - 2 \Sigma (a_3 b_1 - a_1 b_3) (a_1 b_2 - a_2 b_1) \cos \alpha}{(a_1 \sin \alpha + a_2 \sin \beta + a_3 \sin \gamma)^2 (b_1 \sin \alpha + \dots)^2},$$

may also be expressed as the product of the determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ -1 & e^{i\gamma} & e^{-i\beta} \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ -1 & e^{-i\gamma} & e^{i\beta} \end{vmatrix}$$

or (abi) , (abj) ; so that

$$\overline{ab}^3 = 16 \frac{(abi)(abj)}{(aij)^2 (bij)^2} \quad \text{or} \quad \overline{ab} = 4 \frac{\sqrt{(abi)(abj)}}{(aij)(bij)}.$$

Again the distance of a from B is

$$\frac{B_1 a_1 + B_2 a_2 + B_3 a_3}{\sqrt{B_1^2 + B_2^2 + B_3^2 - 2B_2 B_3 \cos \alpha - \dots}},$$

and the denominator may be written as the square root of the product

$$(-B_1 + B_2 e^{i\gamma} + B_3 e^{-i\beta}) (-B_1 + B_2 e^{-i\gamma} + B_3 e^{i\beta});$$

thus

$$\text{distance } aB = \frac{aB}{\sqrt{iB \cdot jB}}.$$

The well-known formula for $\sin AB$ is

$$\frac{\Sigma (A_2 B_3 - A_3 B_2) \sin \alpha}{\sqrt{iA \cdot jA \cdot iB \cdot jB}} \text{ or } \frac{AB \bar{ij}}{\dots},$$

wherein \bar{ij} stands for the line containing i, j , viz., as has been shewn, the line

$$x_1 \sin \alpha + x_2 \sin \beta + x_3 \sin \gamma = 0.$$

Also

$$\cos AB = \frac{A_1 B_1 + A_2 B_2 + A_3 B_3 - (A_2 B_3 + A_3 B_2) \cos \alpha - \dots}{\dots} = \frac{\frac{1}{2} \{ (-A_1 + A_2 e^{i\gamma} + A_3 e^{-i\beta}) (-B_1 + B_2 e^{-i\gamma} + B_3 e^{i\beta}) + (-A_1 + A_2 e^{-i\gamma} + A_3 e^{i\beta}) (-B_1 + \dots) \}}{\dots},$$

or

$$2 \cos AB = \frac{Ai \cdot Bj + Aj \cdot Bi}{\sqrt{Ai \cdot Aj \cdot Bi \cdot Bj}}.$$

I may also state that I have recently received lecture-notes on the subject from a pupil of Prof. Clifford's at University College*, who has also furnished me with the accompanying proof, given also in lecture, of Ivory's Theorem†.

Def. Corresponding points on two confocal ellipsoids are such as coincide when either ellipsoid is deformed by a homogeneous strain, so as to coincide with the other.

Statement. Let corresponding points Pp be taken on two homogeneous confocal ellipsoids Ee . The x component of the attraction of E on p is to that of e on P as the area of the section of E by the plane yz is to the area of the coplanar section of e , that is in the ratio $\beta\gamma : \alpha\alpha$.

This theorem is true for any law of force.

First we show that for any law of force the attraction of a straight line AB of uniform thickness and density on an external point P depends only on the distances PA, PB . (Fig. 125.)

$$\frac{dr}{dx} = \cos PMB.$$

Thus, when $f(r)$ determines the law of force, attraction of element dx resolved parallel to $AB = f(r) \rho dx \cos PMB$.

Thus whole attraction parallel to AB is

$$\int_a^b f(r) \rho dr = \rho \{F(v) - F(u)\}.$$

Divide up both ellipsoids (Fig. 126) into strips parallel to the axis of x .

Let Qq be corresponding points.

Then clearly $Q'q'$ are also corresponding points.

* Mr G. W. von Tunzelmann, who says that Prof. Clifford informed him that he had found it easy to extend the method to higher plane curves, but that surfaces were much more difficult and he had not made much progress in its application to them. This pupil has also put at my service other Notes which I hope to make use of hereafter.

† See *Lectures and Essays*, Vol. I. p. 4.

The attraction of the strip qq' on any point P on the outer ellipsoidal B depends on the thickness of the strip, and on Pq, Pq' ; and attraction of qq' on corresponding point on inner ellipsoid e , depends on the thickness of the strip and on $Qp, Q'p$. But $Qp = Pq$ and $Q'p = Pq'$.

Therefore the whole attractions are in the ratios of the sections by the yz . Therefore the Theorem is proved.

XXV. is referred to in Maxwell's *Electricity*, Vol. I. p. 171. Nothing on the subject has been met with in the papers that have been submitted.

*XXXVII. On a loose sheet I find some work slightly differing from the text, with the result

$$\cos(2x \cos \phi) = f(-x^2) - 2x^2 f_2(-x^2) \cos 2\phi + 2x^4 f_4(-x^2) \cos 4\phi \dots$$

$$\therefore \int_0^\pi \cos(2x \cos \phi) \cos 2n\phi d\phi = (-)^n \pi x^{2n} f_{2n}(-x^2).$$

The following notes are given here as additional illustrations to *XI. II., p. 171.

[i] If a line meeting two polar lines be displaced equally along both, or through an angle α , the two positions of the line will be called *parallel* and *antiparallel* to have the same direction. The normal distance from any point on each of them to the other is α . Two lines parallel to the same line are parallel to each other, and if a line meet two parallel lines it meets them at equal angles. According as the twist converting a line into a parallel line is right handed or left-handed, the common direction of the two lines will be called a *right* or *left* direction; thus every line has two directions, and through an arbitrary point two lines may be drawn having with it right and left parallelism respectively. All lines having the same right direction meet the same two generators of the system of the absolute; all having the same left direction meet two generators of the other system. If a series of parallel lines be drawn through all the points of any line, they will trace out a surface of zero curvature and first order extent, which is in fact a quadric having quadruple contact with the absolute or meeting it in four straight lines. Starting with three lines meeting at right angles, we may determine a triple series of such surfaces, intersecting each other at right angles in lines parallel to the original three; thus every point in space will be characterized by co-ordinates measured parallel to these three axes. If any solid body receive simultaneous equal rotations about two parallel lines, all the points of the body will move in parallel lines, and any series of these lines may be regarded as an axis of the motion, viz. the body has equal rotations about this line and its polar. Such a motion may be called a *translation*; it may be represented by a finite straight line having given magnitude, direction, and will be a *right* or *left* vector according as the parallelism is right or left. The ratio of two vectors of the same side (i.e. both right or both left) is a *quaternion* of the same side; viz., one can be converted into the other by a certain rotation about an axis perpendicular to both but of indeterminate position. Every motor is the sum of a right and a left vector; for every motor is identically

$$A = \frac{1+\omega}{2} A + \frac{1-\omega}{2} A,$$

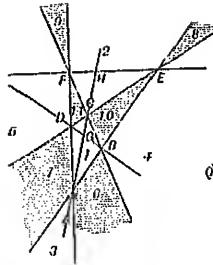


Fig. 120.

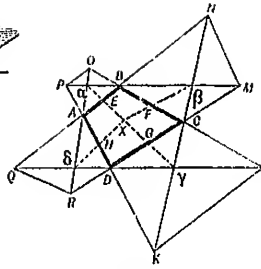


Fig. 121.

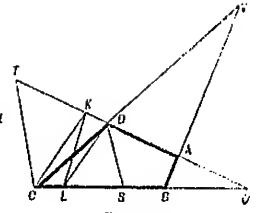
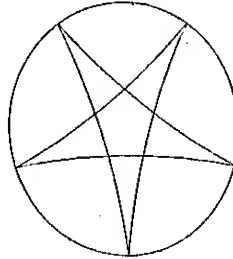
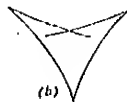


Fig. 122.

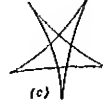
Fig. 123.



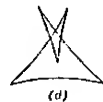
(a)



(b)



(c)



(d)



(e)

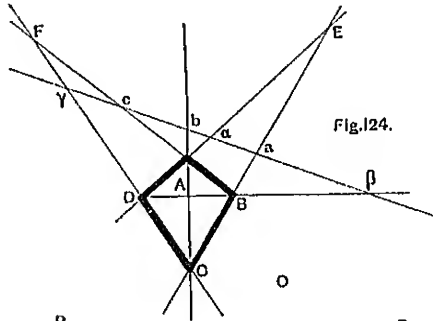


Fig. 124.

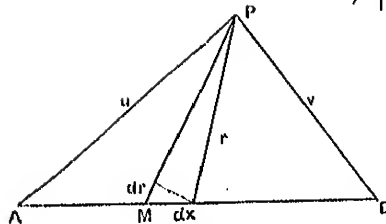


Fig. 125.

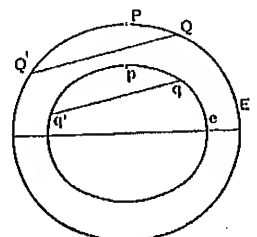


Fig. 126.

viz. these vector parts are the half sum and half differences of the motor A and its polar motor ωA . If we write $\xi = \frac{1}{2}(1 + \omega)$, $\eta = \frac{1}{2}(1 - \omega)$, then

$$\xi^2 = \xi, \quad \eta^2 = \eta, \quad \xi\eta = 0,$$

and we have, if $\alpha \beta \gamma \delta$ are rotors through a fixed point,

$$\frac{\xi\alpha + \eta\beta}{\xi\gamma + \eta\delta} = \xi \frac{\alpha}{\gamma} + \eta \frac{\beta}{\delta},$$

whereby the quotient of any two motors is expressed as a biquaternion. For on substituting for $\xi \eta$ their values, the expression becomes $q + \omega r$, where q and r are the quaternions

$$\frac{1}{2} \left(\frac{\alpha}{\gamma} + \frac{\beta}{\delta} \right), \quad \frac{1}{2} \left(\frac{\alpha}{\gamma} - \frac{\beta}{\delta} \right).$$

[ii] The co-ordinates of a straight line being $p_{12} p_{23} p_{31} p_{14} p_{24} p_{34}$, let

$$M = p_{12} p_{34} + p_{13} p_{42} + p_{14} p_{23},$$

and let $\Omega = 0$ be the condition that this line touches the absolute or has no length. If we now write

$$\alpha = a_{12} \partial_{p_{12}} + \dots = \Sigma a_{ij} \partial_{p_{ij}},$$

then $\alpha\beta M = 0$ is the condition that the lines α, β (i.e. the lines whose equations are $\alpha M = 0, \beta M = 0$) should meet. $\alpha\Omega = 0$ is the equation of the polar line of α .

In general there are two straight lines which meet two lines and their polars. For the five equations $\alpha M = 0, \beta M = 0, \alpha\Omega = 0, \beta\Omega = 0, M = 0$, determine quadricly the ratios of the p . Now these two lines are polars. If γ be one, we have $\alpha\gamma M = 0, \beta\gamma M = 0, \alpha\gamma\Omega = 0, \beta\gamma\Omega = 0, \gamma^2 M = 0$.

$$\text{Moment of two motors} = \frac{\alpha\beta M_3}{\sqrt{\alpha^2 \Omega_2 \cdot \beta^2 \Omega_2}}.$$

[iii] $(\xi q + \eta r)(\xi A + \eta A) =$ a pure motor if $S\xi q A = 0, S\eta r A = 0$, or if we write

$$A = \xi\alpha + \eta\beta,$$

then q must directly operate on α and r on β .

Axes of $\xi\alpha + \eta\beta = \frac{1}{2}(T\alpha \pm T\beta)(\xi U\alpha \pm \eta U\beta)$,

$$\xi\alpha + \eta\beta = \frac{\alpha + \beta}{2} + \omega \frac{\alpha - \beta}{2}.$$

This is a rotor if $S \cdot \overline{\alpha + \beta} \cdot \overline{\alpha - \beta} = 0$, or if $\alpha^2 - \beta^2 = 0$. Hence the sum of two right and left vectors of equal length is a pure rotor.

Or we may say the axes of A are $(T\xi A \pm T\eta A)(\xi U\xi A \pm \eta U\eta A)$.

Observe that $\omega(\xi A + \eta B) = \xi A - \eta B$. The general expression for rotor right parallel to i is $\xi i + \eta\beta$, β being of unit length; the axis is $\xi i + \eta\beta$ as it should be, or $\frac{1}{2}(i + \beta) + \frac{1}{2}\omega(i - \beta)$; therefore if ρ meets this, $\xi(i + k\rho) + \eta(\beta + k\rho)$ is a rotor; or

$$T(i + k\rho) = T(\beta + k\rho),$$

$$S i \rho = S \beta \rho,$$

$$\therefore S(i - \beta)\rho = 0.$$

Hence ρ is in a plane equally inclined to i and β .

The following fragments have some points of interest, and I print them in their incomplete state as they may suggest lines of working to some readers. The portion on curves [v] is headed *Appendix*: it looks at first sight as if it were intended to be an appendix to Frost's *Curve Tracing*.

[iv] *Geometry in a Quadric Space Q_2 .*

Space-sections are represented by quadric-surfaces passing through a fixed conic ω_2 . The points of this conic represent a cone K_2 lying entirely in the space Q_2 . The vertex o of the cone is represented by the plane O of the conic ω_2 .

A Plane represents a quadric surface containing two lines of K_2 , since it meets ω_2 in two points.

The complete section of Q_2 by a space of order n passing a times through o is represented by a surface of order $2n - a$, passing $n - a$ times through the conic ω_2 .

A surface of order μ passing b times through ω_2 represents a surface in Q_2 of order $2\mu - 2b$, having o for a conical point of order $\mu - 2b$.

Q_2 is cut by a surface of order r passing a times through o in a curve represented by a curve of order $2r - a$ meeting ω in $2r - 2a$ points.

And a curve of order ρ meeting ω in b points represents a curve of order $2\rho - b$ passing $\rho - b$ times through o .

Thus e.g. a quadriquadric surface is represented by a surface of the fourth order passing twice through ω_2 ; and is therefore unicursal.

A Quadric space can contain no surface of odd order, unless its discriminant vanish. In this case the conic ω_2 is replaced by two lines $\omega\omega'$, and a surface passing a, b times through these represents a surface of order $2\mu - a - b$. Thus the quadric space now contains two pencils of planes represented by the planes through these lines ω, ω' . And a [quadri]quadric surface passing through one of them represents a skew cubic surface, passing once through o , meeting each of the planes of its own system in a line, and each of the others in two points. This cubic surface is ruled, then; and the other lines of the representative quadric represent a pencil of conics upon it, passing through o , but otherwise not meeting each other, but each meeting all the lines. Plane sections of the quadric represent skew cubic curves on the surface, namely its space-sections made by spaces passing through o . But the general space-sections are represented by skew cubics, meeting the conics twice and the lines once. We may generally represent the skew cubic surface upon a plane in the following manner. Its space-sections are represented by conics passing through one point a ; sections by space of n th order, curves $2n$, n times through a ; curves μ , a times through a represent curves of order $2\mu - a$. Thus the point a itself and the lines through it are the only straight lines on the surface; all other lines in the plane represent a doubly infinite system of conics, any two meeting in one point; each of these is met twice by a space-section; and so on.

Cubic Space C_3 .

Space-sections are cubic surfaces any three of which meet in three points; i.e. they have common a cubic curve of deficiency 2 and one fixed point, or else a quartic curve of deficiency 0 and two fixed points.

[v] *On the general shapes of Algebraic Curves.*

Those who have studied the preceding chapters of this work must still be convinced that, even with the facility which such study has given them, the operation of tracing a curve from its equation is one which requires some time and much close attention for its successful accomplishment. They will readily admit that if, in order to know the chief varieties of form among curves of the first four or five orders, it were necessary to trace one after another the curves represented by all varieties of equations of those degrees, by such processes as have already been explained; this work would be enormous, and beyond the reach of any but a lifelong study. Accordingly, it has never been attempted in the case of any degree exceeding the second. In the enumeration of lines of the third order, either the equation has been reduced by some preliminary discussion (to be presently mentioned) to a very special form, or its consideration has been evaded altogether. Of quartic curves the enumeration, so far as I know, has not yet been systematically attempted; but such general results as have been arrived at were obtained by considerations and methods other than those which the equation itself suggests.

To the explanation of such considerations and methods the present appendix is devoted. The study of them presents many direct advantages to the mathematician, besides the indirect ones of the help they afford him in the tracing of curves and the investigation of their properties. These methods are exceedingly simple and easy of application; they partake more of the nature of a manual craft than of a purely intellectual occupation, and may so be used as a rest from severer studies; and—as we can only imagine things of which we have seen the like—by appealing directly to the senses they extend those powers of concrete realization which the growing complication of modern analysis renders daily more desirable.

They consist of rules for transformation, by the aid of which the figures of a vast variety of curves may be obtained from a few simple types. These rules will be explained under three heads. In the first section is considered Projection, a process by which no alteration is made in the order, the class, nor in any other purely descriptive property of a curve. In the second section those modifications of form are described which leave the *order* of a curve unaltered. In the third, mention is made of those changes which exercise no effect upon the class. In the two latter sections a process will be described which may be called the *composition* of curves; by which a curve of any order or class may be built up out of the simplest elements.

Section 1. Projection.

The shadow of a hyperbola may be an ellipse, if it is properly held; and conversely, an ellipse may be so placed as to have a hyperbola for its shadow. More generally, the shadow of a curve may have more, or fewer, infinite branches than the curve itself. Now the fewer infinite branches a curve has, the easier it is to get a general idea of its shape; thus, to return to our first example, the shape of an ellipse is easier to grasp and remember than that of a hyperbola. The object of the method of Projection is to substitute for any given curve that shadow of it which has the least possible number of infinite branches; and thus to classify the enormous variety of shapes as the shadow of a comparatively small number of simple forms.

[vi] *The One-Two Correspondence of Two Planes.*

1. I consider two planes X and Ξ , the points of which are related in such a way that to every point x of X correspond in general two points ξ and ξ' of Ξ , while to every point ξ of Ξ corresponds in general one and only one point x of X .
2. Let the correspondent of a line L in X be a curve Γ_m in Ξ . I shall now prove that the correspondent of a line Λ in Ξ is a curve C_m in X ; the curves C_m and Γ_m being both of the same order m . The curves Γ_m must be such that two of them intersect in two points (variable), and that only one can be drawn through two assigned points. Γ_m meets Λ in m points, and to each of these corresponds one point in X , m is therefore the number of intersections of L and the correspondent of Λ , which consequently is of order m as asserted.
3. The two representatives on Ξ of a point on X have between themselves a (1, 1) correspondence; let this be of order μ . Then the entire correspondent of C_m is not merely the line Λ but besides a curve Δ_μ which is unicursal and passes through a set of Cremona's principal points. The intersections of C_m , C'_m are represented then by (1) the point $\Lambda\Lambda'$; (2) the point Δ_μ , Δ'_μ ; (3) the 2μ points Λ , Δ'_μ and Λ' , Δ_μ —in all $2+2\mu$ points. So then two curves, C_m , C'_m intersect in $\mu+1$ variable points. They are unicursal, as corresponding uniquely to straight lines. And through two arbitrary points ab four of them can be drawn; viz., the representatives of the lines $\alpha\beta$, $\alpha\beta'$, $\alpha'\beta$, $\alpha'\beta'$.
4. The curves Γ_m pass through certain fixed points. To single points correspond lines in X , to double ones conics, &c., exactly as in Cremona's theory. These curves are all unicursal. But to common points of the C_m may correspond curves which are not unicursal.
5. The case $m=2$ is of little interest and easily studied. The Γ_2 are conics through two fixed points, with a linear relation; their Jacobian is then a conic K_2 and a line Λ , both passing through the two fixed points. μ is equal to 2, and the Δ_2 are conics through the fixed points and the pole of Λ in respect of K_2 . The C_2 are conics through a fixed point correspondent of Λ having double contact with a fixed conic, the correspondent of K_2 . To the fixed points on Ξ correspond the tangents from the point to the conic on X .

6. The case $m=3$ gives us two systems: (1) cubics with a node and three points fixed, satisfying besides a linear relation; (2) cubics with seven fixed points. The first case is obtained by quadric transformation from the case $m=2$, whereby we see that the Jacobian is a cubic of the system (not satisfying the linear relation) and the straight lines joining the node to the fixed points. $\mu=3$, and the Δ_3 are cubics through the fixed points and another. The C_3 are cubics having triple contact with a fixed conic and a node at a certain fixed point.

The second system clearly inverts into itself. The Jacobian is a sextic S_6 with seven nodes at the fixed points. The correspondent of this is a curve, N_{12} .

[vii] *On the (1, n) Correspondence of Two Planes.*

1. I consider planes X, Ξ so related that to every point x on the first plane correspond in general n points, $\xi^1, \xi^2, \dots, \xi^n$, on the second, while to every point ξ of the second plane corresponds in general only one point x of the first.

2. Let the correspondent of a line L in X be a curve Γ_m in Ξ . I shall now prove that the correspondent of a line Λ in Ξ is a curve C_m in X ; the curves C_m and Γ_m being both of the same order m . The curves Γ_m must be such that two of them intersect in n variable points, and that only one can in general be drawn through two assigned points. Γ_m meets Λ in m points, and to each of these corresponds one point in X ; m is therefore the number of intersections of L and the correspondent of Λ , which consequently is of order m as asserted.

3. Two representatives on Ξ of the same point on X have between them a $(n-1, n-1)$ correspondence; let this be of order μ . Then the entire correspondent of C_m is not merely the line Λ , but besides a curve Δ_μ . Two curves Δ_μ, Δ'_μ intersect in $n-1$ variable points. The intersections of C_m, C'_m are represented then by (1) the point $\Lambda\Lambda'$; (2) the $n-1$ points Δ_μ, Δ'_μ ; (3) the 2μ points Λ, Δ'_μ and Λ', Δ_μ —in all $n+2\mu$ points. Thus the number of variable intersections of two curves C_m, C'_m is $1+2\frac{\mu}{n}$, and consequently (except when $n=2$), μ must be a multiple of n . The curves C_m are unicursal, as corresponding uniquely to straight lines. And through two arbitrary points a, b there may be drawn n^2 of them; viz., those corresponding to the n^2 lines joining the n correspondents of a, a^1, a^2, \dots, a^n to the n correspondents of $b, \beta^1, \beta^2, \dots, \beta^n$.

[viii] *Geometry on a Cubic Surface.*

1. A system of values of the coordinates $X^{(1)}, X^{(2)}, X^{(3)}$ in general determines one point on the surface. There are however six systems of values which determine not points but loci (viz. one-half of a double-sixer). These systems are to be denoted by $a^{(1)}, a^{(2)}, \dots, a^{(6)}$, and called the Extension-Absolute (as distinguished from the Measure-Absolute, which is another set of six values).

2. Let $K_2^{(1)}, K_2^{(2)}, K_2^{(3)}$ be quadric functions of the X which are satisfied by the systems $a^{(1)}, a^{(2)}, a^{(3)}$. In general, a system of ratios of the K_2 determines one point on the surface; but the system $K_2^{(1)} = 0, K_2^{(2)} = 0, K_2^{(3)} = 0$ has a meaning, viz. the loci $a^{(1)}, a^{(2)}, a^{(3)}$.

Three systems of values of the K_2 (viz. the other three points a) now represent loci and not points. But besides these there are three other systems which represent the lines $a^{(1)}a^{(2)}, a^{(2)}a^{(3)}, a^{(1)}a^{(3)}$. I call these six value-systems $k^{(1)} \dots k^{(6)}$. The K -coordinates then are precisely equivalent in their properties to the X , except that they vanish all together on the locus $a^{(1)}, a^{(2)}, a^{(3)}$.

But now the point $a^{(1)}$ may be approached in various ways; and it will be found that though the K_2 all vanish at that point, yet the ratios in which they vanish depend on the direction in which the point is approached. In fact, the vanishing ratios of the K_2 enable us to discriminate between the points of the locus $a^{(1)}$. We may then, if we like, consider only their ratios, deprive the system $K_2 = 0$ of its meaning, and so arrive at a new coordinate-system exactly similar to the one with which we started.

In fact, if $A^{(i)}$ denote $a^{(1)}a^{(2)},$ etc. we can find quadric functions of the K_2 respectively equal to $X^{(1)}A^{(1)}A^{(2)}A^{(3)}, X^{(2)}A^{(1)}A^{(2)}A^{(3)}, X^{(3)}A^{(1)}A^{(2)}A^{(3)}$.

Here it is clear that $A^{(i)} = 0$ makes these three simultaneously vanish, but their ratios are those of the X .

3. I now take three cubic functions $C_3^{(1)}, C_3^{(2)}, C_3^{(3)}$ of the X , which are all satisfied doubly by $a^{(1)}$ and singly by all the other points a except $a^{(1)}$. A system of ratios of the C_3 will in general determine one point of the surface; but the system $C_3^{(1)} = 0, C_3^{(2)} = 0, C_3^{(3)} = 0$ has a meaning; viz. the locus $a^{(1)}$ twice, and the other loci a except $a^{(1)}$ once.

The system of values $a^{(1)}$ of the C_3 now represents a locus and not a point. But there are systems of values which represent (1) the lines joining $a^{(1)}$ to the other a except $a^{(1)}$ and (2) the conic passing through all the a but $a^{(1)}$. I call these six values $c^{(1)} \dots c^{(6)}$. The C_3 -coordinates are thus again equivalent to the former systems, with the exception that they all vanish together on the locus $\Sigma a - a^{(1)}$.

The ratios of the C_3 in the immediate neighbourhood of the point $a^{(1)}$ will be found to acquire two distinct value-systems for every direction of approach. The locus $a^{(1)}$ is therefore represented by a quadric equation in the C_3 . The remaining loci a are represented by linear equations as before.

4. Similar remarks are to be made on quartic functions $Q_4^{(1)}, Q_4^{(2)}, Q_4^{(3)}$ having nodes at $a^{(1)}, a^{(2)}, a^{(3)}$ and single points at the other three. They remove entirely the pre-existing absolute, and substitute a new one, $q^{(1)} \dots q^{(6)}$ consisting of Cremona's well-known principal system. So further the quintics $P_5^{(1)}, P_5^{(2)}, P_5^{(3)}$ which have a node at each a , substitute a new absolute representing the six conics each through five a points. Altogether we have $1 + 20 + 30 + 20 + 1 = 72$ equivalent linear systems.

Plane sections of cubic
skew cubics

cubics through 6 points a
straight lines on plane

skew cubics generally

- (1) straight lines
- (2) conics through 8 points a
- (3) cubics with node at a_1 , passing through
 a_2, a_3, a_4, a_5
- (4) quartics with nodes at a_1, a_2, a_3 , and single
points a_4, a_5, a_6
- (5) quintics with a node at each point a .

Any one of Clebsch's 72 nets of skew cubics may be taken for a linear system on the cubic surface. The net cuts twice 6 lines on the cubic; the other half of the double-six forms the absolute. Analytically there are six sets of ratios $X : Y : Z$ which mean not points but these six lines; I call these quasi-points (a, b, c, f, g, h) . A line $lX + mY + nZ$ which satisfies the condition of passing through one of these say, a , breaks up into the line a and a conic through a' , the corresponding line of the double-six. The line ab contains these two lines and the line which meets ab' , $a'b$. Abstractions made of the absolute, then, the lines ab, \dots are the other 15 st. lines of the cubic. The lines a', \dots are represented by the conics through each five of the points.

When the surface has a node, the linear system is that of the plane sections through the node. The points $abc fgh$ are then upon one conic.

$$\begin{array}{llll} K_2^{(1)} \equiv X^{(2)} X^{(3)} \text{ the point } K_2^{(2)} = 0, & K_2^{(3)} \equiv 0 & \text{gives merely } X^{(1)} = 0 \\ K_2^{(2)} \equiv X^{(3)} X^{(1)} & ,, & K_2^{(3)} = 0, & K_2^{(1)} \equiv 0 & ,, & X^{(2)} = 0 \\ K_2^{(3)} \equiv X^{(1)} X^{(2)} & ,, & K_2^{(1)} = 0, & K_2^{(2)} \equiv 0 & ,, & X^{(3)} = 0 \end{array}$$

[ix] *On the Correspondence between a Doubled Line and a Cubic.*

If I take a fixed point o on a cubic curve C_3 , I can draw through it lin which univocally correspond to the points on a straight line X . I have th for every point x on the line a pair of points c, c' on the cubic. But now I ma suppose this line to be doubled, call it XX' , and then the two (coincident points xx' may be held either to correspond as a whole to the pair cc' , or x may be held to correspond to c , and x' to c' . I shall now take instead of the doubled line XX' , a doubled circle, so as to have the whole of it within reach. But now from the fixed point o four tangents can be drawn to the cubic; let us assume to begin with that they are all real. Then (calling them $ABCD$) if between A and B the points cc' are real, they must also be real between C and D , but imaginary between B and C and between D and A . Thus the real part of the cubic may be represented by two portions of a doubled circle*.

We have in fact substituted for the cubic an unalgebraic quartic on the point of becoming a doubled circle. If two of the tangents are imaginary we

* [There are three figures, (3) is a complete ring, (4) consists of two parts $(AB), (CD)$ of the same ring, and (2) is a part of a ring, semicircular (AB) .]

may suppose B and C to coincide, and the curve to be represented by one very thin oval; if they are all imaginary, the curve is represented by the entire contour of two indefinitely near circles.

(MS. ends.)

[x] *Syllabus of Lectures**.

| | |
|------------------|--|
| History Space | General Principle—all the properties of a geometric form depend on its Order. Hence begin by establishing theory of imaginaries, on which that of the order depends. Thus, 1. Historique. 2. Fundamental Hypotheses. Continuity of Space. Motion without change of size. Infinite extent. Definitions of Line and Plane, perpendicular and parallel. 3. Calculus of Ratios and Position. Primary theorem of number. Deduction of arithmetical rules. Operations. Algebraic calculus of numbers. Primary theorem of continuous quantity. Rules analogous to arithmetical operations. Algebraic calculus of ratios. Calculus of Position in one dimension. 4. Position in two dimensions. Gaussian Formulæ. Discontinuity of the reversor symbol, is made continuous by calculus of position in two dimensions. Idea of functional correspondence, monodromie, similarity of the smallest parts. Every equation of n th order has exactly n roots. |
| Quantity | 5. Position in two dimensions. Cartesian formulæ. Co-ordinates of a point, of a straight line, distance between given points, angle between given lines. |
| Imaginaries | Equation of straight line, of circle, ellipse, hyperbola, parabola. |
| Equations | 6. Projection and Linear Transformation. Passage to homogeneous form. Line at infinity. Projection equivalent to Linear Transformation. Order of curve. Number of points in which it is met by <i>any</i> line whatever. General notion of invariant. Invariants of points and lines. GRASSMANN notation for these. ARONHOLOD notation for invariants in general. Duality, contravariant symbols, contravariant differentiation. |
| Invariance | 7. Correspondence in one dimension. (1,1) correspondence, anharmonics, harmonics, involution. Harmonics and involution of higher orders. |
| Correspondence | 9. Plücker's equations, the Deficiency. Correspondence on a curve. |
| | 10. General theory of Polars. Special application to conics. Conjugierte Kerncurven. |

[xi] *Geometric Analysis.*

1. The calculus of Ratios, and of onefold Position.
2. The calculus of twofold Position: (1) the Cartesian; (2) Gauss' plane of numbers.
3. The simpler Cartesian formulæ for geometrical magnitudes.
4. Equation in general. Forms of equation of straight line.
5. Equations of circle and conic sections.

* These are evidently the notes of the lectures printed above (pp. 531--530), and I give them as they show what were the subjects of the missing articles on p. 528.

6. Equation of a curve of any order.
7. Passage to homogeneous co-ordinates. The co-ordinates of a geometrical form in general.
8. The GRASSMANN notation.
9. The imaginary in geometry.
10. Systematic geometry of one dimension. Harmonics and Anharmonics.
11. The Polar Theory of conics.
12. The Bitangent circles of conics (pp. 543--5).
13. General Theory of the circle: powers of circles.
14. Theory of anallagmatic curves (pp. 546--555).
15. Extension of the GRASSMANN notation. (General theory of distance.
16. Plücker's Equations. The Deficiency.
17. Polar theory of curves of the n th order.
18. General theorems relating to cubics.
19. The Polar theory of cubics.
20. Passage from the extended GRASSMANN notation to the symbolic form of covariants.

[xii]

The following are the 'heads' of two of OXFORD'S lectures on Quaternions (pp. 479--515). They will show how little of *written* prepared matter (L. and B. Vol. I. p. 9) he took to his lectures.

Whole numbers; scale of them. Steps of addition and subtraction; sum of any number of steps independent of their order.

Multiplication of numbers; in $2 \times 3 = 6$, 2 is an *operator*. Other interpretation; 2 and 3 are both operators; then 6 is one also.

Multiplication of steps; $3 \times (-3) = -9$; only one interpretation as yet.

Retention and reversal of step; symbols k, r ;

$$k3 \times (-3) = -9, r3 \times (-3) = +9.$$

Product of operations on steps; $k3 \times r3 = r6$, $r2 \times r3 = k6$. Product independent of order of factors.

Analogy with multiplication of steps; + and - used for k and r ; double meaning of + and -.

Quantities; all continuous quantities *must* be specified by lines or angles, and angles are conveniently specified by lines. Scale of quantities on straight line.

Ratios of quantities. Ratio as operator. Product of ratios. Double meaning of equation $ab = c$.

Ancient ideas about product of quantities. Arabic solution of quadratic equations. Cardan and Tartaglia. Vieta's scale of dimensions. Introduction of ratios by Descartes. His non-recognition of negative quantities. Geometrical view of product.

Addition and subtraction of quantities. Steps on straight line. Ratio of steps. Signed or *scalar* numbers. Hamilton's view of algebra as science of pure time. Reference to geometry and kinematics implied in all the ordinary algebra.

Steps in plane and space. Addition, subtraction. Multiplication by scalars.

Applications. Theory of mass-centre. Equations of uniform and parabolic motion. Mass-centre of number of falling bodies.

Flux of a vector. Flux of product of vector and scalar. Hodograph. Acceleration. Curvature. Tangential and normal acceleration.

Ratio of steps in one plane. Tensor \times versor. Scalar $+$ rectangular versor.

Distributive, associative, commutative laws.

Complex numbers. Exponential. Meaning of $e^{i\theta} = \cos \theta + i \sin \theta$.

Correspondence of points on the planes by complex function. Equation of n^{th} order has n roots. Systems of orthogonal curves.

Ratio of steps in space. Scalar $+$ rectangular versor.

Representation of rectangular versor, or handle. Addition of two. Associative and commutative laws for addition of three or more. Distributive law for operation on sum of vectors.

Product of rectangular versors. Scalar and versor part. Distributive law of product.

Expression in terms of unit rectangular versors at right angles, I, J, K . Laws of multiplication. Product of two versors. Verification. Quaternions.

Product of two quaternions. Associative law. Spherical theorem equivalent to it.

Comparison of multiplication of versors with multiplication of vectors by versors.

Replacement of I, J, K by i, j, k ; double meaning of certain expressions.

Geometrical view of scalar and vector products. Applications: velocity-system of a rotation; composition of rotations; moment of momentum; work.

Linear function of a vector. Strain, homogeneous. Representation by conic or quadric when irrotational. Strain-flux due to given displacement. Moment of momentum is pure function of rotation. Motion of a body under no forces.

Slope of a function; condensation, convergence, curl. Rotation in liquid is curl of velocity. Force is slope of potential.

INDEX.

This Table applies to the Clifford Papers only, and the references are made generally to the pages, but the numerous Problems are referred to by their number, and listed in Table 2, that number being quoted which gives Clifford's own problem, and the first place on to which the solution is appended.

[illegible]

Answer: Queen

Fourmelt, 433
 Frabin, 239
 Franklin, 530
 Gaines, 56, 60, 63, 271, 526, 529
 Gapsel, 352, 370
 Gardan, 353
 Garsmann, 130, 267, 530
 Gieser, 39, 603
 Gindelfinger, 211, 221
 Habsted, 55
 Hamilton, 269, 406
 Hankel, 399
 Hayward, 292
 Hebedodtz, 407
 Henrich, 117, 118, 307
 Hermite, 117, 221, 223
 Heuss, 119, 200
 Hirst, 51, 313
 Hopkinson, 16
 Jacobi, 121, 205, 208, 467
 Jevons, 1, 10
 Joachimstal, 283
 Klein, 152, 189, 191, 193, 236, 270,
 362
 Kummer, 610
 Ledbaitz, 161, 524
 Lée, 139
 Lindemann, 323
 Lobatschewsky, 101, 631

AUTHORS QUOTED

Litroth, 229, 235
 Magnus, 81
 Malet, 22
 Marey, 431
 Maxwell, 269
 Morin, 428
 Moutard, 552, 610
 Newton, 69
 Peirce, 273, &c.
 Plücker, 89, 43, 121
 Poncelet, 17, 81, 208, 209, 413
 Puiseux, 241
 Purkiss, 45
 Rayleigh, 348
 Riemann, 21, 329, 524
 Roberts, S., 542
 Rosanes, 315
 Rosenhain, 240, 350, 471
 Routh, 34
 Salmon, 27, 30, 33, 35—8, 43, 78,
 117, 127, 228, 229, 302, 306,
 617
 Schröter, 312, 470
 Serret, Paul, 119, 123, 127
 Siebeck, 45
 Smith, H. J. S., 453
 Spottiswoode, 286, &c.
 Steiner, 45
 Stokes, 407
 Sylvester, 20, 119, 124, 192, 196,
 234, 266, 322, 409, 414
 Thomson, W., 234
 Todhunter, 349
 Voss, 300
 Weber, 329, 364
 Wolstenholme, 589

B.

Booth's NEW GEOMETRICAL METHODS,
 Review of, 562—564
 BOUNDARIES, 628
 BRIDE'S CHAIR, 638, 637

C.

CALCULATION

first principles of, 632
 CHAIN (not defined), 201

CHARACTERISTICS

Sylvester's theory of, 132
 Chasles, Fundamental Proposition,
 415

CLOSED CONTOUR, number of roots in,
 528

COMBINATION OF R. J. P., laws of,
 574

COMPLEX NUMBERS

Modulus, Argument of, 526
 addition, multiplication, 527

CONIC—PROPERTIES

In- and circum-scribed polygon
 (Poncelet), 17, 209, 532

Bitangent circles, 543—5, 552

Jacobian of two conics, discriminant
 of, 29

Rectangular hyperbola, &c., 410

CUBANGLE, 414

CUBIC

Jacobian of, 27

properties of, 532—4

F. D. Thomson's, 416

CURVE—PROPERTIES

Double parabola, 47—50; $2n$ -lines
 determine $2n$ -circles meeting in a
 point and for $2n+1$ lines the
 $2n+1$ points so found lie on a
 circle, 52—4; n -fold parabola,
 pedal of with regard to focus, 53

Three-cusped Quartic, 413

Trinodal Quartic, 413

D.

DEFINITIONS

Statement, 1, 14

simple, 1; inverse, 14

compound (two-fold, &c.), 1, 2;
 (pure), 3; (similar), 2

complementary, obverse, 2

type, 2; (complementary), 2;
 (same), 14

marks, 1; (cross-divisions), 356;
 distance of, 2; obverse, 2; ori-
 gin, proximates, mediates, ulti-
 mates, 2

groups, proper, improper, 4

DEFINITIONS

- distance, 11, distance of, 11
 Jacobian, 24, 26, 505
 extended meaning of, 23; polar
 opposite, 24, 57
 Distance, 111, 132, &c., 112, 141-6,
 150, 151, 161, 191, 230, 613
 of nodes, 2; between two points,
 25, 61, 83
 Euclidean Geometry (under project),
 30
 Curve, 30, 306, 307
 meaning of (Plucker), 30
 order, class, deficiency, 40, 41, 131
 ideal tangent (Poncelet), 43
 Polar, harmonic and complement-
 ary, 143; focus, 47
 Cross, Node, 526 (3255)
 Double Parabola, 46
 Cayley Invariant, 54
 Quanta, 57
 Manifolds—continuous, discrete,
 52, 60, 69
 Unboundedness and infinite extent,
 distinction between, 67
 Bifurcal, incanta, ascanta of the,
 73
 Euler, 73
 Octahedral, axis of, 77
 Octahed, 73
 Geometry, Metric, Graphic, 60, 190
 Absolute, The, 61, 63, 132, 163, 177,
 190, 191
 Best, second, 547
 Cosymetric Functions, 65
 Perimeter of a plane triangle, 111
 Plane of three lines and of three
 planes, 111
 Homomorphically situated, 112-4
 Graphic surface, Rank (or line)
 equation, 126
 Polyhedron, 120; Index-surface of,
 121
 Pyramids, 173
 Vertices, 246, 526
 Hamilton's, 101, 197; right, left,
 191
 Maxwell, Grassmann, 197

DEFINITIONS

- Rotom, 162
 addition of, 162, 260
 Motor, 163
 Quaternion, 183, 507, 509
 Tensor-twist, 185
 Biquaternion, 189, 270, 394
 Geometry
 Parabolic, Elliptic, Hyperbolic,
 163, 169, 191, 236, 270
 chromatic, power, 190, 191
 Biquo
 unimodal, algebraic, 180; linear,
 190
 projective-connection of, 180; me-
 tric geometry of, 190
 Binar, 516
 Elasticity, 517
 Parallel, right, left, 193
 Area (Hayward's), 203
 Momentum, 237
 Quadrant, 230, 353
 Surface, 306, 307
 Riemann's, loop, 243; chainwise
 connection, 249; canonical form
 of, 249; transformation of, 249
 354; circuit (reducible, irre-
 ducible, ungatory), 252-4; ca-
 nonical dissection of, 253
 Alternate Numbers, 256
 Bond, 256, 259, Quadratic Form,
 256
 Multiplication (polar, outer, in-
 ner), 266, 267, 495
 Term, form, quantity, 272
 Algebra, linear, n -way geometric,
 273
 Loops, two-way, or two-spread, 306
 Net, 306, 307
 Spheres, power of, 332
 Matrix, 337
 Periodic, optand, 360; half, 360
 Second moment, 378
 Mass-centre, Octahedron, 400
 Symmetry
 of the right angle (or rectangular),
 415
 Quantic, Quintic, 414

DEFINITIONS

- Measurement, 419
 of Time, 421; Force, 421, 437
 Mass, Weight, 421, 436
 Planimeter, 420
 Rigid Bodies, Translation of, 426,
 510, 517
 Harmonic Motion, amplitude, pe-
 riod, epoch, periodic motion, 430,
 517
 Theta series, 443
 Harmonic Centre, 577; Triad, 613
 Anallagmatic, 624 (1929), 625
 principal circles of, 552
 Quadrilateral, self-conjugate, 599
 Conic, 603 (prime, regular, rect-
 angular), 603
 Trizomal curve, 610
 Conic Tangential, 624
 Cycloid, Tore, 625
 DERIVED POINTS, Sylvester's theory of,
 322
 DERIVED SURFACES AND CURVES, theory
 of, 574
 DYNAMICS, 425, 435; velocity, 520,
 (instantaneous), 521

E.

- EDUCATIONAL TIMES, 33
 ELLIPTIC FUNCTIONS, defn. and proper-
 ties of, 443, &c.
 Addition Theorem, 451—5 (Abeli-
 an form of, 457)
 Product of four θ -functions, 453
 of the second kind, 452, 6
 of the third kind, 457
 θ -functions as infinite products, 459
 doubly infinite factorials, Cayley's
 theory of, 463
 linear transformations, problem of,
 446
 Transformation, general problem of
 (Jacobi's Theorem for product of
 n θ -functions), 467
 Grassmann's construction, 459
 Schröter's theorem for product of
 2 θ -functions, 470

ELLIPTIC FUNCTIONS

- Rosenthal's functions and integrals
 of third kind, 471
 Inversion Problem, Jacobi's form of,
 472
 Infinite Series, multiplication of,
 474
 ENERGY, conservation of, 439
 EQUATION, every rational, has a root,
 20
 EQUILATERAL TRIANGLE, projective pro-
 perty of, 412
 "ERGÄNZUNG," 269
 EXPANSION OF $F(x+y)$, 527

F.

FIGURE—PROPERTIES

- Octolateral, 75 (inounts and exounts
 of), 77, 78, (satellites of), 79
 FLUXIONS, 428; elliptic, 431
 FORMULAE
 Adaptation (of) in Analytical Metrics,
 103
 Distance from Plano-conic, 132, &c.
 FOURIER'S THEOREM, 431, 519
 $F(x)=0$, n -roots, 528

G.

- GEOMETRICAL PROBLEMS, order of, 637
 GRAPHS (defn.), 286
 quantification of, 257
 how to find algebraic content of,
 257
 GRASSMANN'S NOTATION, 180

H.

- HEXAHEDRON, Conjugate, 127
 HYPOTHESES, of Continuity, Rigid mo-
 tion, Infinite extent, 524
 Synopsis of, 70

I.

- INVERSION, Tangential, 575

J.

- JACOBIANS (in four variables), 81

DEFINITIONS

- Measurement, 419
 of Time, 421; Force, 421, 437
 Mass, Weight, 421, 436
 Planimeter, 420
 Rigid Bodies, Translation of, 426,
 516, 517
 Harmonic Motion, amplitude, pe-
 riod, epoch, periodic motion, 430,
 517
 Theta series, 443
 Harmonic Centre, 577; Triad, 613
 Anallagmatic, 624 (1929), 625
 principal circles of, 552
 Quadrilateral, self-conjugate, 599
 Confiner, 603 (prime, regular, rect-
 angular), 603
 Trizomal curve, 610
 Conic Tangential, 624
 Cyclide, Toric, 625
 DERIVED POINTS, Sylvester's theory of,
 322
 DERIVED SURFACES AND CURVES, theory
 of, 574
 DYNAMICS, 425, 435; velocity, 520,
 (instantaneous), 521

E.

- EDUCATIONAL TIMES, 33
 ELLIPTIC FUNCTIONS, defn. and proper-
 ties of, 443, &c.
 Addition Theorem, 451—5 (Abeli-
 an form of, 457)
 Product of four θ -functions, 453
 of the second kind, 452, 6
 of the third kind, 457
 θ -functions as infinite products, 459
 doubly infinite factorials, Cayley's
 theory of, 463
 linear transformations, problem of,
 446
 Transformation, general problem of
 (Jacobi's Theorem for product of
 n θ -functions), 467
 Grassmann's construction, 459
 Schröter's theorem for product of
 2 θ -functions, 470

ELLIPTIC FUNCTIONS

- Rosenhain's functions and integrals
 of third kind, 471
 Inversion Problem, Jacobi's form of,
 472
 Infinite Series, multiplication of,
 474
 ENERGY, conservation of, 439
 EQUATION, every rational, has a root,
 20
 EQUILATERAL TRIANGLE, projective pro-
 perty of, 412
 "ERGÄNZUNG," 260
 EXPANSION OF $F'(x+y)$, 527

F.

FIGURE—PROPERTIES

- Octolateral, 75 (incunants and ex-
 cunants of), 77, 78, (satellites of), 79
 FLUXIONS, 428; elliptic, 431
 FORMULAE
 Adaptation (of) in Analytical Metrics,
 103
 Distance from Plane-conic, 132, &c.
 FOURIER'S THEOREM, 431, 519
 $F'(x)=0$, n -roots, 528

G.

- GEOMETRICAL PROBLEMS, order of, 637
 GRAPHS (defn.), 236
 quantification of, 257
 how to find algebraic content of,
 257
 GRASSMANN'S NOTATION, 130

H.

- HEXAHEDRON, Conjugate, 127
 HYPOTHESES, of Continuity, Rigid mo-
 tion, Infinite extent, 524
 Synopsis of, 70

I.

- INVERSION, Tangential, 575

J.

- JACOBIANS (in four variables), 31

A.

Asymptotes, 434, 436
 Asymptotic Curves, 423, 416

B.

Barycentric Coordinates, 434
 Barycentric Coordinates, Theory of, 467
 Barycentric, 434
 Barycentric Coordinates of Equilateral
 Triangle, 412; of Regular Tetra-
 hedron, 411

C.

Cayley, Alfred, Theory of, 24
 Cayley's
 (C, D, Correspondence, 17
 Cayley's Theorem, 491
 Cayley's
 Cayley's, 437, 438
 Cayley's, 439, 417
 Cayley's Theorem, 439, 417
 Cayley's of (Pascal's, 434, 433;
 Cayley's, Cayley, De Hall, 431
 Cayley's Theorem, Cayley's, 433

D.

Determinants, rate of deter-
 mination of, 421
 Determinants, Cayley's plane of, 426

E.

Euclidean, Properties of, 24, &c.

F.

Fermat's, Double, Geometrical pro-
 perty of focus, 17-40; form of foci
 of, which touch the fixed line in a
 circle, 40
 Fermat's, Budget of, Review of,
 439-441
 Fermat's, Perpendicular, 435
 Fermat's, Conjugate, 437
 Fermat's, 435-47; mixed, 438
 Fermat's Theorem, Results in theory of,
 43, 471
 Fermat's on a straight line, Analysis
 of, 435; on a plane, 426

Power-Coordinates in general, 446-
 456
 Power, Theory of, 455-48
 Projective Properties of Equilateral
 Triangle, 412; of Regular Tetra-
 hedron, 411
 Pythagorean Theorem of, 438

Q.

QUANTITY AND MEASURE, 425

R.

Raman, Calculus of, 425
 Raman, Double, 23
 Raman, Theorem
 (Proposed)
 Cayley, 446
 Clifford, 445, 446, 447, 448, 449,
 449, 449, 449, 449, 449, 449, 449,
 449, 449-9, 449-14, 449, 449,
 449, 449-9
 Collins, M., 449
 Crook, 449, 449
 Griffiths, 449
 Hall, 449, 449
 de Jonquieres, 449
 Miller, 449
 Roberts, R., 449
 Sylvester, 449, 449, 449, 449
 Thomson, F. D., 449, 449
 Tucker, R., 449 (1418)
 Whitworth, 449
 Wilkinson, T. T., 449
 Wilson, J. R., 449
 Wolstenholme, 449, 449
 (Subjects)
 Algebra, functions, 420 (1900)
 Probability, 449 (1878)
 Analytical Geometry, triangle, 449
 (1407), 449 (1783)
 quadrangle, 449 (1760)
 Plane Geometry, 449 (1652), 449
 (1433)
 Solid Geometry, 449 (1810), 449
 (1421), 449 (1617), 449 (1775, 1820),
 449 (2108), 449 (4097), 449 (1507,
 1606, 1691), 449 (1748), 449 (3901)

- REPRINT PROBLEMS—*Subjects*
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 Cubics, 567 (1379), 568 (1389), 588 (1486, 1519), 600 (1688, 1888), 608 (2743, 2776), 609 (2793), 611 (2446), 613 (2979, 3885, 3876), 617 (4236), 618 (4641), 620 (4972), 624 (1918), 625 (2510)
 Pedal curves, 574 (1442), 584 (1479), 591 (1394), 621 (5304)
 Quartics, 609 (2932), 610 (2023), 621 (5626), 624 (1929), 625 (2229), 626 (3308)
 General curves, 614 (4069), 626 (3255)
 Epi-hypo-cycloids, 607 (2732), 613 (3197), 614 (4069), 617 (4069)
 Spherical curve (class n), 626 (4819)
 Sphero-conic, 612 (3021)
 Surface, Quadric, 614 (4010, 4034), 618 (4199), 627 (4843, 4923)
 Surface, Cubic, 614 (4034)
 Surface (order n), 611 (2960)
 Elliptic Functions, 627 (5457)
 Cubic Functions, 621 (4871)
 Definite Integral, 623 (1423)
 Matrices, 627 (4950)
 Statics, 571 (1393), 599 (1795), 613 (3197), 621 (4143), 623 (1448), 627 (5457)
 RIEMANN'S PAPER, synopsis of, 70
 RIGID BODIES, Translation of, 426, 516, 517
 ROTATION, 432, 630
 S.
 SHADOWS OF A CIRCLE, 636
 SIMILAR FIGURES, 631
 SIMILARITY OF SMALLEST PARTS, 527
 SLOPES, Physical examples of, 514
 SPHERE, 108
 SURFACES, plano, and STRAIGHT LINES, 629
 SURFACES OF CONSTANT CURVATURE, 65
 SYMMETRY
 Quartic, Quintic, 414
 Rectangular, 415
 T.
 TABLES
 Marks, 3
 Compound statements, Number of types of, 12, 13
 Statements of each type, Number of, 16
 Syzygetic relations, 120, 120
 First, second, third Process (173), 174, 175
 Biquaternions, 188, 404
 Koenigsberger's, 446
 Multiplication of Versors, 506
 Science of Motion, 517
 TETRAHEDRON, Regular, projective, 414; property of, 109, 414
 THOMSON'S TIDAL CLOCK, 431, 518
 TRANSFORMATION
 Cremona's, 538—542
 $z = F(x)$, 527

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